Research Article

Boundedness of Fractional Integral Operators Containing Mittag-Leffler Function via Exponentially s -Convex Functions

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The main objective of this paper is to obtain the fractional integral operator inequalities which provide bounds of the sum of these operators at an arbitrary point. These inequalities are derived for s -exponentially convex functions. Furthermore, a Hadamard inequality is obtained for fractional integrals by using exponentially symmetric functions. The results of this paper contain several such consequences for known fractional integrals and functions which are convex, exponentially convex, and s -convex.

1. Introduction

Convex functions play an important role in many areas of mathematics. They are important especially in the study of optimization problems, theory of inequalities, mathematical analysis, statistical analysis, operation research, and so on. The analytical definition of convex function motivated the authors to define more such functions theoretically; for example, the terms quasi-convex, m -convex, s -convex, h -convex, (α, m) -convex, and $(h - m)$ -convex functions [1] are defined by extending or generalizing inequality (1). For this paper, we will use exponentially s -convex functions which include exponentially convex, s -convex, and convex functions.

Definition 1. A function $f: K \subseteq \mathbb{R} \rightarrow \mathbb{R}$, where K is an interval in $\mathbb{R}$, is said to be convex function if the following inequality holds:

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b), \quad (1)$$

for all $a, b \in K$ and $t \in [0, 1]$.

Definition 2 (see [2]). A function $f: K \subseteq \mathbb{R} \rightarrow \mathbb{R}$, where K is an interval in $\mathbb{R}$, is said to be exponentially convex function if

$$f(ta + (1-t)b) \leq t \frac{f(a)}{e^{a\alpha}} + (1-t) \frac{f(b)}{e^{b\alpha}} \quad (2)$$

holds for all $a, b \in K$, $t \in [0, 1]$ and $\alpha \in \mathbb{R}$. If the inequality in (2) is reversed, then f is called exponentially concave.

A generalization of convex function defined on the right half of the real line is the s -convex function defined as follows:

Definition 3 (see [3]). Let $s \in [0, 1]$. A function $f: [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex function in the second sense if

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b) \quad (3)$$

holds for all $a, b \in [0, \infty)$ and $t \in [0, 1]$.

It is noted that I-convex function is convex. A further generalization of the above defined functions is given as follows:

Definition 4 (see [4]). Let $s \in (0, 1]$ and $K \subseteq [0, \infty)$ be an interval. A function $f: K \rightarrow \mathbb{R}$ is said to be exponentially s -convex in the second sense if

$$f(ta + (1-t)b) \leq t^s \frac{f(a)}{e^{\alpha a}} + (1-t)^s \frac{f(b)}{e^{\alpha b}} \quad (4)$$

holds for all $a, b \in K, t \in [0, 1]$ and $\alpha \in \mathbb{R}$. If the inequality in (4) is reversed, then f is called exponentially s -concave.

For utilizations of exponentially convex functions, one can see [2, 4–7]. Our aim in this paper is to utilize exponentially s -convex functions for establishing bounds of fractional integral operators with kernel Mittag-Leffler function. The Mittag-Leffler function denoted by $E_\sigma(t)$ was introduced by Mittag-Leffler [8] in 1903:

$$E_\sigma(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\sigma n + 1)}, \quad (5)$$

where $\sigma, t \in \mathbb{C}$, $\Gamma(\cdot)$ is the gamma function, and $\Re(\sigma) \geq 0$.

The Mittag-Leffler function is a direct generalization of the exponential function to which it reduces for $\sigma = 1$. In the solution of fractional integral and fractional differential equations, it arises naturally. Due to its importance and utilizations, Mittag-Leffler function has been generalized by many authors. By direct involvement in the problems of physics, chemistry, biology, engineering, and other applied sciences, Mittag-Leffler function and its generalizations have successful applications. Recently, in [9], Andrić et al. introduced an extended generalized Mittag-Leffler function which is defined as follows:

Definition 5. Let $\mu, \sigma, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, and $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Then, the extended generalized Mittag-Leffler function is defined by

$$E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(t; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \sigma)} \frac{t^n}{(l)_{n\delta}}, \quad (6)$$

where β_p is the generalized beta function defined as follows:

$$\beta_p(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-(p/t)(1-t)} dt, \quad (7)$$

and $(c)_{nk}$ is the Pochhammer symbol defined by $(c)_{nk} = (\Gamma(c + nk)/\Gamma(c))$.

From (4), one can obtain the Mittag-Leffler functions defined in [10–14], see Remark 1.3 of [32].

Lemma 1 (see [9]). If $m \in \mathbb{N}$ and $\omega, \mu, \sigma, l, \gamma, c \in \mathbb{C}$ such that $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k < \delta + \Re(\mu)$, then

$$\left(\frac{d}{dt} \right)^m \left[t^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega t^\mu; p) \right] = t^{\sigma-m-1} E_{\mu, \sigma-m, l}^{\gamma, \delta, k, c}(\omega t^\mu; p), \quad (8)$$

$\Re(\sigma) > m.$

Fractional integral operators are very useful in mathematical inequalities. Some authors have established fractional integral inequalities due to different fractional integral operators, see [15–30]. Many authors have used Mittag-Leffler

function to define fractional integral operators. The fractional integral operators corresponding to Mittag-Leffler function (4) are defined as follows:

Definition 6 (see [9]). Let $\omega, \mu, \sigma, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, and $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a, b]$ and $x \in [a, b]$. Then, the generalized fractional integral operators containing Mittag-Leffler function are defined by

$$\left(\epsilon_{\mu, \sigma, l, \omega, a^+}^{\gamma, \delta, k, c} f \right)(x; p) = \int_a^x (x-t)^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-t)^\mu; p) f(t) dt, \quad (9)$$

and

$$\left(\epsilon_{\mu, \sigma, l, \omega, b^-}^{\gamma, \delta, k, c} f \right)(x; p) = \int_x^b (t-x)^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(t-x)^\mu; p) f(t) dt, \quad (10)$$

where $E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\cdot)$ is the Mittag-Leffler function defined in (6).

From (9) and (10), one can obtain the fractional integral operators defined in [10–14], see Remark 1.4 of [32]. In [20], Farid et al. proved that

$$\left(\epsilon_{\mu, \sigma, l, \omega, a^+}^{\gamma, \delta, k, c} 1 \right)(x; p) = (x-a)^\sigma E_{\mu, \sigma+1, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p), \quad (11)$$

and

$$\left(\epsilon_{\mu, \tau, l, \omega, b^-}^{\gamma, \delta, k, c} 1 \right)(x; p) = (b-x)^\tau E_{\mu, \tau+1, l}^{\gamma, \delta, k, c}(\omega(b-x)^\mu; p). \quad (12)$$

We will follow the upcoming notations in the main results:

$$D_{\sigma, \omega, a^+}(x; p) = \left(\epsilon_{\mu, \sigma, l, \omega, a^+}^{\gamma, \delta, k, c} 1 \right)(x; p), \quad (13)$$

$$D_{\tau, \omega, b^-}(x; p) = \left(\epsilon_{\mu, \tau, l, \omega, b^-}^{\gamma, \delta, k, c} 1 \right)(x; p). \quad (14)$$

The upcoming section contains the main results which address the bounds of fractional integral operators containing Mittag-Leffler functions for exponentially s -convex functions. The continuity of such fractional integrals is proved. A Hadamard inequality is established that leads to several Hadamard inequalities for convex, exponentially convex, and s -convex functions. Moreover, the results of papers [31, 32] can be obtained in particular.

2. Main Results

Theorem 1. Let $f: [a, b] \rightarrow \mathbb{R}$ be a real-valued function. If f is positive and exponentially s -convex, then for $\sigma, \tau \geq 1$, the following upper bound for generalized integral operators holds:

$$\begin{aligned} & \left(\epsilon_{\mu, \sigma, l, \omega, a^+}^{\gamma, \delta, k, c} f \right)(x; p) + \left(\epsilon_{\mu, \tau, l, \omega, b^-}^{\gamma, \delta, k, c} f \right)(x; p) \\ & \leq \left(\frac{f(a)}{e^{\alpha a}} + \frac{f(x)}{e^{\alpha x}} \right) \frac{(x-a) D_{\sigma-1, a^+}(x; p)}{s+1} \\ & \quad + \left(\frac{f(b)}{e^{\beta b}} + \frac{f(x)}{e^{\beta x}} \right) \frac{(b-x) D_{\tau-1, b^-}(x; p)}{s+1}, \\ & \quad x \in [a, b], \alpha, \beta \in \mathbb{R}. \end{aligned} \quad (15)$$

Proof. Let $x \in [a, b]$. Then, for $t \in [a, x]$ and $\sigma \geq 1$, we have the following inequality:

$$\begin{aligned} & (x-t)^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-t)^\mu; p) \\ & \leq (x-a)^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p). \end{aligned} \quad (16)$$

As f is exponentially s -convex, therefore, one can obtain

$$f(t) \leq \left(\frac{x-t}{x-a} \right)^s \frac{f(a)}{e^{aa}} + \left(\frac{t-a}{x-a} \right)^s \frac{f(x)}{e^{ax}}. \quad (17)$$

By multiplying (13) and (14) and then integrating over $[a, x]$, we get

$$\begin{aligned} & \int_a^x (x-t)^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-t)^\mu; p) f(t) dt \\ & \leq \frac{(x-a)^{\alpha-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p)}{(x-a)^s} \\ & \cdot \left(\frac{f(a)}{e^{aa}} \int_a^x (x-t)^s dt + \frac{f(x)}{e^{ax}} \int_a^x (t-a)^s dt \right). \end{aligned} \quad (18)$$

The left integral operator follows the upcoming inequality:

$$\left(\epsilon_{\mu, \sigma, l, \omega, a^+}^{\gamma, \delta, k, c} f \right)(x; p) \leq \frac{(x-a) D_{\sigma-1, a^+}(x; p)}{s+1} \left(\frac{f(a)}{e^{aa}} + \frac{f(x)}{e^{ax}} \right). \quad (19)$$

Now, on the contrary, for $t \in (x, b]$ and $\tau \geq 1$, we have the following inequality:

$$\begin{aligned} & (t-x)^{\tau-1} E_{\mu, \tau, l}^{\gamma, \delta, k, c}(\omega(t-x)^\mu; p) \\ & \leq (b-x)^{\tau-1} E_{\mu, \tau, l}^{\gamma, \delta, k, c}(\omega(b-x)^\mu; p). \end{aligned} \quad (20)$$

Again from exponentially s -convexity of f , we have

$$f(t) \leq \left(\frac{t-x}{b-x} \right)^s \frac{f(b)}{e^{\beta b}} + \left(\frac{b-t}{b-x} \right)^s \frac{f(x)}{e^{\beta x}}. \quad (21)$$

By multiplying (20) and (21) and then integrating over $[x, b]$, we get

$$\begin{aligned} & \int_x^b (t-x)^{\tau-1} E_{\mu, \tau, l}^{\gamma, \delta, k, c}(\omega(t-x)^\mu; p) f(t) dt \\ & \leq \frac{(b-x)^{\tau-1} E_{\mu, \tau, l}^{\gamma, \delta, k, c}(\omega(b-x)^\mu; p)}{(b-x)^s} \\ & \cdot \left(\frac{f(b)}{e^{\beta b}} \int_x^b (t-x)^s dt + \frac{f(x)}{e^{\beta x}} \int_x^b (b-t)^s dt \right). \end{aligned} \quad (22)$$

The right integral operator satisfies the following inequality:

$$\left(\epsilon_{\mu, \tau, l, \omega, b^-}^{\gamma, \delta, k, c} f \right)(x; p) \leq \frac{(b-x) D_{\tau-1, b^-}(x; p)}{s+1} \left(\frac{f(b)}{e^{\beta b}} + \frac{f(x)}{e^{\beta x}} \right). \quad (23)$$

By adding (19) and (23), required inequality (15) can be obtained. $\square$

Corollary 1. If we set $\sigma = \tau$ in (15), then the following inequality is obtained:

$$\begin{aligned} & \left(\epsilon_{\mu, \sigma, l, \omega, a^+}^{\gamma, \delta, k, c} f \right)(x; p) + \left(\epsilon_{\mu, \sigma, l, \omega, b^-}^{\gamma, \delta, k, c} f \right)(x; p) \\ & \leq \left(\frac{f(a)}{e^{aa}} + \frac{f(x)}{e^{ax}} \right) \frac{(x-a) D_{\sigma-1, a^+}(x; p)}{s+1} \\ & + \left(\frac{f(b)}{e^{\beta b}} + \frac{f(x)}{e^{\beta x}} \right) \frac{(b-x) D_{\sigma-1, b^-}(x; p)}{s+1}, \quad x \in [a, b]. \end{aligned} \quad (24)$$

Corollary 2. Along with assumption of Theorem 1, if $f \in L_\infty[a, b]$, then the following inequality is obtained:

$$\begin{aligned} & \left(\epsilon_{\mu, \sigma, l, \omega, a^+}^{\gamma, \delta, k, c} f \right)(x; p) + \left(\epsilon_{\mu, \tau, l, \omega, b^-}^{\gamma, \delta, k, c} f \right)(x; p) \\ & \leq \frac{\|f\|_\infty}{s+1} \left[\left(\frac{1}{e^{aa}} + \frac{1}{e^{ax}} \right) (x-a) D_{\sigma-1, a^+}(x; p) \right. \\ & \quad \left. + \left(\frac{1}{e^{\beta b}} + \frac{1}{e^{\beta x}} \right) (b-x) D_{\tau-1, b^-}(x; p) \right]. \end{aligned} \quad (25)$$

Corollary 3. For $\sigma = \tau$ in (25), we get the following result:

$$\begin{aligned} & \left(\epsilon_{\mu, \sigma, l, \omega, a^+}^{\gamma, \delta, k, c} f \right)(x; p) + \left(\epsilon_{\mu, \sigma, l, \omega, b^-}^{\gamma, \delta, k, c} f \right)(x; p) \\ & \leq \frac{\|f\|_\infty}{s+1} \left[\left(\frac{1}{e^{aa}} + \frac{1}{e^{ax}} \right) (x-a) D_{\sigma-1, a^+}(x; p) \right. \\ & \quad \left. + \left(\frac{1}{e^{\beta b}} + \frac{1}{e^{\beta x}} \right) (b-x) D_{\sigma-1, b^-}(x; p) \right]. \end{aligned} \quad (26)$$

Corollary 4. For $s = 1$ in (25), we get the following result for exponentially convex functions:

$$\begin{aligned} & \left(\epsilon_{\mu, \alpha, l, \omega, a^+}^{\gamma, \delta, k, c} f \right)(x; p) + \left(\epsilon_{\mu, \beta, l, \omega, b^-}^{\gamma, \delta, k, c} f \right)(x; p) \\ & \leq \frac{\|f\|_\infty}{2} \left[\left(\frac{1}{e^{aa}} + \frac{1}{e^{ax}} \right) (x-a) D_{\alpha-1, a^+}(x; p) \right. \\ & \quad \left. + \left(\frac{1}{e^{\beta b}} + \frac{1}{e^{\beta x}} \right) (b-x) D_{\beta-1, b^-}(x; p) \right]. \end{aligned} \quad (27)$$

Theorem 2. With the assumptions of Theorem 1, if $f \in L_\infty[a, b]$, then operators defined in (9) and (10) are bounded and continuous.

Proof. If $f \in L_\infty[a, b]$, then from (19), we have

$$\begin{aligned} & \left| \left(\epsilon_{\mu, \sigma, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x; p) \right| \\ & \leq \frac{\|f\|_\infty (x-a) D_{\sigma-1, a^+}(x; p)}{s+1} \left(\frac{1}{e^{a\alpha}} + \frac{1}{e^{a\chi}} \right) \\ & \leq \frac{2\|f\|_\infty (b-a) D_{\sigma-1, a^+}(b; p)}{(s+1)e^{a\alpha}}, \end{aligned} \quad (28)$$

that is,

$$\left| \left(\epsilon_{\mu, \sigma, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x; p) \right| \leq M \|f\|_\infty, \quad (29)$$

where $M = (2(b-a)D_{\sigma-1, a^+}(b; p)/(s+1)e^{a\alpha})$. Therefore, $(\epsilon_{\mu, \sigma, l, \omega, a^+}^{\gamma, \delta, k, c} f)(x; p)$ is bounded, and also, it is easy to see that it is linear; hence, this is a continuous operator. Also, on the contrary, from (23), one can obtain

$$\left| \left(\epsilon_{\mu, \tau, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x; p) \right| \leq K \|f\|_\infty, \quad (30)$$

where $K = (2(b-a)D_{\tau-1, b^-}(a; p)/(s+1)e^{b\alpha})$. Therefore, $(\epsilon_{\mu, \tau, l, \omega, b^-}^{\gamma, \delta, k, c} f)(x; p)$ is bounded and also it is linear, hence continuous.

The next result provides boundedness of the left and the right fractional integrals at an arbitrary point for functions whose derivatives in absolute values are exponentially s -convex. $\square$

Theorem 3. Let $f: [a, b] \rightarrow \mathbb{R}$ be a real-valued function. If f is differentiable and $|f'|$ is exponentially s -convex, then for $\sigma, \tau \geq 1$, the following fractional integral inequality for generalized integral operators (9) and (10) holds:

$$\begin{aligned} & \left| \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x; p) + \left(\epsilon_{\mu, \tau+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x; p) \right. \\ & \quad \left. - (D_{\sigma-1, a^+}(x; p)f(a) + D_{\tau-1, b^-}(x; p)f(b)) \right| \\ & \leq \left(\frac{|f'(a)|}{e^{a\alpha}} + \frac{|f'(x)|}{e^{a\chi}} \right) \frac{(x-a)D_{\sigma-1, a^+}(x; p)}{s+1} \\ & \quad + \left(\frac{|f'(b)|}{e^{b\beta}} + \frac{|f'(x)|}{e^{b\chi}} \right) \frac{(b-x)D_{\tau-1, b^-}(x; p)}{s+1}, \quad x \in [a, b]. \end{aligned} \quad (31)$$

Proof. Let $x \in [a, b]$ and $t \in [a, x]$; by using exponentially s -convexity of $|f'|$, we have

$$|f'(t)| \leq \left(\frac{x-t}{x-a} \right)^s \frac{|f'(a)|}{e^{a\alpha}} + \left(\frac{t-a}{x-a} \right)^s \frac{|f'(x)|}{e^{a\chi}}. \quad (32)$$

From (32), one can have

$$f'(t) \leq \left(\frac{x-t}{x-a} \right)^s \frac{|f'(a)|}{e^{a\alpha}} + \left(\frac{t-a}{x-a} \right)^s \frac{|f'(x)|}{e^{a\chi}}. \quad (33)$$

The product of (16) and (33) gives the following inequality:

$$\begin{aligned} & (x-t)^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-t)^\mu; p) f'(t) dt \\ & \leq (x-a)^{\sigma-1-s} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p) \\ & \quad \cdot \left(\frac{|f'(a)|}{e^{a\alpha}} (x-t)^s + \frac{|f'(x)|}{e^{a\chi}} (t-a)^s \right). \end{aligned} \quad (34)$$

After integrating the above inequality over $[a, x]$, we get

$$\begin{aligned} & \int_a^x (x-t)^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-t)^\mu; p) f'(t) dt \\ & \leq (x-a)^{\sigma-1-s} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p) \\ & \quad \cdot \left(\frac{|f'(a)|}{e^{a\alpha}} \int_a^x (x-t)^s dt + \frac{|f'(x)|}{e^{a\chi}} \int_a^x (t-a)^s dt \right) \\ & = \frac{(x-a)^\sigma E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p)}{s+1} \left(\frac{|f'(a)|}{e^{a\alpha}} + \frac{|f'(x)|}{e^{a\chi}} \right). \end{aligned} \quad (35)$$

The left-hand side of (35) is calculated as follows:

$$\int_a^x (x-t)^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-t)^\mu; p) f'(t) dt. \quad (36)$$

Putting $x-t=z$, that is, $t=x-z$ and also utilizing (8), we have

$$\begin{aligned} & \int_0^{x-a} z^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega z^\mu; p) f'(x-z) dz \\ & = (x-a)^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p) f(a) \\ & \quad - \int_0^{x-a} z^{\sigma-2} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega z^\mu; p) f(x-z) dz. \end{aligned} \quad (37)$$

Now, putting $x-z=t$ in the second term of the right-hand side of the above equation and then using (9), we get

$$\begin{aligned} & \int_0^{x-a} z^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega z^\mu; p) f'(x-z) dz \\ & = (x-a)^{\sigma-1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p) f(a) \\ & \quad - \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x; p). \end{aligned} \quad (38)$$

Therefore, (35) takes the following form:

$$\begin{aligned} & (D_{\sigma-1, a^+}(x; p)f(a) - \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x; p)) \\ & \leq \frac{(x-a)D_{\sigma-1, a^+}(x; p)}{s+1} \left(\frac{|f'(a)|}{e^{a\alpha}} + \frac{|f'(x)|}{e^{a\chi}} \right). \end{aligned} \quad (39)$$

Also, from (32), one can have

$$f'(t) \geq - \left(\left(\frac{x-t}{x-a} \right)^s \frac{|f'(a)|}{e^{a\alpha}} + \left(\frac{t-a}{x-a} \right)^s \frac{|f'(x)|}{e^{a\chi}} \right). \quad (40)$$

Following the same procedure as we did for (33), one can obtain

$$\begin{aligned} & \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x; p) - D_{\sigma-1, a^+} (x; p) f(a) \\ & \leq \frac{(x-a) D_{\sigma-1, a^+} (x; p)}{s+1} \left(\frac{|f'(a)|}{e^{a\alpha}} + \frac{|f'(x)|}{e^{\alpha x}} \right). \end{aligned} \quad (41)$$

From (39) and (41), we get

$$\begin{aligned} & \left| \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x; p) - D_{\sigma-1, a^+} (x; p) f(a) \right| \\ & \leq \frac{(x-a) D_{\sigma-1, a^+} (x; p)}{s+1} \left(\frac{|f'(a)|}{e^{a\alpha}} + \frac{|f'(x)|}{e^{\alpha x}} \right). \end{aligned} \quad (42)$$

Now, we let $x \in [a, b]$ and $t \in [x, b]$. Then, by exponentially s -convexity of $|f'|$, we have

$$|f'(t)| \leq \left(\frac{t-x}{b-x} \right)^s \frac{|f'(b)|}{e^{\beta b}} + \left(\frac{b-t}{b-x} \right)^s \frac{|f'(x)|}{e^{\beta x}}. \quad (43)$$

On the same lines as we have done for (16), (33), and (40), one can get from (20) and (43) the following inequality:

$$\begin{aligned} & \left| \left(\epsilon_{\mu, \tau+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x; p) - D_{\tau-1, b^-} (x; p) f(b) \right| \\ & \leq \frac{(b-x) D_{\tau-1, b^-} (x; p)}{s+1} \left(\frac{|f'(b)|}{e^{\beta b}} + \frac{|f'(x)|}{e^{\beta x}} \right). \end{aligned} \quad (44)$$

From inequalities (42) and (44), via triangular inequality, (28) can be obtained.

The following results hold as special cases. $\square$

Corollary 5. If we put $\sigma = \tau$ in (28), then the following inequality is obtained:

$$\begin{aligned} & \left| \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x; p) + \left(\epsilon_{\mu, \sigma+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x; p) \right. \\ & \quad \left. - (D_{\sigma-1, a^+} (x; p) f(a) + D_{\sigma-1, b^-} (x; p) f(b)) \right| \\ & \leq \left(\frac{|f'(a)|}{e^{a\alpha}} + \frac{|f'(x)|}{e^{\alpha x}} \right) \frac{(x-a) D_{\sigma-1, a^+} (x; p)}{s+1} \\ & \quad + \left(\frac{|f'(b)|}{e^{\beta b}} + \frac{|f'(x)|}{e^{\beta x}} \right) \frac{(b-x) D_{\sigma-1, b^-} (x; p)}{s+1}, \quad x \in [a, b]. \end{aligned} \quad (45)$$

Definition 7. Let $f: [a, b] \rightarrow \mathbb{R}$ be a function; we say f is exponentially symmetric about $((a+b)/2)$ if

$$\frac{f(x)}{e^{\alpha x}} = \frac{f(a+b-x)}{e^{\alpha(a+b-x)}}, \quad \alpha \in \mathbb{R}. \quad (46)$$

It is required to give the following lemma which will be helpful to produce Hadamard-type estimations.

Lemma 2. Let $f: [a, b] \rightarrow \mathbb{R}$ be an exponentially s -convex function. If f is exponentially symmetric, then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(x)}{2^{s-1} e^{\alpha x}}. \quad (47)$$

Proof. For $[a, b] \subset \mathbb{R}$ be a closed interval, $t \in [0, 1]$, and $\alpha \in \mathbb{R}$, we have

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{at + (1-t)b}{2} + \frac{a(1-t) + bt}{2}\right). \quad (48)$$

Since f is exponentially s -convex, so

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(at + (1-t)b)}{2^s e^{\alpha(at + (1-t)b)}} + \frac{f(a(1-t) + bt)}{2^s e^{\alpha(a(1-t) + bt)}}. \quad (49)$$

Let $x = at + (1-t)b$, where $x \in [a, b]$. Then, we have $a + b - x = bt + (1-t)a$, and we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(x)}{2^s e^{\alpha(x)}} + \frac{f(a+b-x)}{2^s e^{\alpha(a+b-x)}}. \quad (50)$$

Now, using the fact of exponentially symmetric, we will get (47). $\square$

Theorem 4. Let $f: [a, b] \rightarrow \mathbb{R}$, $a > b$, be a real-valued function. If f is positive, exponentially s -convex, and symmetric about $((a+b)/2)$, then for $\sigma, \tau > 0$, the following fractional integral inequality for generalized integral operators (9) and (10) holds:

$$\begin{aligned} & 2^{s-1} e^{\alpha x} f\left(\frac{a+b}{2}\right) [D_{\tau+1, b^-} (a; p) + D_{\sigma+1, a^+} (b; p)] \\ & \leq \left(\epsilon_{\mu, \tau+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (a; p) + \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (b; p) \\ & \leq [D_{\tau-1, b^-} (a; p) + D_{\sigma-1, a^+} (b; p)] \frac{(b-a)^2}{s+1} \left(\frac{f(a)}{e^{a\alpha}} + \frac{f(b)}{e^{\beta b}} \right). \end{aligned} \quad (51)$$

Proof. For $x \in [a, b]$, we have

$$\begin{aligned} & (x-a)^\tau E_{\mu, \tau, l}^{\gamma, \delta, k, c} (\omega(x-a)^\mu; p) \\ & \leq (b-a)^\tau E_{\mu, \tau, l}^{\gamma, \delta, k, c} (\omega(b-a)^\mu; p), \quad \tau > 0. \end{aligned} \quad (52)$$

As f is exponentially s -convex, so for $x \in [a, b]$, we have

$$f(x) \leq \left(\frac{x-a}{b-a} \right)^s \frac{f(b)}{e^{\beta b}} + \left(\frac{b-x}{b-a} \right)^s \frac{f(a)}{e^{a\alpha}}. \quad (53)$$

By multiplying (52) and (53) and then integrating over $[a, b]$, we get

$$\begin{aligned} & \int_a^b (x-a)^\tau E_{\mu, \tau, l}^{\gamma, \delta, k, c} (\omega(x-a)^\mu; p) f(x) dx \\ & \leq (b-a)^{\tau-s} E_{\mu, \tau, l}^{\gamma, \delta, k, c} (\omega(b-a)^\mu; p) \\ & \quad \cdot \left(\frac{f(b)}{e^{\beta b}} \int_a^b (x-a)^s dx + \frac{f(a)}{e^{a\alpha}} \int_a^b (b-x)^s dx \right), \end{aligned} \quad (54)$$

from which we have

$$\begin{aligned} \left(\epsilon_{\mu, \tau+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (a; p) &\leq \frac{(b-a)^{\tau+1} E_{\mu, \tau, l}^{\gamma, \delta, k, c} (\omega (b-a)^\mu; p)}{s+1} \\ &\quad \cdot \left(\frac{f(a)}{e^{a\alpha}} + \frac{f(b)}{e^{b\beta}} \right), \end{aligned} \quad (55)$$

$$\left(\epsilon_{\mu, \tau+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (a; p) \leq \frac{(b-a)^2}{s+1} D_{\tau-1, b^-} (a; p) \left(\frac{f(a)}{e^{a\alpha}} + \frac{f(b)}{e^{b\beta}} \right). \quad (56)$$

Now, on the contrary, for $x \in [a, b]$, we have

$$\begin{aligned} (b-x)^\sigma E_{\mu, \sigma, l}^{\gamma, \delta, k, c} (\omega (b-x)^\mu; p) \\ \leq (b-a)^\sigma E_{\mu, \sigma, l}^{\gamma, \delta, k, c} (\omega (b-a)^\mu; p), \quad \alpha > 0. \end{aligned} \quad (57)$$

By multiplying (53) and (57) and then integrating over $[a, b]$, we get

$$\begin{aligned} \int_a^b (b-x)^\sigma E_{\mu, \sigma, l}^{\gamma, \delta, k, c} (\omega (b-x)^\mu; p) f(x) dx \\ \leq (b-a)^{\sigma-s} E_{\mu, \sigma, l}^{\gamma, \delta, k, c} (\omega (b-a)^\mu; p) \\ \cdot \left(\frac{f(b)}{e^{b\beta}} \int_a^b (x-a)^s dx + \frac{f(a)}{e^{a\alpha}} \int_a^b (b-x)^s dx \right), \end{aligned} \quad (58)$$

from which we have

$$\begin{aligned} \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (b; p) &\leq \frac{(b-a)^{\sigma+1} E_{\mu, \sigma, l}^{\gamma, \delta, k, c} (\omega (b-a)^\mu; p)}{s+1} \\ &\quad \cdot \left(\frac{f(a)}{e^{a\alpha}} + \frac{f(b)}{e^{b\beta}} \right), \end{aligned} \quad (59)$$

$$\left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (b; p) \leq \frac{(b-a)^2}{s+1} D_{\sigma-1, a^+} (b; p) \left(\frac{f(a)}{e^{a\alpha}} + \frac{f(b)}{e^{b\beta}} \right). \quad (60)$$

Adding (56) and (60), we get

$$\begin{aligned} \left(\epsilon_{\mu, \tau+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (a; p) + \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (b; p) \\ \leq [D_{\tau-1, b^-} (a; p) + D_{\sigma-1, a^+} (b; p)] \frac{(b-a)^2}{s+1} \left(\frac{f(a)}{e^{a\alpha}} + \frac{f(b)}{e^{b\beta}} \right). \end{aligned} \quad (61)$$

Multiplying (47) with $(x-a)^\tau E_{\mu, \tau, l}^{\gamma, \delta, k, c} (\omega (x-a)^\mu; p)$ and integrating over $[a, b]$, we get

$$\begin{aligned} f \left(\frac{a+b}{2} \right) \int_a^b (x-a)^\tau E_{\mu, \tau, l}^{\gamma, \delta, k, c} (\omega (x-a)^\mu; p) dx \\ \leq \frac{1}{2^{s-1} e^{\alpha x}} \int_a^b (x-a)^\tau E_{\mu, \tau, l}^{\gamma, \delta, k, c} (\omega (x-a)^\mu; p) f(x) dx. \end{aligned} \quad (62)$$

By using (10) and (14), we get

$$f \left(\frac{a+b}{2} \right) D_{\tau+1, b^-} (a; p) \leq \frac{1}{2^{s-1} e^{\alpha x}} \left(\epsilon_{\mu, \tau+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (a; p). \quad (63)$$

Multiplying (47) with $(b-x)^\sigma E_{\mu, \sigma, l}^{\gamma, \delta, k, c} (\omega (b-x)^\mu; p)$, integrating over $[a, b]$, and also using (6) and (10), we get

$$f \left(\frac{a+b}{2} \right) D_{\sigma+1, a^+} (b; p) \leq \frac{1}{2^{s-1} e^{\alpha x}} \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (b; p). \quad (64)$$

By adding (63) and (64), we get

$$\begin{aligned} 2^{s-1} e^{\alpha x} f \left(\frac{a+b}{2} \right) [D_{\tau+1, b^-} (a; p) + D_{\sigma+1, a^+} (b; p)] \\ \leq \left(\epsilon_{\mu, \tau+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (a; p) + \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (b; p). \end{aligned} \quad (65)$$

By combining (61) and (65), inequality (51) can be obtained. $\square$

Corollary 6. If we put $\sigma = \tau$ in (51), then the following inequality is obtained:

$$\begin{aligned} 2^{s-1} e^{\alpha x} f \left(\frac{a+b}{2} \right) [D_{\sigma+1, b^-} (a; p) + D_{\sigma+1, a^+} (b; p)] \\ \leq \left(\epsilon_{\mu, \sigma+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (a; p) + \left(\epsilon_{\mu, \sigma+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (b; p) \\ \leq [D_{\sigma-1, b^-} (a; p) + D_{\sigma-1, a^+} (b; p)] \frac{(b-a)^2}{s+1} \left(\frac{f(a)}{e^{a\alpha}} + \frac{f(b)}{e^{b\beta}} \right). \end{aligned} \quad (66)$$

3. Concluding Remarks

We have established the general fractional integral inequalities by using exponentially s -convex functions. By selecting particular values at the place of parameters, a variety of results can be obtained. For example, the reader can obtain bounds for fractional integral operators defined by Salim and Faraj in [12] by selecting $p = 0$, bounds for fractional integral operators defined by Rahman et al. in [11] by selecting $l = \delta = 1$, bounds for fractional integral operators defined by Shukla and Prajapati in [13] by selecting $p = 0$ and $l = \delta = 1$, bounds for fractional integral operators defined by Prabhakar in [10] by selecting $p = 0$ and $l = \delta = k = 1$, and bounds for Riemann–Liouville fractional integrals by selecting $p = \omega = 0$.

Data Availability

All the data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors do not have any conflicts of interest.

Authors' Contributions

All authors contributed equally to this paper.

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