Hindawi
Journal of Mathematics
Volume 2020, Article ID 2460702, 9 pages
https://doi.org/10.1155/2020/2460702

Research Article

Cyclic b-Multiplicative (A, B)-Hardy–Rogers-Type Local Contraction and Related Results in b-Multiplicative and b-Metric Spaces

Abdullah Eqal Al-Mazrooei[1], **Abdullah Shoaib**[2], **and Jamshaid Ahmad**[1]

[1] *Department of Mathematics, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematics & Statistics, Riphah International University, Islamabad, Pakistan*

Correspondence should be addressed to Jamshaid Ahmad; jamshaid_jasim@yahoo.com

Received 2 June 2020; Revised 28 July 2020; Accepted 10 August 2020; Published 12 October 2020

Academic Editor: Nan-Jing Huang

The aim of this paper is to define cyclic b-multiplicative Hardy–Rogers-type local contraction in the context of generalized spaces named as b-multiplicative spaces to extend various results of the literature including the main results of Yamaod et al. In this way, we apply a new generalized contractive condition only on a closed set instead of a whole set and by using b-multiplicative space instead of multiplicative metric space. We apply our results to obtain new results in b-metric spaces. Examples are given to show the usability of our results, when others cannot.

1. Introduction and Preliminaries

Bakhtin [1] was the first who gave the idea of b-metric. After that, Czerwik [2] gave an axiom and formally defined a b-metric space. For further results on the b-metric space, see [3, 4]. Ozaksar and Cevical [5] investigated the multiplicative metric space and proved its topological properties. Mongkolkeha and Sintunavarat [6] described the concept of multiplicative proximal contraction mapping and proved some results for such mappings. Recently, Abbas et al. [7] proved some common fixed point results of quasi-weak commutative mappings on a closed ball in the setting of multiplicative metric spaces. For further results on the multiplicative metric space, see [8–12]. In 2017, Ali et al. [13] introduced the notion of the b-multiplicative space and proved some fixed point results. As an application, they established an existence theorem for the solution of a system of Fredholm multiplicative integral equations. For further results on the b-multiplicative space, see [14]. Shoaib et al. [4] discussed some results for the mappings satisfying contraction condition on a closed ball in a b-metric space. For further results on a closed ball, see [15–25]. In this paper, we generalized the results in [12] by using cyclic b-multiplicative (A, B)-Hardy–Rogers-type local contraction on a closed ball in a b-multiplicative space. Moreover, we show that our results can be applied on those mappings where the other results cannot be applied. The following definitions and results will be used to understand this paper.

Definition 1 (see [13]). Let W be a nonempty set, and let $s \geq 1$ be a given real number. A mapping $m_b\colon W \times W \longrightarrow [1, \infty)$ is called b-multiplicative with coefficient s, if the following conditions hold:

(i) $m_b(w, \mu) > 1$ for all $w, \mu \in W$ with $w \neq \mu$ and $m_b(w, \mu) = 1$ if and only if $w = \mu$

(ii) $m_b(w, \mu) = m_b(\mu, w)$ for all $w, \mu \in W$

(iii) $m_b(w, z) \leq [m_b(w, \mu).m_b(\mu, z)]^s$ for all $w, \mu, z \in W$

The pair (W, m_b) is called a b-multiplicative space. If $r > 1$, $u \in W$, then $\overline{B_{m_b}(u, r)} = \{v\colon m_b(u, v) \leq r\}$ and $B_{m_b}(u, r) = \{v\colon m_b(u, v) < r\}$ are called the closed ball and the open ball in (W, m_b), respectively. Note that if $r \leq 1$, then we obtain empty open balls because $m_b(u, v) \geq 1$.

Example 1. Let $W = [0, \infty)$. Define a mapping $m_a: W \times W \longrightarrow [1, \infty)$:

$$m_a(w, \mu) = a^{(w,\mu)^2}, \quad (1)$$

where $a > 1$ is any fixed real number. Then, for each a, m_a is b-multiplicative on W with $s = 2$. Note that m_a is not a multiplicative metric on W. Considering $a = 2$, $r = 2^{16}$, and $u = 1$, then $\overline{B_{m_b}(u, r)} = [0, 5]$ is a closed ball in W.

Definition 2 (see [13]). Let (W, m_b) be a b-multiplicative space.

(i) A sequence $\{w_n\}$ is called b-multiplicative Cauchy iff

$$m_b(w_m, w_n) \longrightarrow 1, \text{ as } m, n \longrightarrow +\infty. \quad (2)$$

(ii) A sequence $\{w_n\}$ is b-multiplicative convergent iff there exists $w \in W$ such that

$$m_b(w_n, w) \longrightarrow 1, \text{ as } n \longrightarrow +\infty. \quad (3)$$

(iii) A b-multiplicative space (W, m_b) is said to be b-multiplicative complete if every b-multiplicative Cauchy sequence in W is b-multiplicative convergent to some $\mu \in W$.

Example 2. The space (W, m_b) defined in Example 1 is a b-multiplicative complete space. Taking $\{w_n\} = \{1/n\}$, then $m_b((1/m), (1/n)) \longrightarrow 1$, as $m, n \longrightarrow +\infty$. Hence, $\{w_n\}$ is a b-multiplicative Cauchy sequence. Now, $m_b((1/n), 0) \longrightarrow 1$, as $n \longrightarrow +\infty$ implies that $\{w_n\}$ is b-multiplicative convergent to $0 \in W$.

Definition 3 (see [3]). Let $W \neq \phi$ and $s \geq 1$ be a real number. A mapping $d: W \times W \longrightarrow \mathbb{R}^+ \cup \{0\}$ is said to be b-metric with coefficient "s," if for all $w, \mu, z \in W$, the following assertions hold:

(i) $d(w, \mu) = 0 \Longleftrightarrow w = \mu$

(ii) $d(w, \mu) = d(\mu, w)$

(iii) $d(w, z) \leq [d(w, \mu) + d(\mu, z)]$

The pair (W, d) is said to be a b-metric space. If $r > 0$, $u \in W$, then $\overline{B_d(u, r)} = \{v: m_b(u, v) < r\}$ is called a closed ball in (W, d).

Remark 1 (see [13]). Every b-metric space (W, d) generates a b-multiplicative space (W, m_b) defined as

$$m_b(þ, \mu) = e^{d(þ,\mu)}. \quad (4)$$

Remark 2. Let (W, m_b) be a b-multiplicative space generated by a b-metric space (W, d), $r > 0$ and $þ_\circ \in W$. If $\overline{B_d(þ_\circ, r)}$ and $\overline{B_{m_b}(þ_\circ, e^r)}$ are closed balls in (W, d) and (W, m_b), respectively, then $\overline{B_d(þ_\circ, r)} = \overline{B_{m_b}(þ_\circ, e^r)}$.

2. Results on b-Multiplicative Spaces

Definition 4. Let $K \neq \phi$, $A, B: K \longrightarrow [0, \infty)$, and $g: K \longrightarrow K$. We say that g is a locally cyclic (A, B)-admissible mapping on set H, if

$$\begin{aligned} þ \in H, &\quad A(þ) \geq 1 \Longrightarrow B(gþ) \geq 1, \\ þ \in H, &\quad B(þ) \geq 1 \Longrightarrow A(gþ) \geq 1. \end{aligned} \quad (5)$$

Definition 5. Let (K, m_b) be a b-multiplicative space with $s \geq 1$ and $\overline{B_{m_b}(þ_\circ, r)}$ be a closed set. A self-mapping g is said to be cyclic b-multiplicative (A, B)-Hardy–Rogers-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$, if the following conditions hold:

(1) $A(þ_\circ) \geq 1$ and $B(þ_\circ) \geq 1$

(2) g is a locally cyclic (A, B)-admissible mapping on $\overline{B_{m_b}(þ_\circ, r)}$

(3) $A(þ)B(\mu) \geq 1$ implies

$$\begin{aligned} m_b(gþ, g\mu) \leq\ & m_b(þ, \mu)^\lambda \left[m_b(þ, gþ).m_b(\mu, g\mu)\right]^\theta \\ & \cdot \left[m_b(þ, g\mu).m_b(\mu, gþ)\right]^\nu, \end{aligned} \quad (6)$$

for $þ, \mu \in \overline{B_{m_b}(þ_\circ, r)}$, where $\lambda, \theta, \nu \in [0, 1)$ and $s\lambda + (s+1)\theta + (s^2 + s)\nu < 1$.

Example 3. Let $K = [0, \infty)$ and $g: X \longrightarrow X$ be a mapping defined as

$$gx = \begin{cases} \dfrac{3x}{10}, & \text{if } 0 \leq x \leq 3, \\ x^7 + \sqrt{x} + 6, & \text{otherwise.} \end{cases} \quad (7)$$

Define a complete b-multiplicative metric with $s = 2$ as

$$m_b(x, y) = \begin{Bmatrix} 1 & \text{if } x = y \\ 2^{(x+y)^2} & \text{if } x \neq y \end{Bmatrix}. \quad (8)$$

Considering $þ_\circ = 1$ and $r = 2^{16}$, then $\overline{B_{m_b}(þ_\circ, r)} = [0, 3]$. $A, B: K \longrightarrow [0, +\infty)$ is defined by

$$\begin{aligned} A(þ) &= \begin{Bmatrix} 1 & \text{if } þ \in [0, 30] \\ 0 & \text{otherwise} \end{Bmatrix}, \\ B(þ) &= \begin{Bmatrix} 1 & \text{if } þ \in [0, 3] \\ 0 & \text{otherwise} \end{Bmatrix}, \end{aligned} \quad (9)$$

now, $A(þ_\circ) \geq 1$ and $B(þ_\circ) \geq 1$. Clearly, g is a locally cyclic (A, B)-admissible mapping on $\overline{B_{m_b}(þ_\circ, r)}$. Note that, if we take $þ = 20$, then $A(þ) \geq 1$ but $B(gþ) \ngeq 1(gþ)$ so g is a not-cyclic (A, B)-admissible mapping on K, and the results in [12] cannot be applied. Taking $\lambda = (9/100)$, $\theta = (1/40)$, and $\nu = (3/70)$, now, $\lambda, \theta, \nu \in [0, 1)$ and $s\lambda + (s+1)\theta + (s^2 + s)\nu < 1$. For each $þ, \mu \in \overline{B_{m_b}(þ_\circ, r)}$ with $A(þ)B(\mu) \geq 1$, we have

$$2^{((3þ/10)+(3\mu/10))^2} \leq \left(2^{(þ+\mu)^2}\right)^{(9/100)} . \left[2^{(þ+(3þ/10))^2} . 2^{(\mu+(3\mu/10))^2}\right]^{(1/40)} . \left[2^{(þ+(3\mu/10))^2} . 2^{(\mu+(3þ/10))^2}\right]^{(3/70)}$$
$$\text{or } m_b(gþ, g\mu) \leq m_b(þ, \mu)^{\lambda}\left[m_b(þ, gþ).m_b(\mu, g\mu)\right]^{\theta}\left[m_b(þ, g\mu).m_b(\mu, gþ)\right]^{\nu}. \tag{10}$$

Thus, all conditions of Definition 5 hold. Therefore, g is a cyclic b-multiplicative (A, B)-Hardy–Rogers-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$. Note that

$$m_b(gþ, g\mu) \nleq m_b(þ, \mu)^{\lambda}\left[m_b(þ, gþ).m_b(\mu, g\mu)\right]^{\theta} \left[m_b(þ, g\mu).m_b(\mu, gþ)\right]^{\nu}, \tag{11}$$

for all $þ, \mu \in K$, so again the results in [12] cannot be applied because g cannot satisfy any definition in [12].

Definition 6. If we take $\theta = \nu = 0$ in Definition 5, then we say it is cyclic b-multiplicative (A, B)-Banach-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$. If we take $\lambda = \nu = 0$ in Definition 5, then we say it is cyclic b-multiplicative (A, B)-Kannan-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$. If we take $\theta = \lambda = 0$ in Definition 5, then we say it is b-multiplicative cyclic (A, B)-Chatterjea-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$. If we take $s = 1$ in Definition 5, then we say it is cyclic multiplicative (A, B)-Hardy–Rogers-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$. If we exclude the role of functions A and B in Definition 5, that is, if we exclude conditions (1) and (2) and the restriction $A(þ)B(\mu) \geq 1$ from Definition 5, then we say it is b-multiplicative Hardy–Rogers-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$.

Example 4. If we take $\theta = \nu = 0$ in Example 3, then we obtain an example of cyclic b-multiplicative (A, B)-Banach-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$. If we take $\lambda = \nu = 0$ in Example 3, then we obtain an example of cyclic b-multiplicative (A, B)-Kannan-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$. If we take $\theta = \lambda = 0$ in Example 3, then we obtain an example of cyclic b-multiplicative (A, B)-Chatterjea-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$. If we take $s = 1$ in Example 3, then we obtain an example of cyclic multiplicative (A, B)-Hardy–Rogers-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$. If we exclude the role of functions A and B in Example 3, then we obtain an example of b-multiplicative Hardy–Rogers-type local contraction on $\overline{B_{m_b}(þ_\circ, r)}$.

Definition 7. Let (K, m_b) be a b-multiplicative complete space with coefficient $s \geq 1$ and $g\colon K \longrightarrow K$ and $\rho\colon K \longrightarrow [0, \infty)$ be two mappings. We say that g is cyclic regular on a closed ball $\overline{B_{m_b}(þ_\circ, r)}$ with respect to ρ, if one of the following conditions holds:

(a) If $\overline{B_{m_b}(þ_\circ, r)}$ contains a sequence $\{P_n\}$ such that $\rho(P_n) \geq 1$ for all $n \in \mathbb{N}$ and $\{þ_n\} \longrightarrow þ^* \in \overline{B_{m_b}(þ_\circ, r)}$ as $n \longrightarrow \infty$, then $\rho(þ^*) \geq 1$

(b) g is continuous on $\overline{B_{m_b}(þ_\circ, r)}$

Example 5. The mapping g in Example 3 is cyclic regular on a closed ball $\overline{B_{m_b}(þ_\circ, r)} = [0, 3]$ with respect to $B(þ) = \left\{\begin{matrix} 1 & \text{if } þ \in [0, 3] \\ 0 & \text{otherwise} \end{matrix}\right\}$ because for any sequence $\{þ_n\}$ in $[0, 3]$ such that $B(þ_n) \geq 1$ for all $n \in \mathbb{N}$ and $\{þ_n\} \longrightarrow þ^* \in \overline{B_{m_b}(þ_\circ, r)}$ as $n \longrightarrow \infty$, then $B(þ^*) \geq 1$. Also, g is continuous on $[0, 3]$.

Theorem 1. *Let (K, m_b) be a b-multiplicative complete space with coefficient $s \geq 1$ and $g\colon K \longrightarrow K$ be a cyclic b-multiplicative (A, B)-Hardy–Roger-type local contraction mapping on $\overline{B_{m_b}(þ_\circ, r)}$. Suppose that*

$$m_b\left(þ_\circ, gþ_\circ\right) \leq r^{(((1-\mathrm{sh}))/S)}, \tag{12}$$

where $h = (\lambda + \theta + s\nu/1 - \theta - s\nu)$. Then, there exists a b-multiplicative convergent sequence in $\overline{B_{m_b}(þ_\circ, r)}$. Also, if g is cyclic regular on $\overline{B_{m_b}(þ_\circ, r)}$ with respect to B, then there exists a fixed point of g in $\overline{B_{m_b}(þ_\circ, r)}$. Moreover, if $B(z) \geq 1$ and $A(z) \geq 1$, for all z in the set of fixed points of g, then the fixed point of g will be unique.

Proof. Consider that $A(þ_\circ) \geq 1$ and $B(þ_\circ) \geq 1$ and $þ_\circ \in \overline{B_{m_b}(þ_\circ, r)}$. As g is a cyclic (A, B) admissible mapping on $\overline{B_{m_b}(þ_\circ, r)}$, we have

$$\begin{aligned} A\left(þ_\circ\right) \geq 1 &\Longrightarrow B(þ_1) = B\left(gþ_\circ\right) \geq 1, \\ B\left(þ_\circ\right) \geq 1 &\Longrightarrow A(þ_1) = A\left(gþ_\circ\right) \geq 1. \end{aligned} \tag{13}$$

Note that

$$m_b\left(þ_\circ, þ_1\right) = m_b\left(þ_\circ, gþ_\circ\right) \leq r^{(1-\mathrm{sh}/s)}. \tag{14}$$

By assumption, we have

$$\begin{aligned} &s\lambda + (s+1)\theta + \left(s^2 + s\right)\nu < 1 \\ &\text{or } s\lambda + s\theta + s^2\nu < 1 - \theta - s\nu \\ &\text{or } s\left(\frac{\lambda + \theta + s\nu}{1 - \theta - s\nu}\right) < 1. \end{aligned} \tag{15}$$

Since $h = (\lambda + \theta + s\nu/1 - \theta - s\nu)$, so $\mathrm{sh} < 1$. Now, $(1 - \mathrm{sh}/s) < 1$ and $r > 1$ implies $r^{(1-\mathrm{sh}/s)} < r$. Hence, $þ_1 \in \overline{B_{m_b}(þ_\circ, r)}$. As

$$\begin{aligned} A(þ_1) \geq 1 &\Longrightarrow B(þ_2) = B(gþ_1) \geq 1, \\ B(þ_1) \geq 1 &\Longrightarrow A(þ_2) = A(gþ_1) \geq 1. \end{aligned} \tag{16}$$

Assume that $þ_2, þ_3, \ldots, þ_j \in\in \overline{B_{m_b}(þ_\circ, r)}$ for some $j \in \mathbb{N}$. By a similar method, as above for $þ_1$ and $þ_2$, we get

$$A(þ_i) \geq 1 \text{ and } B(þ_i) \geq 1, \quad \text{for } i = 1, 2, \ldots, j+1. \tag{17}$$

This implies that

$$A\left(þ_{j-1}\right)B\left(þ_j\right) \geq 1, \quad \text{for } i = 1, 2, \ldots, j+1. \tag{18}$$

By using (6), we have

$$
\begin{aligned}
m_b(þ_j, þ_{j+1}) &= m_b(gþ_{j-1}, gþ_j) \\
&\leq A(þ_{j-1})B(þ_j).m_b(gþ_{j-1}, gþ_j) \\
&\leq [m_b(þ_{j-1}, þ_j)]^{\lambda}[m_b(þ_{j-1}, gþ_{j-1}).m_b(þ_j, gþ_j)]^{\theta} \\
&\quad [m_b(þ_{j-1}, gþ_j).m_b(þ_j, gþ_{j-1})]^{\nu} \\
&\leq [m_b(þ_{j-1}, þ_j)]^{\lambda}[m_b(þ_{j-1}, þ_j).m_b(þ_j, þ_{j+1})]^{\theta} \\
&\quad [m_b(þ_{j-1}, þ_{j+1}).m_b(þ_j, þ_j)]^{\nu} \\
&\leq [m_b(þ_{j-1}, þ_j)]^{\lambda}[m_b(þ_{j-1}, þ_j).m_b(þ_j, þ_{j+1})]^{\theta} \\
&\quad [m_b(þ_{j-1}, þ_j)^{s}.m_b(þ_j, þ_{j+1})^{s}]^{\nu} \\
m_b(þ_j, þ_{j+1})^{1-\theta-s\nu} &\leq m_b(þ_{j-1}, þ_j)^{\lambda+\theta+s\nu} \\
m_b(þ_j, þ_{j+1}) &\leq m_b(þ_{j-1}, þ_j)^{(\lambda+\theta+s\nu/1-\theta-s\nu)} \\
m_b(þ_j, þ_{j+1}) &\leq m_b(þ_{j-1}, þ_j)^{h}, \quad \text{where } h = \frac{\lambda+\theta+s\nu}{1-\theta-s\nu}.
\end{aligned}
\tag{19}
$$

Now, using inequality (19), we get

$$
m_b(þ_j, þ_{j+1}) \leq m_b(þ_{j-1}, þ_j)^{h} \leq m_b(þ_{j-2}, þ_{j-1})^{h^2} \leq \cdots \leq m_b(þ_\circ, þ_1)^{h^j}. \tag{20}
$$

By using triangle inequality and inequality (20), we get

$$
\begin{aligned}
m_b(þ_\circ, þ_{j+1}) &\leq m_b(þ_\circ, þ_1)^{s}.m_b(þ_1, þ_2)^{s^2}, \ldots, m_b(þ_j, þ_{j+1})^{s^{j+1}} \\
&\leq m_b(þ_\circ, þ_1)^{s}.m_b(þ_\circ, þ_1)^{s^2h}, \ldots, m_b(þ_\circ, þ_1)^{s^{j+1}h^j} \\
&\leq m_b(þ_\circ, þ_1)^{s(1+sh+\cdots+s^jh^j)} \\
m_b(þ_\circ, þ_{j+1}) &\leq m_b(þ_\circ, þ_1)^{s(1-(sh)^j/1-sh).}
\end{aligned}
\tag{21}
$$

By using inequality (12), we have

$$
\begin{aligned}
m_b(þ_\circ, þ_{j+1}) &\leq r^{((1-sh)/s).s(1-(sh)^j/1-sh)} \\
m_b(þ_\circ, þ_{j+1}) &\leq r^{1-(sh)^j} \leq r, \quad \text{for all } j \in \mathbb{N}.
\end{aligned}
\tag{22}
$$

This implies that $þ_{j+1} \in \overline{B_{m_b}(þ_\circ, r)}$. By induction on n, we conclude that $\{þ_n\} \in \overline{B_{m_b}(þ_\circ, r)}$ for all $n \in \mathbb{N}$. By a similar method, for all $n \in \mathbb{N}$, we get

$$
\begin{aligned}
&A(þ_n) \geq 1, \\
&B(þ_n) \geq 1, \quad \text{for all } n \in \mathbb{N}.
\end{aligned}
\tag{23}
$$

This implies that

$$
A(þ_{n-1})B(þ_n) \geq 1, \quad \text{for all } n \in \mathbb{N}. \tag{24}
$$

Now, inequality (20) implies that

$$
m_b(þ_n, þ_{n+1}) \leq m_b(þ_\circ, þ_1)^{h^n}. \tag{25}
$$

Now, we prove that $\{þ_n\}$ is a b-multiplicative Cauchy sequence in K. Let $m > n$, so $m = n + p$; $p \in \mathbb{N}$. By using the triangle inequality, we have

$$
\begin{aligned}
m_b(þ_n, þ_m) \leq m_b(þ_n, þ_{n+1})^{s}.m_b(þ_{n+1}, þ_{n+2})^{s^2}, \ldots, \\
m_b(þ_{n+p-1}, þ_{n+p})^{s^p}.
\end{aligned}
\tag{26}
$$

By using inequality (25), we get

$$
\begin{aligned}
m_b(þ_n, þ_m) &\leq m_b(þ_\circ, þ_1)^{sh^n}.m_b(þ_\circ, þ_1)^{s^2h^{n+1}}, \ldots, m_b(þ_\circ, þ_1)^{s^ph^{n+p-1}} \\
&\leq m_b(þ_\circ, þ_1)^{sh^n(1+sh+\cdots+s^{p-1}h^{p-1})} \\
&\leq m_b(þ_\circ, þ_1)^{sh^n(1+sh+\cdots+(sh)^{p-1})} \\
&< m_b(þ_\circ, þ_1)^{sh^n(1+sh+\cdots)} \\
m_b(þ_n, þ_m) &< m_b(þ_\circ, þ_1)^{(sh^n/1-sh).}
\end{aligned}
\tag{27}
$$

Taking limit as $m, n \longrightarrow \infty$, we get $m_b(þ_n, þ_m) \longrightarrow 1$. Hence, the sequence $\{þ_n\}$ is a b-multiplicative Cauchy sequence. By the completeness of (K, m_b), it follows that $þ_n \longrightarrow þ^* \in \overline{B_{m_b}(þ_\circ, r)}$. Suppose that g is continuous. Thus, we get $þ^* = \lim_{n\longrightarrow\infty}þ_{n+1} = \lim_{n\longrightarrow\infty}gþ_n = g(\lim_{n\longrightarrow\infty}þ_n) = gþ^*$. Now, we assume that condition (a) of Definition 7 holds. As $B(þ_n) \geq 1$ and $þ_n \longrightarrow þ^* \in \overline{B_{m_b}(þ_\circ, r)}$, so $B(þ^*) \geq 1$. Then, we have

$$
\begin{aligned}
m_b(gþ^*, þ^*) &\leq m_b(gþ^*, gþ_n)^{s}.m_b(gþ_n, þ^*)^{s} \\
&\leq [A(þ_n)B(þ^*).m_b(gþ_n, gþ^*)]^{s}.m_b(gþ_n, þ^*)^{s} \\
&\leq [m_b(þ_n, þ^*)]^{s\lambda}[m_b(þ_n, gþ_n).m_b(þ^*, gþ^*)]^{s\theta} \\
&\quad [m_b(þ_n, gþ^*).m_b(þ^*, gþ_n)]^{s\nu}.m_b(gþ_n, þ^*)^{s} \\
&\leq [m_b(þ_n, þ^*)]^{s\lambda}[m_b(þ_n, þ_{n+1}).m_b(þ^*, gþ^*)]^{s\theta} \\
&\quad [m_b(þ_n, gþ^*).m_b(þ^*, þ_{n+1})]^{s\nu}.m_b(þ_{n+1}, þ^*)^{s}.
\end{aligned}
\tag{28}
$$

Letting $n \longrightarrow \infty$, we get

$$
\begin{aligned}
m_b(gþ^*, þ^*) &\leq [m_b(þ^*, gþ^*)]^{s\theta}[m_b(þ^*, gþ^*)]^{s\nu} \\
m_b(gþ^*, þ^*)^{1-s\theta-s\nu} &\leq 1 \\
m_b(gþ^*, þ^*) &\leq (1)^{(1/1-s\theta-s\nu)} = 1.
\end{aligned}
\tag{29}
$$

Hence, $m_b(qþ^*, þ^*) = 1$, that is, $gþ^* = þ^*$. This proves that $þ^*$ is a fixed point of g. Eventually we prove that $þ^*$ is the unique fixed point of g. Suppose that μ is another fixed point of g. By the hypothesis, we find that $A(þ^*) \geq 1$ and $B(\mu) \geq 1$. Thus,

$$\begin{aligned}m_b(\text{þ}^*,\mu) &= m_b(g\text{þ}^*,g\mu)\\ &\le A(\text{þ}^*)B(\mu).m_b(g\text{þ}^*,g\mu)\\ &\le m_b(\text{þ}^*,\mu)^{\lambda}\left[m_b(\text{þ}^*,g\text{þ}^*).m_b(\mu,g\mu)\right]^{\theta}\left[m_b(\text{þ}^*,g\mu).m_b(\mu,g\text{þ}^*)\right]^{\nu}\\ &\le m_b(\text{þ}^*,\mu)^{\lambda}\left[m_b(\text{þ}^*,\text{þ}^*).m_b(\mu,\mu)\right]^{\theta}\left[m_b(\text{þ}^*,\mu).m_b(\mu,\text{þ}^*)\right]^{\nu}\\ m_b(\text{þ}^*,\mu)^{1-\lambda-2\nu} &\le 1\\ m_b(\text{þ}^*,\mu) &\le (1)^{(1/1-\lambda-2\nu)} = 1.\end{aligned} \tag{30}$$

This proves that $m_b(\text{þ}^*,\mu)=1$ and then $\text{þ}^*=\mu$. Thus, þ^* is the unique fixed point of g. □

Example 6. In Example 3, we have proved that g is a cyclic b-multiplicative (A,B)-Hardy–Rogers-type local contraction on $\overline{B_{m_b}(\text{þ}_\circ,r)}$. It has been proved in Example 5 that the mapping g in Example 3 is cyclic regular on a closed ball $\overline{B_{m_b}(\text{þ}_\circ,r)}=[0,3]$ with respect to $B(\text{þ})=\left\{\begin{matrix}1 & \text{if } \text{þ}\in[0,3]\\ 0 & \text{otherwise}\end{matrix}\right\}$. Now,

$$h=\frac{\lambda+\theta+s\nu}{1-\theta-s\nu}=\frac{(9/100)+(1/40)+2(3/70)}{1-(1/40)-2(3/70)}=\frac{281}{1245}. \tag{31}$$

Now,

$$\begin{aligned}m_b\left(\text{þ}_\circ,g\text{þ}_\circ\right) &= m_b(1,g1)=m_b(1,(3/10))\\ &= 2^{(1,(3/10))^2}\approx 3.27<20.95\approx r^{((1-\text{sh})/s)}.\end{aligned} \tag{32}$$

Hence, all the conditions of Theorem 1 are satisfied, and zero is the unique fixed point of the mapping g. Note that the results in [12] cannot ensure the existence of a fixed point of mapping g because g cannot satisfy the contractive condition of any theorem in [12].

The following results for various other contractions on b-multiplicative spaces can be proved by following the proof of Theorem 1.

Theorem 2. *Let (K,m_b) be a b-multiplicative complete space with coefficient $s\ge 1$ and $g\colon K\longrightarrow K$ be a cyclic b-multiplicative (A,B)-Banach-type local contraction mapping on $\overline{B_{m_b}(\text{þ}_\circ,r)}$. Suppose that*

$$m_b\left(\text{þ}_\circ,g\text{þ}_\circ\right)\le r^{((1-s\lambda)/s)}. \tag{33}$$

Then, there exists a b-multiplicative convergent sequence in $\overline{B_{m_b}(\text{þ}_\circ,r)}$. Also, if g is cyclic regular on $\overline{B_{m_b}(\text{þ}_\circ,r)}$, then there exists a fixed point of g in $\overline{B_{m_b}(\text{þ}_\circ,r)}$. Moreover, if $B(z)\ge 1$ and $A(z)\ge 1$, for all z in the set of fixed points of g, then the fixed point of g will be unique.

Theorem 3. *Let (K,m_b) be a b-multiplicative complete space with coefficient $s\ge 1$ and $g\colon K\longrightarrow K$ be a cyclic b-multiplicative (A,B)-Kannan-type local contraction mapping on $\overline{B_{m_b}(\text{þ}_\circ,r)}$. Suppose that*

$$m_b\left(\text{þ}_\circ,g\text{þ}_\circ\right)\le r^{((1-\text{sh})/s)}, \tag{34}$$

where $h=(\theta/1-\theta)$. Then, there exists a b-multiplicative convergent sequence in $\overline{B_{m_b}(\text{þ}_\circ,r)}$. Also, if g is cyclic regular on $\overline{B_{m_b}(p_\circ,r)}$, then there exists a fixed point of g in $\overline{B_{m_b}(\text{þ}_\circ,r)}$. Moreover, if $B(z)\ge 1$ and $A(z)\ge 1$, for all z in the set of fixed points of g, then the fixed point of g will be unique.

Theorem 4. *Let (K,m_b) be a b-multiplicative complete space with coefficient $s\ge 1$ and $g\colon K\longrightarrow K$ be a cyclic b-multiplicative (A,B)-Chatterjea-type local contraction mapping on $\overline{B_{m_b}(\text{þ}_\circ,r)}$. Suppose that*

$$m_b\left(\text{þ}_\circ,g\text{þ}_\circ\right)\le r^{((1-\text{sh})/s)}, \tag{35}$$

where $h=(s\nu/1-s\nu)$. Then, there exists a b-multiplicative convergent sequence in $\overline{B_{m_b}(\text{þ}_\circ,r)}$. Also, if g is cyclic regular on $\overline{B_{m_b}(\text{þ}_\circ,r)}$, then there exists a fixed point of g in $\overline{B_{m_b}(\text{þ}_\circ,r)}$. Moreover, if $B(z)\ge 1$ and $A(z)\ge 1$, for all z in the set of fixed points of g, then the fixed point of g will be unique.

Theorem 5. *Let (K,m_b) be a b-multiplicative complete space with coefficient $s\ge 1$ and $g\colon K\longrightarrow K$ be a b-multiplicative Hardy–Roger-type local contraction mapping on $\overline{B_{m_b}(\text{þ}_\circ,r)}$. Suppose that*

$$m_b\left(\text{þ}_\circ,g\text{þ}_\circ\right)\le r^{((1-\text{sh})/s)},\quad \text{where } h=\frac{\lambda+\theta+s\nu}{1-\theta-s\nu}. \tag{36}$$

Then, there exists a unique fixed point in $\overline{B_{m_b}(\text{þ}_\circ,r)}$.

Theorem 6. *Let (K,m_b) be a b-multiplicative complete space with coefficient $s\ge 1$ and $g\colon K\longrightarrow K$ be a cyclic b-multiplicative (A,B)-Hardy–Roger-type local contraction mapping on K. Then, there exists a b-multiplicative convergent sequence in K. Also, if g is cyclic regular on K, then there exists a fixed point of g in K. Moreover, if $B(z)\ge 1$ and $A(z)\ge 1$, for all z in the set of fixed points of g, then the fixed point of g will be unique.*

The following result is a multiplicative metric version of Theorem 1.

Theorem 7. *Let (K,m) be a complete multiplicative space and $g\colon K\longrightarrow K$ be a multiplicative (A,B)-Hardy–Roger-type local contraction mapping on $\overline{B_{m_b}(\text{þ}_\circ,r)}$. Suppose that*

$$m_b\left(\text{þ}_\circ,g\text{þ}_\circ\right)\le r^{(1-h)}, \tag{37}$$

where $h = (\lambda + \theta + \nu/1 - \theta - \nu)$. Then, there exists a multiplicative convergent sequence in $\overline{B_b(þ_\circ, r)}$. Also, if one of the following conditions holds:

(a) *If $\overline{B_m(þ_\circ, r)}$ contains a sequence $\{þ_n\}$ such that $B(þ_n) \geq 1$ for all $n \in \mathbb{N}$ and $\{þ_n\} \longrightarrow þ^* \in \overline{B_m(þ_\circ, r)}$ as $n \longrightarrow \infty$, then $B(þ^*) \geq 1$*

(b) *g is continuous on $\overline{B_m(þ_\circ, r)}$*

Then, there exists a fixed point of g in $\overline{B_m(þ_\circ, r)}$. Moreover, if $B(z) \geq 1$ and $A(z) \geq 1$, for all z in the set of fixed points of g, then the fixed point of g will be unique.

As an application, we give an existence theorem for the Fredholm multiplicative integral equation of the following type:

$$g(u) = \int_a^b Q(u, w, g(w))^{\mathrm{d}w}, \quad u, w \in [a, b], \tag{38}$$

where Q: $[a, b] \times [a, b] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an integrable function.

Let $K = C([a, b], \mathbb{R}_+)$, $a > 0$ and $\mathbb{R}_+ = (0, \infty)$, be the space of all positive, continuous real-valued functions, endowed with the b-multiplicative:

$$m_b(g, h) = \sup_{u \in [a,b]} \left\{ \max\left\{ \left| \frac{g(u)}{h(u)} \right|^2, \left| \frac{h(u)}{g(u)} \right|^2 \right\} \right\}. \tag{39}$$

Clearly, the set $E_{g_0,r} = \{h(u)\text{: } \sup_{u \in [a,b]}\{\max\{|(g_0(u)/h(u))|^2, |(h(u)/g_0(u))|^2\}\} \leq r\}$ is a closed ball $\overline{B_{m_b}(g_0, r)}$ in (K, m_b).

Theorem 8. *Let $K = C([a, b], \mathbb{R}_+)$, $g_0(u) \in K$, $r > 1$, $a > 0$, A, B: $K \longrightarrow [0, \infty)$, and S: $K \longrightarrow K$:*

$$Sg(u) = \int_a^b Q(u, w, g(w))^{\mathrm{d}w}, \tag{40}$$

where Q: $[a, b] \times [a, b] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an integrable function. Assume that the following conditions hold:

(1) *$A(_{g^\circ}(u)) \geq 1$ and $B(_{g^\circ}(u)) \geq 1$;*

(2) *S is a cyclic (A, B)-admissible mapping on $E_{g_0,r}$;*

(3) *For each $u, w \in [a, b]$ and for $g(u)$, $h(u)$ belongs to closed set $E_{g_0,r}$, such that $A(g(u))B(h(u)) \geq 1$; then, this implies*

$$\left| \frac{Q(u, w, g(w))}{Q(u, w, h, (w))} \right| \leq \left(\left| \frac{g(w)}{h(w)} \right| \right)^\lambda; \tag{41}$$

(4) *The constant λ is such that $2\lambda < (1/b - a)$ and*

$$\sup_{u \in [a,b]} \left\{ \max\left\{ \left| \frac{g_0(u)}{g_1(u)} \right|^2, \left| \frac{g_1(u)}{g_0(u)} \right|^2 \right\} \right\} \leq r^{(1 - 2\lambda(b-a)/2)}; \tag{42}$$

Also, if one of the following conditions holds:

(5) *S is continuous on $E_{g_0,r}$ or*

(6) *If $\{g_n(u)\}$ is a sequence in $E_{g_0,r}$ such that $\{g_n(u)\} \longrightarrow g^*(u) \in E_{g_0,r}$ as $n \longrightarrow \infty$ and $B(g_n(u)) \geq 1$ for all $n \in \mathbb{N}$, then $B(g^*(u)) \geq 1$.*

Then, the integral equation (38) has a solution. Moreover, if $A(g) \geq 1$ and $B(g) \geq 1$ for all g in the set of fixed points of S, then equation (38) has a unique solution.

Proof. Let $g(u), h(u) \in E_{g_0,r}$. Now, we have

$$\begin{aligned} \left| \frac{Sg(u)}{Sh(u)} \right|^2 &\leq \left(\int_a^b \left| \frac{Q(u, w, g(w))}{Q(u, w, h(w))} \right|^{dw} \right)^2 \\ &\leq \left(\int_a^b \left(\left| \frac{g(w)}{h(w)} \right|^\lambda \right)^{dw} \right)^2 \\ &\leq \left(\int_a^b \left(m_b(g, h)^{(\lambda/2)} \right)^{dw} \right)^2 \\ &= \left(\left(m_b(g, h)^{b-a} \right)^{(\lambda/2)} \right)^2 \\ &= m_b(g, h)^{\lambda(b-a)}, \quad \text{for each } u \in [a, b]. \end{aligned} \tag{43}$$

Thus, we get $m_b(Sg, Sh) \leq m_b(g, h)^\alpha$, $\alpha = \lambda(b - a)$. As $2\lambda < (1/b - a)$, so $w\alpha < 1$. Also, hypothesis (4) implies

$$m_b(g_0, Sg_0) \leq r^{(1 - w\alpha/w)}. \tag{44}$$

Therefore, by Theorem 2, there exists a unique fixed point of the operator S. Hence, the integral equation (38) has a unique solution. □

3. Results on *b*-Metric Spaces

Definition 8. Let (K, b) be a b-metric space and $\overline{B_b(þ_\circ, r)}$ be a closed set. A self-mapping g is said to be cyclic b-(A, B)-Hardy–Rogers-type local contraction on $\overline{B_b(þ_\circ, r)}$, if the following conditions hold:

(1) $A(þ_\circ) \geq 1$ and $B(þ_\circ) \geq 1$

(2) g is a locally cyclic (A, B)-admissible mapping on $\overline{B_b(þ_\circ, r)}$

(3) $A(þ)B(\mu) \geq 1$ implies

$$\begin{aligned} b(gþ, g\mu) \leq{}& \lambda b(þ, \mu) + \theta[b(þ, gþ) + b(\mu, g\mu)] \\ &+ \nu[b(þ, g\mu) + b(\mu, gþ)], \end{aligned} \tag{45}$$

for $þ, \mu \in \overline{B_{m_b}(þ_\circ, r)}$, where $\lambda, \theta, \nu \in [0, 1)$ and $s\lambda + (s + 1)\theta + (s^2 + s)\nu < 1$.

Theorem 9. *Let (K, b) be a complete b-metric space with coefficient $s \geq 1$ and g: $K \longrightarrow K$ be a cyclic b-(A, B)-Hardy–Roger-type local contraction mapping on $\overline{B_b(þ_\circ, r)}$. Suppose that*

$$b(þ_\circ, gþ_\circ) \leq \frac{r(1 - \mathrm{sh})}{s}, \tag{46}$$

where $h = (\lambda + \theta + s\nu/1 - \theta - s\nu)$. Then, there exists a convergent sequence in $\overline{B_b(þ_\circ, r)}$. Also, if one of the following conditions holds:

(a) *If $\overline{B_b(þ_\circ, r)}$ contains a sequence $\{þ_n\}$ such that $B(þ_n) \geq 1$ for all $n \in \mathbb{N}$ and $\{þ_n\} \longrightarrow þ^* \in \overline{B_b(þ_\circ, r)}$ as $n \longrightarrow \infty$, then $B(þ^*) \geq 1$*

(b) *g is continuous on $\overline{B_b(þ_\circ, r)}$.*

Then, there exists a fixed point of g in $\overline{B_b(þ_\circ, r)}$. Moreover, if $B(z) \geq 1$ and $A(z) \geq 1$, for all z in the set of fixed points of g, then the fixed point of g will be unique.

Proof. Defining $m_b(þ, \mu) = e^{b(þ,\mu)}$. Then, by Remark 1 (W, m_b) is a b-multiplicative space. By taking exponential on both sides of inequality (46), we have

$$\begin{aligned} &e^{b(þ_\circ, gþ_\circ)} \leq e^{(r(1-\mathrm{sh})/s)} \\ &\text{or } m_b\left(þ_\circ, gþ_\circ\right) \leq \varepsilon^{(1-\mathrm{sh}/s)}, \end{aligned} \tag{47}$$

where $\varepsilon = e^r > 1$. Now, by taking exponential on both sides of inequality (45) and by using Remark 2, we have

$$e^{b(gþ,g\mu)} \leq e^{\lambda b(þ,\mu)} e^{\theta[b(þ,gþ)+b(\mu,g\mu)]} e^{\nu[b(þ,g\mu)+b(\mu,gþ)]}, \tag{48}$$

for all $þ, \mu$ belong to the closed set $\overline{B_b(þ_\circ, r)}$. Now, by using Remarks 1 and 2, we have

$$\begin{aligned} &m_b(gþ, g\mu) \leq m_b(þ, \mu)^\lambda \left[m_b(þ, gþ).m_b(\mu, g\mu)\right]^\theta \\ &\left[m_b(þ, g\mu).m_b(\mu, gþ)\right]^\nu, \end{aligned} \tag{49}$$

for all $þ, \mu$ belong to the closed set $\overline{B_{m_b}(þ_\circ, \varepsilon)}$. Now, by Theorem 1, g has a unique fixed point in $\overline{B_{m_b}(þ_\circ, \varepsilon)}$ or $\overline{B_b(þ_\circ, r)}$. □

Example 7. Let $K = \mathbb{R}$ endowed with the b-metric $b(þ, \mu) = |þ - \mu|$ for all $þ, \mu \in K$ and $g\colon K \longrightarrow K$ be defined by

$$gþ = \begin{Bmatrix} -\frac{1}{2}þ & \text{if } þ \in \left[-\frac{1}{3}, \frac{1}{3}\right] \\ 2þ & \text{if } þ \in \mathbb{R}\backslash\left[-\frac{1}{3}, \frac{1}{3}\right] \end{Bmatrix}, \tag{50}$$

and $A, B\colon K \longrightarrow [0, +\infty)$ be given by

$$\begin{aligned} A(þ) &= \begin{Bmatrix} 1 & \text{if } þ \in [-3, 0] \\ 0 & \text{otherwise} \end{Bmatrix}, \\ B(þ) &= \begin{Bmatrix} 1 & \text{if } þ \in [0, 1] \\ 0 & \text{otherwise} \end{Bmatrix}. \end{aligned} \tag{51}$$

Let $r = (1/3)$ and $þ_\circ = 0$, then $\overline{B_b(þ_\circ, r)} = [-(1/3), (1/3)]$ is closed. Now, $A(þ_\circ) \geq 1$ and $B(þ_\circ) \geq 1$. Also, g is a locally (A, B)-admissible mapping on $\overline{B_b(þ_\circ, r)}$. If $þ, \mu$ belong to $\overline{B_b(þ_\circ, r)}$ such that $A(þ)B(\mu) \geq 1$, then $þ \in [-(1/3), 0]$ and $\mu \in [0, (1/3)]$. Taking $\lambda = (1/2)$, $\theta = (1/9)$, and $\nu = (1/18)$, we have

$$\begin{aligned} b(gþ, g\mu) \leq{}& \lambda b(þ, \mu) + \theta[b(þ, gþ) + b(\mu, g\mu)] \\ &+ \nu[b(þ, g\mu) + b(\mu, gþ)]. \end{aligned} \tag{52}$$

So, g is cyclic b-(A, B)-Hardy–Rogers-type local contraction on $\overline{B_b(þ_\circ, r)}$. Also,

$$b\left(þ_\circ, gþ_\circ\right) \leq \frac{r(1 - \mathrm{sh})}{s}. \tag{53}$$

Now, if $\{þ_n\}$ is a sequence in $\overline{B_b(þ_\circ, r)}$ such that $B(þ_n) \geq 1$ and $þ_n \longrightarrow þ$ as $n \longrightarrow \infty$. Then, $þ_n \in [0, (1/3)]$. Hence, $þ \in [0, (1/3)]$ and $B(þ) \geq 1$. So, all hypotheses of Theorem 9 are satisfied, and therefore g has a unique fixed point.

Note that the results of Alizadeh et al. [26] and other results for (A, B)-admissible mapping cannot be applied. Since $A(-3) \geq 1$, but $B(g(-3)) = B(-6)0 \ngeq 1$. Also, $B(1) \geq 1$, but $A(g(1)) = A(2) = 0 \ngeq 1$. Therefore, g is not a cyclic (A, B)-admissible mapping.

Now, we give an example of a mapping g which is a cyclic (A, B)-admissible, but none of the previously defined contractions in other papers holds. Therefore, other results for (A, B)-admissible mapping fail to ensure the existence of a fixed point. However, g has a fixed point, and our result is valid for such mappings.

Example 8. Let $K = \mathbb{R}$ endowed with the b-metric $B(þ, \mu) = |þ - \mu|$ for all $þ, \mu \in K$ and $g\colon K \longrightarrow K$ be defined by

$$gþ = \begin{Bmatrix} -\frac{1}{2}þ & \text{if } þ \in \left[-\frac{1}{3}, \frac{1}{3}\right] \\ -þ & \text{if } þ \in [-3, 3] - \left[-\frac{1}{3}, \frac{1}{3}\right] \\ 2þ & \text{if } þ \in \mathbb{R}\backslash[-3, 3] \end{Bmatrix}, \tag{54}$$

and $A, B\colon K \longrightarrow [0, +\infty)$ be given by

$$\begin{aligned} A(þ) &= \begin{Bmatrix} 1 & \text{if } þ \in [-3, 0] \\ 0 & \text{otherwise} \end{Bmatrix}, \\ B(þ) &= \begin{Bmatrix} 1 & \text{if } þ \in [0, 3] \\ 0 & \text{otherwise} \end{Bmatrix}. \end{aligned} \tag{55}$$

By taking $\lambda = (1/2)$, $\theta = (1/9)$, $\nu = (1/18)$, $r = (1/3)$, and $þ_\circ = 0$, all hypotheses of Theorem 9 are satisfied, and therefore g has a unique fixed point. Note that, g is a cyclic (A, B)-admissible mapping on K, but all other results for (A, B)-admissible mapping cannot be applied. For example, defining $\psi, \varphi\colon [0, \infty) \longrightarrow [0, \infty)$ by $\psi(t) = t$ and $\varphi(t) = (1/4)t$. Let $A(þ)B(\mu) \geq 1$. Then, $þ \in [-3, 0]$ and $\mu \in [0, 3]$. If $þ = 1$ and $\mu = 2$, then

$$\psi(d(g1, g2)) = 1 > \frac{3}{4} = \psi(d(1, 2)) - \varphi(d(1, 2)). \tag{56}$$

That is, Theorem 2.4 of [26] cannot be applied here.

Definition 9. In Definition 8, if

(1) If $\theta = \nu = 0$, then we say it is cyclic b-(A, B)-Banach-type local contraction on $\overline{B_b(þ_\circ, r)}$

(2) If $\lambda = \nu = 0$, then we say it is cyclic b-(A, B)-Kannan-type local contraction on $\overline{B_b(þ_\circ, r)}$

(3) If $\theta = \lambda = 0$, then we say it is cyclic b-(A, B)-Chatterjea-type local contraction on $\overline{B_b(\text{þ}_\circ, r)}$

(4) If $s = 1$, then we say it is cyclic (A, B)-Hardy–Rogers-type local contraction on $\overline{B_b(\text{þ}_\circ, r)}$

(5) We exclude the role of functions A and B; that is, if we exclude conditions (1) and (2) and the restriction $A(\text{þ})B(\mu) \geq 1$ from Definition 8, then we say it is b-Hardy–Rogers-type local contraction on $\overline{B_b(\text{þ}_\circ, r)}$

Example 9. If we consider $K = \mathbb{R}$ endowed with the b-metric $B(\text{þ}, \mu) = |\text{þ} - \mu|$ for all $\text{þ}, \mu \in K$ and define $g\colon K \longrightarrow K$ as in Example 7, then by taking $\lambda = (1/2)$ and $\theta = \nu = 0$, we can get cyclic b-(A, B)-Banach-type local contraction on $\overline{B_b(\text{þ}_\circ, r)}$. Similarly, if we consider $\lambda = \nu = 0$ and $\theta = (1/9)$ in Example 7, we can get cyclic b-(A, B)-Kannan-type local contraction on $\overline{B_b(\text{þ}_\circ, r)}$. Also, if we take, $\theta = \lambda = 0$ and $\nu = (1/18)$, then we can get cyclic b-(A, B)-Chatterjea-type local contraction on $\overline{B_b(\text{þ}_\circ, r)}$. If we exclude the role of functions A and B in Example 7, then we can get an example of b-Hardy–Rogers-type local contraction on $\overline{B_b(\text{þ}_\circ, r)}$.

Remark 3. By using Definition 9, we can make five new theorems in b-metric spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] I. A. Bakhtin, "The contraction mapping principle in almost metric spaces," *Journal of Functional Analysis*, vol. 30, pp. 26–30, 1989.

[2] S. Czerwick, "Contraction mapping in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[3] M. Sarwar and M. Rahman, "Fixed point theorems for ciric's and generalized contractions in b-metric spaces," *International Journal of Analysis and Application*, vol. 7, no. 1, pp. 70–78, 2015.

[4] A. Shoaib, P. Kumam, A. Shahzad, S. phiangsungnoen, and Q. Mahmood, "Fixed point results for fuzzy mappings in a b-metric space," *Fixed Point Theory and Application*, vol. 2018, pp. 1–12, 2018.

[5] M. Ozavsar and A. C. Cervikel, "Fixed points of multiplicative contraction mappings on multiplicative metric spaces," *Journal of Engineering Technology and Applied Sciences*, vol. 2, no. 2, pp. 65–79, 2012.

[6] C. Mongkolkeha and W. Sintunavarat, "Best proximity points for multiplicative proximal contraction mapping on multiplicative metric spaces," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 1134–1140, 2015.

[7] M. Abbas, B. Ali, and Y. Suleiman, "Common fixed points of locally contractive mappings in multiplicative metric spaces with application," *International Journal of Mathematics and Mathematical Sciences*, vol. 2015, Article ID 218683, 7 pages, 2015.

[8] M. Abbas, M. D. Sen, and T. Nazir, "Common fixed points of generalized rational type cocyclic mapping in multiplicative metric spaces," *Discrete Dynamics in Nature and Society*, vol. 2015, Article ID 147303, 10 pages, 2015.

[9] A. E. Al-Mazrooei, D. Lateef, and J. Ahmad, "Common fixed point theorems for generalized contractions," *Journal of Mathematical Analysis*, vol. 8, no. 3, pp. 157–166, 2017.

[10] C. Mongkolkeha and W. Sintunaravat, "Optimal approximate solutions for multiplicative proximal contraction mappings in multiplicative metric spaces," *Proceedings of National Academy of Sciences*, vol. 86, no. 1, pp. 15–20, 2016.

[11] A. Shoaib and Q. Mehmood, "Fixed point results in js multiplicative metric spaces," *Turkish Journal of Analysis and Number Theory*, vol. 6, no. 6, pp. 159–163, 2018.

[12] O. Yamaod and W. Sintunavarat, "Some fixed point results for generlized contraction mappings with cyclic (α, β) admissible mapping in multiplicative metric spaces," *Journal of Inequalities and Applications*, vol. 2014, pp. 1–15, Article ID 488, 2014.

[13] M. U. Ali, T. Kamran, and A. Kurdi, "Fixed point in b-multiplicative spaces," *UPB Scientific Bulletin, Series A: Applied*, vol. 79, no. 3, pp. 107–116, 2017.

[14] M. Mehmood, A. Shoaib, and H. Khalid, "Some fixed point results for multivalued mappings in b multiplicative and b– metric space," *International Journal of Analysis and Applications*, vol. 18, no. 3, pp. 439–447, 2020.

[15] M. Arshad, A. Shoaib, M. Abbas, and A. Azam, "Fixed points of a pair of kannan type mappings on a closed ball in ordered partial metric spaces," *Miskolic Mathematical Notes*, vol. 14, no. 3, pp. 769–784, 2013.

[16] T. Rasham, A. Shoaib, B. Alamri, A. Asif, and M. Arshad, "Fixed point results for $\alpha*$-ψ-Dominated multivalued contractive mappings endowed with graphic structure," *Mathematics*, vol. 7, no. 3, p. 307, 2019.

[17] T. Rasham, A. Shoaib, N. Hussain, B. A. S. Alamri, and M. Arshad, "Multivalued fixed point results in dislocated b-metric spaces with application to the system of nonlinear integral equations," *Symmetry*, vol. 11, no. 1, p. 40, 2019.

[18] A. Shoaib, "Fixed point results for $\alpha*$-ψ -multivalued mappings," *Bulletin of Mathematical Analysis and Applications*, vol. 8, no. 4, pp. 43–55, 2016.

[19] A. Shoaib, A. Hussain, M. Arshad, and A. Azam, "Fixed point results for $\alpha*$-ψ-Ciric type multivalued mappings on an intersection of a closed ball and a sequence with graph," *Journal of Mathematical Analysis*, vol. 7, no. 3, pp. 41–50, 2016.

[20] A. Shoaib, A. Azam, M. Arshad, and A. Shahzad, "Fixed point results for the multivalued mapping on closed ball in dislocated fuzzy metric space," *Journal of Mathematical Analysis*, vol. 8, no. 2, pp. 98–106, 2017.

[21] A. Shoaib, A. Azam, and A. Shahzad, "Common fixed point results for the family of multivalued mappings satisfying contraction on a sequence in Hausdorff fuzzy metric space," *Journal of Computational Analysis and Applications*, vol. 24, no. 4, pp. 692–699, 2018.

[22] A. Shoaib, T. Rasham, N. Hussain, and M. Arshad, "$\alpha*$-dominated set-valued mappings and some generalised fixed point results," *Journal of the National Science Foundation of Sri Lanka*, vol. 47, no. 2, 2019.

[23] A. Shoaib, A. Azam, M. Arshad, and E. Ameer, "Fixed point results for multivalued mappings on a sequence in a closed ball with application," *Journal of Mathematics and Computer Science*, vol. 17, no. 2, pp. 308–316, 2017.

[24] A. Shoaib, M. Arshad, and A. Azam, “Fixed points of a pair of locally contractive mappings in ordered partial metric spaces,” *Matematič ki Vesnik*, vol. 67, no. 1, pp. 26–38, 2015.

[25] A. Shoaib, M. Arshad, and M. A. Kutbi, “Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered partial metric spaces,” *Journal of Computational Analysis and Applications*, vol. 17, no. 2, pp. 266–264, 2014.

[26] S. Alizadeh, F. Moradlou, and P. Salimi, “Some fixed point results for (α, β)-(ψ, Φ)-contractive mappings,” *Filomat*, vol. 28, no. 3, pp. 635–647, 2014.