Research Article

Derivation of Bounds of an Integral Operator via Exponentially Convex Functions

Hong Ye,¹ Ghulam Farid ,² Babar Khan Bangash,² and Lulu Cai ¹

¹School of Electrical and Information Engineering, Quzhou University, Quzhou 324000, China

²Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan

Correspondence should be addressed to Lulu Cai; ysu-fbg@163.com

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In this paper, bounds of fractional and conformable integral operators are established in a compact form. By using exponentially convex functions, certain bounds of these operators are derived and further used to prove their boundedness and continuity. A modulus inequality is established for a differentiable function whose derivative in absolute value is exponentially convex. Upper and lower bounds of these operators are obtained in the form of a Hadamard inequality. Some particular cases of main results are also studied.

1. Introduction

We start with the definition of convex function.

Definition 1 (see [1]). A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad (1)$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$. If inequality (1) is reversed, then the function f will be concave on $[a, b]$.

Convex functions are very useful in many areas of mathematics and other subjects due to their fascinating properties and characterizations. Their geometric and analytic interpretations provide straightforward proofs of many mathematical inequalities including Hadamard, Jensen, Hölder, and Minkowski [1–3]. Theoretically, convex functions have been generalized and extended as h -convex, m -convex, s -convex, (α, m) -convex, $(h-m)$ -convex, (s, m) -convex, etc. Awan et al. [4] defined the function named exponentially convex function as follows:

Definition 2. A function $f: K \subseteq \mathbb{R} \rightarrow \mathbb{R}$, where K is an interval, is said to be an exponentially convex function if

$$f(ta + (1-t)b) \leq t \frac{f(a)}{e^{ta}} + (1-t) \frac{f(b)}{e^{tb}}, \quad (2)$$

holds for all $a, b \in K$, $t \in [0, 1]$ and $\alpha \in \mathbb{R}$. If the inequality in (2) is reversed, then f is called exponentially concave.

If $\alpha = 0$, then (2) gives inequality (1). For some recent citations and utilization of exponentially convex functions, one can see [5–14] and references therein. Our goal in this paper is to prove generalized integral inequalities for exponentially convex functions by using integral operators given in Definition 7. In the following, we give definitions of Riemann–Liouville fractional integrals:

Definition 3. Let $f \in L_1[a, b]$. Then, the left-sided and right-sided Riemann–Liouville fractional integral operators of order $\mu \in \mathbb{C}$ ($\Re(\mu) > 0$) are defined by

$${}^{\mu}I_{a+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t) dt, \quad x > a, \quad (3)$$

$${}^{\mu}I_{b-} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t) dt, \quad x < b. \quad (4)$$

A k -fractional analogue is given as follows:

Definition 4 (see [15]). Let $f \in L_1[a, b]$. Then, for $k > 0$, the k -fractional integral operators of f of order $\mu \in \mathbb{C}$, $\Re(\mu) > 0$ are defined by

$${}^{\mu}I_{a+}^k f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{(\mu/k)-1} f(t)dt, \quad x > a, \quad (5)$$

$${}^{\mu}I_{b-}^k f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{(\mu/k)-1} f(t)dt, \quad x < b. \quad (6)$$

A more general definition of the Riemann–Liouville fractional integral operators is given as follows:

Definition 5 (see [16]). Let $f: [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b) . The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order $\mu \in \mathbb{C}$ ($\Re(\mu) > 0$) are defined by

$${}^{\mu}I_{a+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (g(x) - g(t))^{\mu-1} g'(t) f(t)dt, \quad x > a, \quad (7)$$

$${}^{\mu}I_{b-} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (g(t) - g(x))^{\mu-1} g'(t) f(t)dt, \quad x < b, \quad (8)$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 6. Let $f: [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b) . The left-sided and right-sided k -fractional integral operators, $k > 0$, of a function f with respect to another function g on $[a, b]$ of order $\mu, k \in \mathbb{C}$, $\Re(\mu) > 0$ are defined by

$${}^{\mu}I_{a+}^k f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (g(x) - g(t))^{(\mu/k)-1} g'(t) f(t)dt, \quad x > a, \quad (9)$$

$${}^{\mu}I_{b-}^k f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (g(t) - g(x))^{(\mu/k)-1} g'(t) f(t)dt, \quad x < b, \quad (10)$$

where $\Gamma_k(\cdot)$ is the k -gamma function.

A compact form of integral operators defined above is given as follows:

Definition 7 (see [17]). Let $f, g: [a, b] \rightarrow \mathbb{R}$, $0 < a < b$ be the functions such that f be positive and $f \in L_1[a, b]$ and g be differentiable and strictly increasing. Also, let (ϕ/x) be an increasing function on $[a, \infty)$. Then, for $x \in [a, b]$, the left- and right-sided integral operators are defined by

$$(F_{a+}^{\phi, g} f)(x) = \int_a^x K_g(x, t; \phi) f(t) d(g(t)), \quad x > a, \quad (11)$$

$$(F_{b-}^{\phi, g} f)(x) = \int_x^b K_g(t, x; \phi) f(t) d(g(t)), \quad x < b, \quad (12)$$

where $K_g(x, y; \phi) = (\phi(g(x) - g(y))/(g(x) - g(y)))$.

Integral operators defined in (11) and (12) produce several fractional and conformable integral operators defined in [16, 18–25].

Remark 1. Integral operators given in (11) and (12) produce fractional and conformable integral operators as follows:

- (i) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$, then (11) and (12) integral operators coincide with (9) and (10) fractional integral operators.
- (ii) If we consider $\phi(t) = (t^{\mu}/\Gamma(\mu))$, $\mu > 0$, then (11) and (12) integral operators coincide with (7) and (8) fractional integral operators.
- (iii) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and g as identity function (11) and (12), integral operators coincide with (5 and 6) fractional integral operators.
- (iv) If we consider $\phi(t) = (t^{\mu}/\Gamma(\mu))$, $\mu > 0$, and along with g as identity function (11) and (12), integral operators coincide with (3 and 4) fractional integral operators.
- (v) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$, $k = 1$, and $g(x) = (x^{\rho}/\rho)$, $\rho > 0$, then (11) and (12) produce Katugampola fractional integral operators defined by Chen and Katugampola in [18].
- (vi) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$, $k = 1$, and $g(x) = (x^{\tau+s}/(\tau+s))$, $s > 0$, then (11) and (12) produce generalized conformable integral operators defined by Khan and Khan in [22].
- (vii) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = ((x-a)^s/s)$, $s > 0$, in (11) and $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = -((b-x)^s/s)$, $s > 0$, in (12), respectively, then conformable (k, s) -fractional integrals are achieved as defined by Habib et al. in [20].
- (viii) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = (x^{1+s}/(1+s))$, then (11) and (12) produce conformable fractional integrals defined by Sarikaya et al. in [24].
- (ix) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = ((x-a)^s/s)$, $s > 0$, in (11) and $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = -((b-x)^s/s)$, $s > 0$, in (12) with $k = 1$, respectively, then conformable fractional integrals are achieved as defined by Jarad et al. in [21].
- (x) If we consider $\phi(t) = t^{\mu/k} \mathcal{F}_{\rho, \lambda}^{\sigma, k}(w(t)^{\rho})$, then (11) and (12) produce generalized k -fractional integral operators defined by Tunc et al. in [25].
- (xi) If we consider $\phi(t) = (\exp(-At)/\mu)$ and $A = ((1-\mu)/\mu)$, $\mu > 0$, then following generalized fractional integral operators with exponential kernel are obtained [19]:

$$\begin{aligned}
{}^{\mu}E_{a^{+}}f(x) &= \frac{1}{\mu} \int_a^x \exp\left(-\frac{1-\mu}{\mu}(g(x)-g(t))\right) f(t) dt, \quad x > a, \\
{}^{\mu}E_{b^{-}}f(x) &= \frac{1}{\mu} \int_x^b \exp\left(-\frac{1-\mu}{\mu}(g(x)-g(t))\right) f(t) dt, \quad x < b.
\end{aligned} \tag{13}$$

- (xii) If we consider $\phi(t) = (t^{\mu}/\Gamma(\mu))$ and $g(t) = \ln t$, then Hadamard fractional integral operators will be obtained [16, 23].
- (xiii) If we consider $\phi(t) = (t^{\mu}/\Gamma(\mu))$ and $g(t) = -t^{-1}$, then Harmonic fractional integral operators given in [16] will be obtained and given as follows:

$$\begin{aligned}
{}^{\mu}R_{a^{+}}f(x) &= \frac{t^{\mu}}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} \frac{f(t)}{t^{\mu+1}} dt, \quad x > a, \\
{}^{\mu}R_{b^{-}}f(x) &= \frac{t^{\mu}}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} \frac{f(t)}{t^{\mu+1}} dt, \quad x < b.
\end{aligned} \tag{14}$$

- (xiv) If we consider $\phi(t) = t^{\mu} \ln t$, then left- and right-sided logarithmic fractional integrals are obtained in [19] and given as follows:

$$\begin{aligned}
{}^{\mu}\mathcal{L}_{a^{+}}f(x) &= \int_a^x (g(x)-g(t))^{\mu-1} \ln(g(x) \\
&\quad - g(t)) g'(t) dt, \quad x > a, \\
{}^{\mu}\mathcal{L}_{b^{-}}f(x) &= \int_x^b (g(t)-g(x))^{\mu-1} \ln(g(x) \\
&\quad - g(t)) g'(t) dt, \quad x < b.
\end{aligned} \tag{15}$$

In the upcoming section, we will derive bounds of sum of the left- and right-sided integral operators defined in (11) and (12) for exponentially convex functions. These bounds lead to produce results associated to several kinds of well-known operators for exponentially convex functions, some of the results are presented in particular cases. Further in Section 3, bounds are presented in the form of a Hadamard inequality; several Hadamard type inequalities are obtained.

2. Bounds of Integral Operators and Their Consequences

Theorem 1. Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive and exponentially convex function and $g: [a, b] \rightarrow \mathbb{R}$ be a differentiable and strictly increasing function. Also, let (ϕ/x) be an increasing function on $[a, b]$. Then, for $x \in [a, b]$ the following inequality for integral operators (11) and (12) holds

$$\begin{aligned}
& (F_{a^{+}}^{\phi, g} f)(x) + (F_{b^{-}}^{\phi, g} f)(x) \\
& \leq \phi(g(x) - g(a)) \left(\frac{f(x)}{e^{\alpha x}} + \frac{f(a)}{e^{\alpha a}} \right) + \phi(g(b) \\
& \quad - g(x)) \left(\frac{f(x)}{e^{\alpha x}} + \frac{f(b)}{e^{\alpha b}} \right).
\end{aligned} \tag{16}$$

Proof. For the kernel of integral operator (11), we have

$$K_g(x, t; \phi) g'(t) \leq K_g(x, a; \phi) g'(t), \quad x \in (a, b] \text{ and } t \in [a, x]. \tag{17}$$

An exponentially convex function satisfies the following inequality:

$$f(t) \leq \left(\frac{x-t}{x-a} \right) \frac{f(a)}{e^{\alpha a}} + \left(\frac{t-a}{x-a} \right) \frac{f(x)}{e^{\alpha x}}. \tag{18}$$

Inequalities (17) and (18) lead to the following integral inequality:

$$\begin{aligned}
& \int_a^x K_g(x, t; \phi) g'(t) f(t) dt \\
& \leq K_g(x, a; \phi) \left(\frac{f(a)}{e^{\alpha a}} \int_a^x \left(\frac{x-t}{x-a} \right) g'(t) dt \right. \\
& \quad \left. + \frac{f(x)}{e^{\alpha x}} \int_a^x \left(\frac{t-a}{x-a} \right) g'(t) dt \right),
\end{aligned} \tag{19}$$

while (19) gives

$$(F_{a^{+}}^{\phi, g} f)(x) \leq \phi(g(x) - g(a)) \left(\frac{f(x)}{e^{\alpha x}} + \frac{f(a)}{e^{\alpha a}} \right). \tag{20}$$

Again, for the kernel of integral operator (12), we have

$$K_g(t, x; \phi) g'(t) \leq K_g(b, x; \phi) g'(t), \quad t \in (x, b] \text{ and } x \in [a, b]. \tag{21}$$

An exponentially convex function satisfies the following inequality:

$$f(t) \leq \left(\frac{t-x}{b-x} \right) \frac{f(b)}{e^{\alpha b}} + \left(\frac{b-t}{b-x} \right) \frac{f(x)}{e^{\alpha x}}. \tag{22}$$

Inequalities (21) and (22) lead to the following integral inequality:

$$\begin{aligned}
& \int_x^b K_g(t, x; \phi) g'(t) f(t) dt \\
& \leq K_g(b, x; \phi) \left(\frac{f(b)}{e^{\alpha b}} \int_x^b \left(\frac{t-x}{b-x} \right) g'(t) dt \right. \\
& \quad \left. + \frac{f(x)}{e^{\alpha x}} \int_x^b \left(\frac{b-t}{b-x} \right) g'(t) dt \right),
\end{aligned} \tag{23}$$

while (23) gives

$$(F_{b^{-}}^{\phi, g} f)(x) \leq \phi(g(b) - g(x)) \left(\frac{f(x)}{e^{\alpha x}} + \frac{f(b)}{e^{\alpha b}} \right). \tag{24}$$

By adding (20) and (24), (16) can be achieved.

The following remark connected the abovementioned theorem with already known results. $\square$

Remark 2

- (1) For $\phi(t) = (t^{\mu}/\Gamma(\mu))$, $\mu > 0$, and $\alpha = 0$ in (16), Corollary 1 in [26] can be achieved.
- (2) For $\phi(t) = (t^{\mu}/\Gamma(\mu))$, $\mu > 0$, $g(x) = x$, and $\alpha = 0$ in (16), Corollary 1 in [27] can be achieved.

(3) For $\alpha = 0$, in (16), Theorem 1 in [28] can be achieved.

Next results indicate upper bounds of several known fractional and conformable integral operators.

Proposition 1. Let $\phi(t) = (t^\mu/\Gamma(\mu))$, $\mu > 0$. Then, (11) and (12) produce the fractional integral operators (7) and (8) as follows:

$$\begin{aligned} \left(F_{a^+}^{(t^\mu/\Gamma(\mu)),g} f\right)(x) &:= {}_g^\mu I_{a^+} f(x), \\ \left(F_{b^-}^{(t^\mu/\Gamma(\mu)),g} f\right)(x) &:= {}_g^\mu I_{b^-} f(x). \end{aligned} \quad (25)$$

Further they satisfy the following bound for $\mu \geq 1$:

$$\begin{aligned} &({}_g^\mu I_{a^+} f)(x) + ({}_g^\mu I_{b^-} f)(x) \\ &\leq \frac{(g(x) - g(a))^\mu}{\Gamma(\mu)} \left(\frac{f(x)}{e^{ax}} + \frac{f(a)}{e^{aa}} \right) \\ &\quad + \frac{(g(b) - g(x))^\mu}{\Gamma(\mu)} \left(\frac{f(x)}{e^{ax}} + \frac{f(b)}{e^{ab}} \right). \end{aligned} \quad (26)$$

Proposition 2. Let $g(x) = I(x) = x$. Then, (11) and (12) produce integral operators defined in [29] as follows:

$$\left(F_{a^+}^{\phi,I} f\right)(x) := ({}_a I_\phi f)(x) = \int_a^x \frac{\phi(x-t)}{(x-t)} f(t) dt, \quad (27)$$

$$\left(F_{b^-}^{\phi,I} f\right)(x) := ({}_b I_\phi f)(x) = \int_x^b \frac{\phi(t-x)}{(t-x)} f(t) dt.$$

Further they satisfy the following bound:

$$\begin{aligned} &({}_a I_\phi f)(x) + ({}_b I_\phi f)(x) \\ &\leq \phi(x-a) \left(\frac{f(x)}{e^{ax}} + \frac{f(a)}{e^{aa}} \right) + \phi(b-x) \left(\frac{f(x)}{e^{ax}} + \frac{f(b)}{e^{ab}} \right). \end{aligned} \quad (28)$$

Corollary 1. If we take $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$, then (11) and (12) produce the fractional integral operators (9) and (10) as follows:

$$\begin{aligned} \left(F_{a^+}^{(t^{\mu/k}/k\Gamma_k(\mu)),g} f\right)(x) &:= {}_g^\mu I_{a^+}^k f(x), \\ \left(F_{b^-}^{(t^{\mu/k}/k\Gamma_k(\mu)),g} f\right)(x) &:= {}_g^\mu I_{b^-}^k f(x). \end{aligned} \quad (29)$$

From (16), the following bound holds for $\mu \geq k$:

$$\begin{aligned} &({}_g^\mu I_{a^+}^k f)(x) + ({}_g^\mu I_{b^-}^k f)(x) \leq \frac{(g(x) - g(a))^{\mu/k}}{k\Gamma_k(\mu)} \left(\frac{f(x)}{e^{ax}} + \frac{f(a)}{e^{aa}} \right) \\ &\quad + \frac{(g(b) - g(x))^{\mu/k}}{k\Gamma_k(\mu)} \left(\frac{f(x)}{e^{ax}} + \frac{f(b)}{e^{ab}} \right). \end{aligned} \quad (30)$$

Corollary 2. If we take $\phi(t) = (t^\mu/\Gamma(\mu))$, $\mu > 0$, and $g(x) = I(x) = x$, then (11) and (12) produce left- and right-sided Riemann–Liouville fractional integral operators (3) and (4) as follows:

$$\begin{aligned} \left(F_{a^+}^{(t^\mu/\Gamma(\mu)),I} f\right)(x) &:= {}^\mu I_{a^+} f(x), \\ \left(F_{b^-}^{(t^\mu/\Gamma(\mu)),I} f\right)(x) &:= {}^\mu I_{b^-} f(x). \end{aligned} \quad (31)$$

From (16), the following bound holds for $\mu \geq 1$:

$$\begin{aligned} &({}^\mu I_{a^+} f)(x) + ({}^\mu I_{b^-} f)(x) \\ &\leq \frac{(x-a)^\mu}{\Gamma(\mu)} \left(\frac{f(x)}{e^{ax}} + \frac{f(a)}{e^{aa}} \right) + \frac{(b-x)^\mu}{\Gamma(\mu)} \left(\frac{f(x)}{e^{ax}} + \frac{f(b)}{e^{ab}} \right). \end{aligned} \quad (32)$$

Corollary 3. If we take $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = I(x) = x$, then (11) and (12) produce the fractional integral operators (5) and (6) as follows:

$$\begin{aligned} \left(F_{a^+}^{(t^{\mu/k}/k\Gamma_k(\mu)),I} f\right)(x) &:= {}^\mu I_{a^+}^k f(x), \\ \left(F_{b^-}^{(t^{\mu/k}/k\Gamma_k(\mu)),I} f\right)(x) &:= {}^\mu I_{b^-}^k f(x). \end{aligned} \quad (33)$$

From (16), the following bound holds for $\mu \geq k$:

$$\begin{aligned} &({}^\mu I_{a^+}^k f)(x) + ({}^\mu I_{b^-}^k f)(x) \\ &\leq \frac{(x-a)^{\mu/k}}{k\Gamma_k(\mu)} \left(\frac{f(x)}{e^{ax}} + \frac{f(a)}{e^{aa}} \right) + \frac{(b-x)^{\mu/k}}{k\Gamma_k(\mu)} \left(\frac{f(x)}{e^{ax}} + \frac{f(b)}{e^{ab}} \right). \end{aligned} \quad (34)$$

Corollary 4. If we take $\phi(t) = (t^\mu/\Gamma(\mu))$, $\mu > 0$ and $g(x) = (x^\rho/\rho)$, $\rho > 0$, then (11) and (12) produce the fractional integral operators defined in [18] as follows:

$$\begin{aligned} \left(F_{a^+}^{(t^\mu/\Gamma(\mu)),g} f\right)(x) &= ({}^\rho I_{a^+}^\mu f)(x) \\ &= \frac{\rho^{1-\mu}}{\Gamma(\mu)} \int_a^x (x^\rho - t^\rho)^{\mu-1} t^{\rho-1} f(t) dt, \\ \left(F_{b^-}^{(t^\mu/\Gamma(\mu)),g} f\right)(x) &= ({}^\rho I_{b^-}^\mu f)(x) \\ &= \frac{\rho^{1-\mu}}{\Gamma(\mu)} \int_x^b (t^\rho - x^\rho)^{\mu-1} t^{\rho-1} f(t) dt. \end{aligned} \quad (35)$$

From (16), they satisfy the following bound:

$$\begin{aligned} &({}^\rho I_{a^+}^\mu f)(x) + ({}^\rho I_{b^-}^\mu f)(x) \\ &\leq \frac{(x^\rho - a^\rho)^\mu}{\Gamma(\mu)(\rho^\mu)} \left(\frac{f(x)}{e^{ax}} + \frac{f(a)}{e^{aa}} \right) + \frac{(b^\rho - x^\rho)^\mu}{\Gamma(\mu)(\rho^\mu)} \left(\frac{f(x)}{e^{ax}} + \frac{f(b)}{e^{ab}} \right). \end{aligned} \quad (36)$$

Corollary 5. If we take $\phi(t) = (t^\mu/\Gamma(\mu))$, $\mu > 0$, and $g(x) = (x^{n+1}/(n+1))$, $n > 0$, then (11) and (12) produce the fractional integral operators defined as follows:

$$\begin{aligned} \left(F_{a^+}^{(\mu/\Gamma(\mu)),g} f\right)(x) &= ({}^n I_{a^+}^\mu f)(x) \\ &= \frac{(n+1)^{1-\mu}}{\Gamma(\mu)} \int_a^x (x^{n+1} - t^{n+1})^{\mu-1} t^n f(t) dt, \\ \left(F_b^{(\mu/\Gamma(\mu)),g} f\right)(x) &= ({}^n I_b^\mu f)(x) \\ &= \frac{(n+1)^{1-\mu}}{\Gamma(\mu)} \int_x^b (t^{n+1} - x^{n+1})^{\mu-1} t^n f(t) dt. \end{aligned} \quad (37)$$

From (16), they satisfy the following bound:

$$\begin{aligned} &({}^n I_{a^+}^\mu f)(x) + ({}^n I_b^\mu f)(x) \\ &\leq \frac{(x^{n+1} - a^{n+1})^\mu}{\Gamma(\mu)(n+1)^\mu} \left(\frac{f(x)}{e^{ax}} + \frac{f(a)}{e^{aa}} \right) \\ &\quad + \frac{(x^{n+1} - a^{n+1})^\mu}{\Gamma(\mu)(n+1)^\mu} \left(\frac{f(x)}{e^{ax}} + \frac{f(b)}{e^{ab}} \right). \end{aligned} \quad (38)$$

Next, we prove boundedness and continuity of integral operators.

Theorem 2. Let the assumptions of Theorem 1 are satisfied. If $f \in L_\infty[a, b]$, then integral operators defined in (11) and (12) are continuous.

Proof. From (20), we have

$$\left(F_{a^+}^{\phi,g} f\right)(x) \leq \phi(g(x) - g(a)) \|f\|_\infty \left(\frac{1}{e^{ax}} + \frac{1}{e^{aa}} \right). \quad (39)$$

It is given that (ϕ/x) is increasing on $[a, b]$:

$$\frac{\phi(g(x) - g(a))}{g(x) - g(a)} \leq \frac{\phi(g(b) - g(a))}{g(b) - g(a)}, \quad (40)$$

$$\phi(g(x) - g(a)) \leq \frac{\phi(g(b) - g(a))g(x) - g(a)}{g(b) - g(a)}.$$

Further g is increasing; therefore, we have

$$\phi(g(x) - g(a)) \leq \phi(g(b) - g(a)). \quad (41)$$

Therefore, (39) gives

$$\left(F_{a^+}^{\phi,g} f\right)(x) \leq \phi(g(b) - g(a)) \|f\|_\infty (e^{-ax} + e^{-aa}). \quad (42)$$

If $\alpha > 0$, then e^{-ax} is decreasing on $[a, b]$ and we get

$$\left(F_{a^+}^{\phi,g} f\right)(x) \leq 2e^{-aa} \phi(g(b) - g(a)) \|f\|_\infty. \quad (43)$$

If $\alpha < 0$, then e^{-ax} is increasing on $[a, b]$ and we get from (42)

$$\left(F_{a^+}^{\phi,g} f\right)(x) \leq 2e^{-ab} \phi(g(b) - g(a)) \|f\|_\infty. \quad (44)$$

Hence, $(F_{a^+}^{\phi,g} f)(x)$ is bounded and it is linear, and therefore, $(F_{a^+}^{\phi,g} f)(x)$ is continuous.

Similarly, continuity of $(F_b^{\phi,g} f)(x)$ can be proved.

For a differentiable function f , as $|f'|$ is exponentially convex, the following result holds: $\square$

Theorem 3. Let $f: I \rightarrow \mathbb{R}$ be a differentiable function. If $|f'|$ is exponentially convex and $g: I \rightarrow \mathbb{R}$ is a differentiable and strictly increasing function. Also, let (ϕ/x) be an increasing function on I , then for $a, b \in I$, $a < b$, the following inequalities for integral operators holds:

$$\left|F_{a^+}^{\phi,g}(f * g)(x)\right| \leq \phi(g(x) - g(a)) \left(\frac{|f'(a)|}{e^{aa}} + \frac{|f'(x)|}{e^{ax}} \right), \quad (45)$$

$$\left|F_b^{\phi,g}(f * g)(x)\right| \leq \phi(g(b) - g(x)) \left(\frac{|f'(b)|}{e^{ab}} + \frac{|f'(x)|}{e^{ax}} \right), \quad (46)$$

where

$$\begin{aligned} F_{a^+}^{\phi,g}(f * g)(x) &= \int_a^x K_g(x, t; \phi) g'(t) f'(t) dt, \\ F_b^{\phi,g}(f * g)(x) &= \int_x^b K_g(t, x; \phi) g'(t) f'(t) dt. \end{aligned} \quad (47)$$

Proof. An exponentially convex function $|f'|$ satisfies the following inequality:

$$|f'(t)| \leq \left(\frac{x-t}{x-a} \right) \frac{|f'(a)|}{e^{aa}} + \left(\frac{t-a}{x-a} \right) \frac{|f'(x)|}{e^{ax}}. \quad (48)$$

From which, we can write

$$f'(t) \leq \left(\frac{x-t}{x-a} \right) \frac{|f'(a)|}{e^{aa}} + \left(\frac{t-a}{x-a} \right) \frac{|f'(x)|}{e^{ax}}. \quad (49)$$

Inequalities (17) and (49) lead to the following integral inequality:

$$\begin{aligned} &\int_a^x K_g(x, t; \phi) g'(t) f'(t) dt \\ &\leq K_g(x, a; \phi) \left(\frac{|f'(a)|}{e^{aa}} \int_a^x \left(\frac{x-t}{x-a} \right) g'(t) dt \right. \\ &\quad \left. + \left(\frac{|f'(x)|}{e^{ax}} \right) \int_a^x \left(\frac{t-a}{x-a} \right) g'(t) dt \right), \end{aligned} \quad (50)$$

while (50) gives

$$F_{a^+}^{\phi,g}(f * g)(x) \leq \phi(g(x) - g(a)) \left[\left(\frac{|f'(a)|}{e^{aa}} + \frac{|f'(x)|}{e^{ax}} \right) \right]. \quad (51)$$

From (48), we can write

$$f'(t) \geq - \left(\left(\frac{x-t}{x-a} \right) \frac{|f'(a)|}{e^{aa}} + \left(\frac{t-a}{x-a} \right) \frac{|f'(x)|}{e^{ax}} \right). \quad (52)$$

Adopting the same method as we did for (49), the following integral inequality holds:

$$F_{a^+}^{\phi, g}(f * g)(x) \geq -\phi(g(x) - g(a)) \left(\frac{|f'(a)|}{e^{aa}} + \frac{|f'(x)|}{e^{ax}} \right). \quad (53)$$

From (51) and (53), (45) can be achieved.

An exponentially convex function $|f'|$ satisfies the following inequality:

$$|f'(t)| \leq \left(\frac{t-x}{b-x} \right) \frac{|f'(b)|}{e^{ab}} + \left(\frac{b-t}{b-x} \right) \frac{|f'(x)|}{e^{ax}}. \quad (54)$$

From which, we can write

$$f'(t) \leq \left(\frac{t-x}{b-x} \right) \frac{|f'(b)|}{e^{ab}} + \left(\frac{b-t}{b-x} \right) \frac{|f'(x)|}{e^{ax}}. \quad (55)$$

Inequalities (21) and (55) lead the following integral inequality:

$$\begin{aligned} & \int_x^b K_g(t, x; \phi) g'(t) f'(t) dt \\ & \leq K_g(b, x; \phi) \left(\frac{|f'(b)|}{e^{ab}} \int_x^b \left(\frac{x-t}{b-x} \right) g'(t) dt \right. \\ & \quad \left. + \frac{|f'(x)|}{e^{ax}} \int_x^b \left(\frac{b-t}{b-x} \right) g'(t) dt \right), \end{aligned} \quad (56)$$

while (56) gives

$$F_{b^-}^{\phi, g}(f * g)(x) \leq \phi(g(b) - g(x)) \left(\frac{|f'(b)|}{e^{ab}} + \frac{|f'(x)|}{e^{ax}} \right). \quad (57)$$

From (54), we can write

$$f'(t) \geq - \left(\left(\frac{t-x}{b-x} \right) \frac{|f'(b)|}{e^{ab}} + \left(\frac{b-t}{b-x} \right) \frac{|f'(x)|}{e^{ax}} \right). \quad (58)$$

Adopting the same method as we did for (55), the following inequality holds:

$$F_{b^-}^{\phi, g}(f * g)(x) \geq -\phi(g(b) - g(x)) \left(\frac{|f'(b)|}{e^{ab}} + \frac{|f'(x)|}{e^{ax}} \right). \quad (59)$$

From (57) and (59), (46) can be achieved. $\square$

3. Hadamard Type Inequalities for Exponentially Convex Function

In this section, we prove the Hadamard type inequality for an exponentially convex function. In order to prove this inequality result, we need the following lemma.

Lemma 1 (see [30]). *Let $f: [a, b] \rightarrow \mathbb{R}$ be an exponentially convex function. If f is exponentially symmetric, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(x)}{e^{ax}}, \quad x \in [a, b]. \quad (60)$$

Theorem 4. *Let $f: [a, b] \rightarrow \mathbb{R}$ be positive, exponentially convex, and symmetric about $((a+b)/2)$ and $g: [a, b] \rightarrow \mathbb{R}$ be a differentiable and strictly increasing function. Also, let (ϕ/x) be an increasing function on $[a, b]$. Then, for $a, b \in I$, $a < b$, the following estimations of Hadamard type are valid.*

$$\begin{aligned} & h(\alpha) f\left(\frac{a+b}{2}\right) \left((F_{b^-}^{\phi, g}(1))(a) + (F_{a^+}^{\phi, g}(1))(b) \right) \\ & \leq \left((F_{b^-}^{\phi, g} f)(a) + (F_{a^+}^{\phi, g} f)(b) \right) \\ & \leq 2\phi(g(b) - g(a)) \left(\frac{f(b)}{e^{ab}} + \frac{f(a)}{e^{aa}} \right), \end{aligned} \quad (61)$$

where $h(\alpha) = e^{ab}$ for $\alpha < 0$ and $h(\alpha) = e^{aa}$ for $\alpha \geq 0$.

Proof. For the kernel of integral operator (11), we have

$$K_g(x, a; \phi) g'(x) \leq K_g(b, a; \phi) g'(x), \quad x \in (a, b]. \quad (62)$$

An exponentially convex function satisfies the following inequality:

$$f(x) \leq \left(\frac{x-a}{b-a} \right) \frac{f(b)}{e^{ab}} + \left(\frac{b-x}{b-a} \right) \frac{f(a)}{e^{aa}}. \quad (63)$$

Inequalities (62) and (63) lead the following integral inequality:

$$\begin{aligned} & \int_a^b K_g(x, a; \phi) g'(x) f(x) dx \\ & \leq K_g(b, a; \phi) \left(\frac{f(b)}{e^{ab}} \int_a^b \left(\frac{x-a}{b-a} \right) g'(x) dx \right. \\ & \quad \left. + \frac{f(a)}{e^{aa}} \int_a^b \left(\frac{b-x}{b-a} \right) g'(x) dx \right), \end{aligned} \quad (64)$$

while (64) gives

$$(F_{b^-}^{\phi, g} f)(a) \leq \phi(g(b) - g(a)) \left(\frac{f(b)}{e^{ab}} + \frac{f(a)}{e^{aa}} \right). \quad (65)$$

On the contrary, for the kernel of integral operator (12), we have

$$K_g(b, x; \phi) g'(x) \leq K_g(b, a; \phi) g'(x). \quad (66)$$

Inequalities (63) and (66) lead the following integral inequality:

$$\begin{aligned} & \int_a^b K_g(b, x; \phi) g'(x) f(x) dx \\ & \leq K_g(b, a; \phi) \left(\frac{f(b)}{e^{ab}} \int_a^b \left(\frac{x-a}{b-a} \right) g'(x) dx \right. \\ & \quad \left. + \frac{f(a)}{e^{aa}} \int_a^b \left(\frac{b-x}{b-a} \right) g'(x) dx \right), \end{aligned} \quad (67)$$

while the abovementioned inequality gives

$$(F_{a^+}^{\phi, g} f)(b) \leq \phi(g(b) - g(a)) \left(\frac{f(b)}{e^{ab}} + \frac{f(a)}{e^{aa}} \right). \quad (68)$$

From (65) and (68), the following inequality can be obtained:

$$(F_{a^+}^{\phi, g} f)(b) + (F_{b^-}^{\phi, g} f)(a) \leq 2\phi(g(b) - g(a)) \left(\frac{f(b)}{e^{ab}} + \frac{f(a)}{e^{aa}} \right). \quad (69)$$

Now, using Lemma 1 and multiplying (60) with $K_g(x, a; \phi) g'(x)$, then integrating over $[a, b]$, we have

$$\begin{aligned} & \int_a^b K_g(x, a; \phi) f\left(\frac{a+b}{2}\right) g'(x) dx \\ & \leq \int_a^b \frac{1}{e^{\alpha x}} K_g(x, a; \phi) g'(x) f(x) dx. \end{aligned} \quad (70)$$

From which, we have

$$f\left(\frac{a+b}{2}\right) (F_{b^-}^{\phi, g}(1))(a) \leq \frac{1}{h(\alpha)} (F_{b^-}^{\phi, g} f)(a). \quad (71)$$

Again using Lemma 1 and multiplying (60) with $K_g(b, x; \phi) g'(x)$, then integrating over $[a, b]$, we have

$$\begin{aligned} & \int_a^b K_g(b, x; \phi) f\left(\frac{a+b}{2}\right) g'(x) dx \\ & \leq \int_a^b \frac{1}{e^{\alpha x}} K_g(b, x; \phi) g'(x) f(x) dx. \end{aligned} \quad (72)$$

From which, we have

$$f\left(\frac{a+b}{2}\right) (F_{a^+}^{\phi, g}(1))(b) \leq \frac{1}{h(\alpha)} (F_{a^+}^{\phi, g} f)(b). \quad (73)$$

From (71) and (73), the following inequality can be achieved:

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) (F_{b^-}^{\phi, g}(1))(a) + (F_{a^+}^{\phi, g}(1))(b) \\ & \leq \frac{1}{h(\alpha)} ((F_{b^-}^{\phi, g} f)(a) + (F_{a^+}^{\phi, g} f)(b)). \end{aligned} \quad (74)$$

From (69) and (74), (61) can be achieved. $\square$

Remark 3. For $\alpha = 0$, in (61), Theorem 3 in [28] can be achieved.

Corollary 6. If we put $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$, then the inequality (61) produces the following Hadamard type inequality:

$$\begin{aligned} & h(\alpha) f\left(\frac{a+b}{2}\right) \left({}^{\mu}I_{b^-}^k(1)(a) + {}^{\mu}I_{a^+}^k(1)(b) \right) \\ & \leq \left({}^{\mu}I_{b^-}^k f(a) + {}^{\mu}I_{a^+}^k f(b) \right) \\ & \leq \frac{2(g(b) - g(a))^{\mu/k}}{(k\Gamma_k(\mu))} \left(\frac{f(b)}{e^{ab}} + \frac{f(a)}{e^{aa}} \right). \end{aligned} \quad (75)$$

Corollary 7. If we put $\phi(t) = (t^{\mu}/\Gamma(\mu))$, then the inequality (61) produces the following Hadamard type inequality:

$$\begin{aligned} & h(\alpha) f\left(\frac{a+b}{2}\right) \left({}^{\mu}I_{b^-}(1)(a) + {}^{\mu}I_{a^+}(1)(b) \right) \\ & \leq \left({}^{\mu}I_{b^-} f(a) + {}^{\mu}I_{a^+} f(b) \right) \\ & \leq \frac{2(g(b) - g(a))^{\mu}}{\Gamma(\mu)} \left(\frac{f(b)}{e^{ab}} + \frac{f(a)}{e^{aa}} \right). \end{aligned} \quad (76)$$

Corollary 8. If we put $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and g as identity function, then the inequality (61) produces the following Hadamard type inequality:

$$\begin{aligned} & h(\alpha) f\left(\frac{a+b}{2}\right) \left({}^{\mu}I_{b^-}^k(1)(a) + {}^{\mu}I_{a^+}^k(1)(b) \right) \\ & \leq \left({}^{\mu}I_{b^-}^k f(a) + {}^{\mu}I_{a^+}^k f(b) \right) \\ & \leq \frac{2(b-a)^{\mu/k}}{k\Gamma_k(\mu)} \left(\frac{f(b)}{e^{ab}} + \frac{f(a)}{e^{aa}} \right). \end{aligned} \quad (77)$$

Corollary 9. If we put $\phi(t) = (t^{\mu}/\Gamma(\mu))$ and g as identity function, then the inequality (61) produces the following Hadamard type inequality:

$$\begin{aligned} & h(\alpha) f\left(\frac{a+b}{2}\right) \left({}^{\mu}I_{b^-}(1)(a) + {}^{\mu}I_{a^+}(1)(b) \right) \\ & \leq \left({}^{\mu}I_{b^-} f(a) + {}^{\mu}I_{a^+} f(b) \right) \\ & \leq \frac{2(b-a)}{\Gamma(\mu)} \left(\frac{f(b)}{e^{ab}} + \frac{f(a)}{e^{aa}} \right). \end{aligned} \quad (78)$$

4. Concluding Remarks

We have studied an integral operator for exponentially convex functions; this operator has direct consequences to several fractional and conformable integral operators. We have obtained bounds of the integral operator in different forms. In Theorem 1, upper bounds of this operator are studied for an exponentially convex function and several

special cases have been presented in the form of propositions and corollaries. The boundedness is studied in Theorem 2. In Theorem 3, we have obtained results for differentiable function f such that $|f'|$ is exponentially convex. A version of the Hadamard inequality is proved in Theorem 4 which leads to its several variants for fractional and conformable integral operators.

Data Availability

No data were used to support this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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