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Research Article

Results on Solutions for Several q-Painlevé Difference Equations concerning Rational Solutions, Zeros, and Poles

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In this article, we discuss the problem about the properties on solutions for several types of q-difference equations and obtain some results on the exceptional values of transcendental meromorphic solutions $f(z)$ with zero order, their q-differences $\Delta_q f(z) = f(qz) - f(z)$, and divided differences $\Delta_q f(z)/f(z)$. In addition, we also investigated the condition on the existence of rational solution for a class of q-difference equations. Our theorems are some extensions and supplement to those results given by Liu and Zhang and Qi and Yang.

1. Introduction and Main Results

All the time, Painlevé equations have attracted much interest due to the reduction of solution equations, which are solvable by inverse scattering transformations, and they often occur in many physical situations: plasma physics, statistical mechanics, and nonlinear waves. The study of Painlevé equations has spanned more than one hundred years (see [1–3]).

Around 2006, Halburd and Korhonen [4, 5] and Ronkainen [6] used Nevanlinna theory to discuss the following equations:

$$\begin{aligned} f(z+1)+f(z-1) &= R(z,f) \\ &= \frac{P(z,f)}{Q(z,f)}, \\ f(z+1)f(z-1) &= R(z,f) \\ &= \frac{P(z,f)}{Q(z,f)}, \end{aligned} \tag{1}$$

where $R(z,f)$ is rational in f and meromorphic in z, respectively, and they singled out the following difference equations:

$$f(z+1)+f(z-1) = \frac{az+b}{f(z)} + c, \tag{2}$$

$$f(z+1)+f(z-1) = \frac{(az+b)f(z)+c}{1-f(z)^2}, \tag{3}$$

$$f(z+1)f(z-1) = \frac{\eta(z)f(z)^2 - \lambda(z)f(z) + \mu(z)}{(f(z)-1)(f(z)-v(z))}, \tag{4}$$

$$f(z+1)f(z-1) = \frac{\eta(z)f(z)^2 - \lambda(z)f(z)}{f(z)-1}, \tag{5}$$

$$f(z+1)f(z-1) = \frac{\eta(z)(f(z)-\lambda(z))}{(f(z)-1)}, \tag{6}$$

$$f(z+1)f(z-1) = h(z)f(z)^m, \tag{7}$$

where $\eta(z), \lambda(z)$, and $v(z)$ satisfy some conditions. In these equations, equation (2) is called as the difference Painlevé I equation, equation (3) is called as the difference Painlevé II equation, and the last four equations are called as the difference Painlevé III equations.

In the last decade or so, there were a lot of papers focusing on the properties of solutions for difference Painlevé I–IV equations (see [7–11]). For example, Chen and Shon [12] in 2010 considered the difference Painlevé I equation (3) and obtained the following theorem.

Theorem 1 (see [12], Theorem 4). *Let a, b, c be constants, where a, b are not both equal to zero. Then, the following holds:*

(i) If $a \neq 0$, then (3) has no rational solution.

(ii) If $a = 0$ and $b \neq 0$, then (3) has a nonzero constant solution $w(z) = A$, where A satisfies $2A^2 - cA - b = 0$.

The other rational solution $w(z)$ satisfies $w(z) = (P(z)/Q(z)) + A$, where $P(z)$ and $Q(z)$ are relatively prime polynomials and satisfy $\deg P < \deg Q$.

In 2013 and 2018, Zhang and Yi [11] and Du et al. [13] studied the difference Painlevé III equations with the constant coefficients and obtained the result as follows.

Theorem 2 (see [11, 13]). *If f is a transcendental finite-order meromorphic solution of*

$$f(z+1)f(z-1)(f(z)-1)^2 = f(z)^2 - \lambda f(z) + \mu, \quad (8)$$

where λ and μ are constants, then the following holds:

(i) $\tau(f) = \sigma(f)$.

(ii) If $\lambda\mu \neq 0$, then $\lambda(f) = \sigma(f)$.

(iii) For any $\eta \in C/\{0\}$, $\tau(f(z+\eta)) = \sigma(f)$.

(iv) $\lambda(1/\Delta f) = \lambda(1/(\Delta f/f)) = \sigma(f)$.

Ramani et al. [14] in 2003 investigated the existence of transcendental solution of equation

$$\begin{aligned}(f(z+1)+f(z))(f(z)+f(z-1)) &= R(z,f)\\ &= \frac{P(z,f)}{Q(z,f)},\end{aligned} \quad (9)$$

which is called as difference Painlevé IV equations and obtained the result as follows.

Theorem 3 (see [14]). *If the second-order difference equation (9) admits a nonrational meromorphic solution of finite order, then $\deg_z P \leq 4$ and $\deg_z Q \leq 2$.*

Of late, many mathematicians paid considerable attention to the value distribution of solutions for complex q-difference equations, which are formed by replacing the q-difference $f(qz)$, $q \in C/\{0,1\}$ with $f(z+c)$ of meromorphic function in some expressions concerning complex difference equations, by utilizing the logarithmic derivative lemma on q-difference operators given by Barnett et al. [15] in 2007 (see [16–26]). For example, Qi and Yang [27] considered the following equation:

$$f(qz) + f\left(\frac{z}{q}\right) = \frac{az+b}{f(z)} + c, \quad (10)$$

which can be seen as q-difference analogues of (2) and obtained the result as follows.

Theorem 4 (see [27], Theorem 1). *Let $f(z)$ be a transcendental meromorphic solution with zero order of equation (10) and a, b, c be three constants such that a, b cannot vanish simultaneously. Then, the following holds:*

(i) $f(z)$ has infinitely many poles.

(ii) If $a \neq 0$ and any $d \in C$, then $f(z) - d$ has infinitely many zeros.

(iii) If $a = 0$ and $f(z)$ takes a finite value A finitely often, then A is a solution of $2z^2 - cz - b = 0$.

In 2018, Liu and Zhang [28] further investigated the following equation:

$$Y(\omega z) + Y(z) + Y\left(\frac{z}{\omega}\right) = \frac{\xi z + o}{Y(z)} + \nu, \quad (11)$$

and obtained the result as follows.

Theorem 5 (see [28], Theorem 1). *Let $Y(z)$ be a transcendental meromorphic solution with zero order of (11) and ξ, o, ν be three constants such that ξ, o cannot vanish simultaneously. Then, the following holds:*

(i) $Y(z)$ has infinitely many poles.

(ii) For any finite value B, if $\xi = 0$, then $Y(z) - B$ has infinitely many zeros.

(iii) If $\xi = 0$ and $Y(z) - A$ has finite zeros, then A is a solution of $3z^2 - o - \nu z = 0$.

Motivated by the idea [27, 28], a natural question is what is the result if we give q-difference analogues of (9). For this question, our main aim of this article is further to investigate some properties of meromorphic solutions for some q-Painlevé difference IV equations. It seems that this topic has never been treated before.

In what follows, it should be assumed that the readers are familiar with the fundamental results and the standard notations in the theory of Nevanlinna value distribution (see Hayman [29], Yang [30], and Yi and Yang [31]). Let f be a meromorphic function, and we denote $\sigma(f), \lambda(f)$, and $\lambda(1/f)$ to be the order, the exponent of convergence of zeros, and the exponent of convergence of poles of $f(z)$, respectively, and denote $\tau(f)$ to be the exponent of convergence of fixed points of $f(z)$, which is defined by

$$\tau(f) = \limsup_{r\longrightarrow+\infty} \frac{\log N(r, 1/(f(z)-z))}{\log r}. \quad (12)$$

In addition, we use $S(r,f)$ denotes any quantity satisfying $S(r,f)=o(T(r,f))$ for all r on a set F of logarithmic density 1, and the logarithmic density of a set F is defined by

$$\lim_{r\longrightarrow\infty}\sup\frac{1}{\log r}\int_{[1,r]\cap F}\frac{1}{t}\mathrm{dt}. \tag{13}$$

Now, our main results are listed as follows.

Theorem 6. *Let $R(z)=A(z)/B(z)$ be an irreducible rational function, and let $q(\neq 0)\in C$ and $|q|\neq 1$, and*

$$\begin{aligned}[f(qz)+f(z)]\left[f(z)+f\left(\frac{z}{q}\right)\right]&=R(z)\\&=\frac{A(z)}{B(z)},\end{aligned} \tag{14}$$

where $A(z),B(z)$ are polynomials with $\deg_z A(z)=a$ *and* $\deg_z B(z)=b$.

(i) Suppose that $a\geq b$ and $a-b$ are even numbers or zero. If equation (14) has an irreducible rational solution $f(z)=\mu(z)/\nu(z)$, where $\mu(z),\nu(z)$ are polynomials with $\deg_z\mu(z)=\mu$ and $\deg_z\nu(z)=\nu$, then the following holds:

$$\mu-\nu=\frac{a-b}{2}. \tag{15}$$

(ii) Suppose that $b\geq a$ and $b-a$ are even numbers or zero. If equation (14) has an irreducible rational solution $f(z)=\mu(z)/\nu(z)$, where $\mu(z),\nu(z)$ are polynomials with $\deg_z\mu(z)=\mu$ and $\deg_z\nu(z)=\nu$, then

$$\nu-\mu=\frac{b-a}{2}. \tag{16}$$

(iii) If $|a-b|$ is an odd number, then equation (14) has no rational solution.

Theorem 7. *For $q(\neq 0)\in C$ and $|q|\neq 1$, let $f(z)$ be a transcendental meromorphic solution with zero order of equation*

$$[f(qz)+f(z)]\left(f(z)+f\left(\frac{z}{q}\right)\right)=af(z)^2, \tag{17}$$

where $a(\neq 0,4)$ is a constant. Let $\Delta_q f=f(qz)-f(z)$. Then, the following holds:

(i) Both f and $\Delta_q f$ have no nonzero finite Nevanlinna exceptional value.

(ii) If $q+(1/q)\neq a-2$, then $f(\eta z)$ and $\Delta_q f(\eta z)$ have infinitely many fixed points and $\tau(f(\eta z))=\tau(\Delta_q f(\eta z))=\sigma(f)$ for any nonzero constant η.

Theorem 8. *For $q(\neq 0)\in C$ and $|q|\neq 1$, and let $f(z)$ be a transcendental meromorphic solution with zero order of equation*

$$[f(qz)+f(z)]\left[f(z)+f\left(\frac{z}{q}\right)\right]=d(z)f(z), \tag{18}$$

where $d(z)$ is a nonconstant rational function satisfying that $d(qz)/d(z)$ is not a constant. Then, the following holds:

(i) Both f and $\Delta_q f/f$ have no Nevanlinna exceptional value.

(ii) $\Delta_q f$ has infinitely many poles and zeros, and $\lambda(1/\Delta_q f)=\lambda(\Delta_q f)=\sigma(f)$.

(iii) $\Delta_q f/f$ has infinitely many fixed points and $\tau(\Delta_q f/f)=\sigma(f)$.

2. Proof of Theorem 6

Proof: assume that (14) has a rational solution $f(z)=\mu(z)/\nu(z)$ and has poles $z_1,z_2,\ldots,z_k$. Then, $f(z)$ can be represented in the following form:

$$\begin{aligned}f(z)=\frac{\mu(z)}{\nu(z)}&=\sum_{j=1}^{k}\left[\frac{c_{j\lambda_j}}{(z-z_j)^{\lambda_j}}+\cdots+\frac{c_{j1}}{(z-z_j)}\right]\\&\quad+a_0+a_1z+\cdots+a_sz^s,\end{aligned} \tag{19}$$

where $c_{j\lambda_j}(\neq 0),\ldots,c_{j1}(j=1,2,\ldots,k)$ and $a_0,a_1,\ldots,a_s$ are constants; $z_j(j=1,2,\ldots,k)$ are poles of $f(z)$ with multiplicity λ_j, respectively.

(i) Suppose that $a>b$ and $a-b$ are even numbers. Then, in view of (14) and (19), it yields

$$\left(\frac{\mu(qz)}{\nu(qz)}+\frac{\mu(z)}{\nu(z)}\right)\left(\frac{\mu(z)}{\nu(z)}+\frac{\mu(z/q)}{\nu(z/q)}\right)=\frac{A(z)}{B(z)}. \tag{20}$$

If $\deg_z\mu(z)=\mu<\nu=\deg_z\nu(z)$, then for $z\longrightarrow\infty$, it follows

$$\begin{aligned}\frac{\mu(qz)}{\nu(qz)}&\longrightarrow 0,\\\frac{\mu(z)}{\nu(z)}&\longrightarrow 0,\\\frac{\mu(z/q)}{\nu(z/q)}&\longrightarrow 0.\end{aligned} \tag{21}$$

However, $A(z)/B(z)\longrightarrow\infty$ as $z\longrightarrow\infty$; thus, from (20), we can get a contradiction easily.

If $\mu=\nu$, then let $z\longrightarrow\infty$, and it leads to

$$\begin{aligned}\frac{\mu(qz)}{\nu(qz)}&\longrightarrow\alpha,\\\frac{\mu(z)}{\nu(z)}&\longrightarrow\alpha,\\\frac{\mu(z/q)}{\nu(z/q)}&\longrightarrow\alpha,\end{aligned} \tag{22}$$

where α is a nonzero constant. Thus, let $z \longrightarrow +\infty$; in view of (20), we also get a contradiction. So, it follows $\mu > \nu$. Thus, assume that $a_s \neq 0$, $(s \geq 1)$, where $s = \mu - \nu$. As $z \longrightarrow \infty$, it yields

$$\begin{aligned} f(z) &= a_s z^s (1 + o(1)),\\ f(qz) &= a_s q^s z^s (1 + o(1)),\\ f\left(\frac{z}{q}\right) &= a_s q^{-s} z^s (1 + o(1)),\\ \frac{A(z)}{B(z)} &= \beta z^{a-b} (1 + o(1)), \end{aligned} \tag{23}$$

where $\beta(\neq 0)$ is a constant, and it follows now in view of (20) that

$$(2 + q^s + q^{-s}) a_s^2 z^{2s} (1 + o(1)) = \beta z^{a-b} (1 + o(1)), \tag{24}$$

as $z \longrightarrow \infty$. Since $|q| \neq 1$, then $q^s + 2 + q^{-s} \neq 0$. Hence, it follows from (24) that

$$\begin{aligned} \mu - \nu &= s\\ &= \frac{a-b}{2}. \end{aligned} \tag{25}$$

Next, assume that $a = b$. As $z \longrightarrow \infty$, it follows

$$\frac{A(z)}{B(z)} = \beta(1 + o(1)), \tag{26}$$

where $\beta(\neq 0)$ is a constant. If $\mu < \nu$, then by using the same argument as above, we get a contradiction. If $\mu \geq \nu$, then we assume that $a_s \neq 0$, $(s \geq 1)$. By using the same argument as above, we conclude

$$[q^s + 2 + q^{-s}] a_s^2 z^{2s} = \beta(1 + o(1)), \tag{27}$$

as $z \longrightarrow \infty$. Thus, if $\mu > \nu$, then in view of (27), we can get a contradiction; if $\mu = \nu$, then we have

$$\begin{aligned} \mu - \nu &= 0\\ &= \frac{a-b}{2}. \end{aligned} \tag{28}$$

(ii) Suppose that $b > a$ and $b - a$ are even numbers. Then, in view of (14) and (19), we get (20).

If $\mu > \nu$, then for $z \longrightarrow \infty$, it leads to

$$\begin{aligned} &\frac{\mu(qz)}{\nu(qz)} \longrightarrow \infty,\\ &\frac{\mu(z)}{\nu(z)} \longrightarrow \infty,\\ &\frac{\mu(z/q)}{\nu(z/q)} \longrightarrow \infty. \end{aligned} \tag{29}$$

However, $A(z)/B(z) \longrightarrow 0$ as $z \longrightarrow \infty$; thus, from (20), we can get a contradiction easily.

If $\mu = \nu$, then let $z \longrightarrow \infty$, it follows

$$\begin{aligned} &\frac{\mu(qz)}{\nu(qz)} \longrightarrow \alpha,\\ &\frac{\mu(z)}{\nu(z)} \longrightarrow \alpha,\\ &\frac{\mu(z/q)}{\nu(z/q)} \longrightarrow \alpha, \end{aligned} \tag{30}$$

where α is a nonzero constant. Thus, let $z \longrightarrow +\infty$; in view of (20), we also get a contradiction. Thus, $\mu < \nu$. We rewrite (14) as the following form:

$$\begin{aligned} &B(z)\left[\mu(qz)\mu(z)\nu\left(\frac{z}{q}\right)\nu(z) + \mu(qz)\mu\left(\frac{z}{q}\right)\nu(z)^2\right.\\ &\qquad \left. + \mu(z)^2\nu(qz)\nu\left(\frac{z}{q}\right) + \mu(z)\mu\left(\frac{z}{q}\right)\nu(qz)\nu(z)\right]\\ &\quad = A(z)\nu(z)^2\nu(qz)\nu\left(\frac{z}{q}\right). \end{aligned} \tag{31}$$

Denote

$$\begin{cases} A(z) = \xi_a z^a + \cdots,\\ B(z) = \delta_b z^b + \cdots,\\ \mu(z) = \gamma_\mu z^\mu + \cdots,\\ \nu(z) = \zeta_\nu z^\nu + \cdots, \end{cases} \tag{32}$$

where $a \geq 1, b \geq 0$, and $\mu \geq 0, \nu \geq 1$ are all nonnegative integers. Thus, in view of (31) and (32), we can deduce

$$(q^{\mu-\nu} + 2 + q^{\nu-\mu})\delta_b \gamma_\mu^2 \zeta_\nu^2 z^{2(\mu+\nu)+b} + \cdots = \xi_a \zeta_\nu^4 z^{4\nu+a}. \tag{33}$$

Since $|q| \neq 1$, then $q^{\mu-\nu} + 2 + q^{\nu-\mu} \neq 0$. Thus, by combining with this and (33), we have

$$2(\mu + \nu) + b = 4\nu + a, \quad \text{that is } \nu - \mu = \frac{b-a}{2}, \tag{34}$$

and $\zeta_\nu^2/\gamma_\mu^2 = (\delta_b/\xi_a)(q^{\mu-\nu} + 2 + q^{\nu-\mu})$.

(iii) If $a > b$, then $|a - b| = a - b$ is an odd number. Assume that $f(z) = \mu(z)/\nu(z)$ is a rational solution of (14). In view of the conclusion of Theorem 6 (i), it follows $\mu - \nu = (a - b)/2$. This means a contradiction with the assumption that $a - b$ is an odd number. Thus, (14) has no rational solution.

If $a < b$, then $|a - b| = b - a$ is an odd number. Similar to the above argument, we also conclude that (14) has no rational solution.

Therefore, this completes the proof of Theorem 6.

3. Proof of Theorem 7

We first introduce some notations and some basic results about Nevanlinna theory, which can be used in Section 3 and Section 4. Let f be a meromorphic function in C, the Nevanlinna characteristic $T(r,f)$, which encodes information about the distribution of values of f on the disk $|z|\le r$, is defined by

$$T(r,f)=m(r,f)+N(r,f). \tag{35}$$

The proximity function $m(r,f)$ is defined by

$$\begin{aligned} m(r,f)&=m(r,\infty,f)\\ &=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(\mathrm{re}^{i\theta}\right)\right|\mathrm{d}\theta, \end{aligned} \tag{36}$$

where $\log^+ x=\max\{0,\log x\}$ and

$$\begin{aligned} N(r,f)&=N(r,\infty,f)\\ &=\int_0^r\frac{n(t,f)-n(0,f)}{t}\mathrm{d}t+n(0,f)\log r, \end{aligned} \tag{37}$$

where $n(r,f)$ is the number of poles of f in the circle $|z|=r$, counted according to multiplicities.

Let $a\in \mathrm{C}\cup\{\infty\}$, and the deficiency of a with respect to $f(z)$ is defined by

$$\begin{aligned} \delta(a,f)&=\liminf_{r\longrightarrow+\infty}\frac{m(r,(1/f-a))}{T(r,f)}\\ &=1-\limsup_{r\longrightarrow+\infty}\frac{N(r,(1/f-a))}{T(r,f)}. \end{aligned} \tag{38}$$

If $\delta(a,f)>0$, then the complex number a is called the Nevanlinna exceptional value. And the order $\sigma(f)$, the exponent of convergence of zeros $\lambda(f)$, and the exponent of convergence of poles $\lambda(1/f)$ of $f(z)$ are defined by

$$\begin{aligned} \sigma(f)&=\limsup_{r\longrightarrow+\infty}\frac{\log^+T(r,f)}{\log r},\\ \lambda(f)&=\limsup_{r\longrightarrow+\infty}\frac{\log^+N(r,1/f)}{\log r},\\ \lambda\left(\frac{1}{f}\right)&=\limsup_{r\longrightarrow+\infty}\frac{\log^+N(r,f)}{\log r}. \end{aligned} \tag{39}$$

Besides, we also use some properties of $T(r,f), m(r,f)$, and $N(r,f)$ such as

$$\begin{aligned} &T\left(r,\frac{1}{f-a}\right)=T(r,f)+O(1),\\ &m\left(r,\prod_{j=1}^p f_j\right)\le\sum_{j=1}^p m\left(r,f_j\right),\ m\left(r,\sum_{j=1}^p f_j\right)\\ &\quad\le\sum_{j=1}^p N\left(r,f_j\right)+\log p,\\ &N\left(r,\prod_{j=1}^p f_j\right)\le\sum_{j=1}^p N\left(r,f_j\right),\ N\left(r,\sum_{j=1}^p f_j\right)\le\sum_{j=1}^p N\left(r,f_j\right),\\ &T\left(r,\prod_{j=1}^p f_j\right)\le\sum_{j=1}^p T\left(r,f_j\right),\ T\left(r,\sum_{j=1}^p f_j\right)\\ &\quad\le\sum_{j=1}^p T\left(r,f_j\right)+\log p, \end{aligned} \tag{40}$$

where $f_j(z)\,(j=1,2,\ldots,p)$ are p meromorphic functions and $a\in \mathrm{C}$ and require some lemmas as follows.

Lemma 1 (see [15], Theorem 2). *Let f be a nonconstant zero-order meromorphic solution of $P(z,f)=0$, where $P(z,f)$ is a q-difference polynomial in $f(z)$. If $P(z,a)\not\equiv 0$ for slowly moving target $a(z)$, then*

$$m\left(r,\frac{1}{f-a}\right)=S(r,f), \tag{41}$$

where $S(r,f)$ denotes any quantity satisfying $S(r,f)=o(T(r,f))$ for all r on a set F of logarithmic density 1.

Remark 1. For $q\in \mathrm{C}/\{0,1\}$, a polynomial in $f(z)$ and finitely many of its q-shifts $f(qz),\ldots,f(q^n z)$ with meromorphic coefficients in the sense that their Nevanlinna characteristic functions are $o(T(r,f))$ on a set F of logarithmic density 1 and can be called as a q-difference polynomial of f.

Lemma 2 (see [24], Theorems 1 and 3). *Let $f(z)$ be a nonconstant zero-order meromorphic function and $q\in C/\{0\}$. Then,*

$$\begin{aligned} T(r,f(qz))&=(1+o(1))T(r,f(z)),\\ N(r,f(qz))&=(1+o(1))N(r,f(z)), \end{aligned} \tag{42}$$

on a set of lower logarithmic density 1.

The proof of Theorem 7: (i) suppose that $f(z)$ is a transcendental meromorphic solution of equation (17), then in view of (17), let

$$P_1(z,f) := [f(qz)+f(z)]\left[f(z)+f\left(\frac{z}{q}\right)\right] - af(z)^2 \equiv 0. \tag{43}$$

For any given constant $d \in C/\{0\}$ and with a view of $a \neq 4$, it follows

$$P_1(z,d) = 4d^2 - ad^2 \neq 0. \tag{44}$$

In view of $P_1(z,d) \not\equiv 0$ and by Lemma 1, we conclude that $m(r, (1/f-d)) = S(r,f)$. This leads to

$$N\left(r,\frac{1}{f-d}\right) = T(r,f) + S(r,f), \tag{45}$$

which implies $\delta(d,f) = 0$. Thus, $f(z)$ has no nonzero finite Nevanlinna exceptional value.

Since $f(z)$ is of zero order and $\Delta_q f = f(qz) - f(z)$, then by Lemma 2, it follows $T(r,\Delta_q f) \leq 2T(r,f) + S(r,f)$, which means that $\Delta_q f$ if of zero order. In view of (17), it follows

$$\left[f\left(q^2 z\right) + f(qz)\right][f(qz)+f(z)] = af(qz)^2. \tag{46}$$

With (17) subtraction, it leads to

$$\begin{aligned}&[f(qz)+f(z)]\left[f\left(q^2z\right)+f(qz)-f(z)-f\left(\frac{z}{q}\right)\right]\\ &\quad = a[f(qz)+f(z)][f(qz)-f(z)].\end{aligned} \tag{47}$$

From (17), we see that $f(qz)+f(z) \equiv 0$. Otherwise, it leads to $f(z) \equiv 0$, a contradiction. Thus, the above equality means

$$f\left(q^2z\right)+f(qz)-f(z)-f\left(\frac{z}{q}\right) = a[f(qz)-f(z)], \tag{48}$$

that is,

$$\Delta_q f(qz) + (2-a)\Delta_q f(z) + \Delta_q f\left(\frac{z}{q}\right) = 0. \tag{49}$$

Denote

$$\begin{aligned}P_2\left(z,\Delta_q f\right) &:= \Delta_q f(qz) + (2-a)\Delta_q f(z)\\ &\quad + \Delta_q f\left(\frac{z}{q}\right) \equiv 0.\end{aligned} \tag{50}$$

For any given constant $d \in C/\{0\}$, then from (50), we have $P_2(z,d) = (4-a)d$. Hence, with a view of $a \neq 4$, it follows $P_2(z,d) \equiv 0$. Thus, by Lemma 1, we have $m(r, 1/\Delta_q f - d) = S(r,\Delta_q f)$, and this leads to

$$N\left(r,\frac{1}{\Delta_q f - d}\right) = T\left(r,\Delta_q f\right) + S\left(r,\Delta_q f\right), \tag{51}$$

which implies that $\delta(d,\Delta_q f) = 0$. Thus, $\Delta_q f$ has no nonzero finite Nevanlinna exceptional value:

(ii) Replacing z by ηz in (17), we have

$$[f(q\eta z)+f(\eta z)]\left[f(\eta z)+f\left(\frac{\eta z}{q}\right)\right] = af(\eta z)^2. \tag{52}$$

Let $g_1(z) = f(\eta z)$, it yields

$$\left[g_1(qz)+g_1(z)\right]\left[g_1(z)+g_1\left(\frac{z}{q}\right)\right] = ag_1(z)^2. \tag{53}$$

Set

$$\begin{aligned}P_3\left(z,g_1\right) &:= \left[g_1(qz)+g_1(z)\right]\\ &\quad \cdot\left[g_1(z)+g_1\left(\frac{z}{q}\right)\right] - ag_1(z)^2 \equiv 0.\end{aligned} \tag{54}$$

Thus, it follows $P_3(z,z) = (((q+1)^2/q) - a)z^2$, and with a view of $q + (1/q) \neq a-2$, we have $P_3(z,z) \equiv 0$. By applying Lemma 1, it yields $m(r, 1/(g_1(z)-z)) = S(r,g_1)$. Thus, in view of Lemma 2, this leads to

$$\begin{aligned}N\left(r,\frac{1}{f(\eta z)-z}\right) &= N\left(r,\frac{1}{g_1(z)-z}\right)\\ &= T\left(r,g_1(z)\right) + S\left(r,g_1\right)\\ &= T(r,f(\eta z)) + S(r,f(\eta z))\\ &= T(r,f) + S(r,f),\end{aligned} \tag{55}$$

which implies that $f(\eta z)$ has infinitely many fixed points and $\tau(f(\eta z)) = \sigma(f)$.

In view of (48), set $g_2(z) = \Delta_q f(\eta z)$ and

$$P_4\left(z,g_2\right) := g_2(qz) + (2-a)g_2(z) + g_2\left(\frac{z}{q}\right) \equiv 0, \tag{56}$$

then $P_4(z,z) = [q + (2-a) + (1/q)]z$. Since $q + (1/q) \neq a-2$, then $P_4(z,z) \equiv 0$. By applying Lemma 1, we have $m(r, (1/(g_2(z)-z))) = S(r,g_2)$. Thus, in view of Lemma 2, this leads to

$$\begin{aligned}N\left(r,\frac{1}{\Delta_q f(\eta z)-z}\right) &= N\left(r,\frac{1}{g_2(z)-z}\right)\\ &= T\left(r,g_2(z)\right) + S\left(r,g_1\right)\\ &= T\left(r,\Delta_q f(\eta z)\right) + S(r,f(\eta z)),\end{aligned} \tag{57}$$

which implies that $\Delta_q f(\eta z)$ has infinitely many fixed points and $\tau(\Delta_q f(\eta z)) = \sigma(f)$.

Therefore, the proof of Theorem 7 is completed.

4. Proof of Theorem 8

Lemma 3 (see [15], Theorem 1). *Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in C/\{0\}$. Then,*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S(r, f). \tag{58}$$

Lemma 4 (see [17], Theorem 2.5). *Let f be a transcendental meromorphic solution of order zero of a q-difference equation of the form*

$$U_q(z, f)P_q(z, f) = Q_q(z, f), \tag{59}$$

where $U_q(z, f), P_q(z, f)$, and $Q_q(z, f)$ are q-difference polynomials such that the total degree $\deg U_q(z, f) = n$ in $f(z)$ and its q-shifts, whereas $\deg Q_q(z, f) \le n$. Moreover, we assume that $U_q(z, f)$ contains just one term of maximal total degree in $f(z)$ and its q-shifts. Then,

$$m\left(r, P_q(z, f)\right) = S(r, f). \tag{60}$$

Proof of Theorem 8:. (i) suppose that $f(z)$ is a transcendental meromorphic solution of equation (18). We firstly prove that $\Delta_q f/f$ has no Nevanlinna exceptional value. Equation (18) can be rewritten as

$$\left(\frac{f(qz)}{f(z)} + 1\right)\left(\frac{f(z/q)}{f(z)} + 1\right) = \frac{d(z)}{f(z)}. \tag{61}$$

Set $g_3(z) = f(qz)/f(z)$. In view of (61) and Lemma 2, it follows

$$\begin{aligned} T(r, f(z)) &= T\left(r, \frac{d(z)}{f(z)}\right) + O(\log r) \le T\left(r, \frac{f(qz)}{f(z)}\right) \\ &\quad + T\left(r, \frac{f(z/q)}{f(z)}\right) + O(\log r) \\ &= T(r, g_3(z)) + T\left(r, g_3\left(\frac{z}{q}\right)\right) + O(\log r) \\ &= 2T(r, g_3) + S(r, g_3) \\ &= 2T\left(r, \frac{\Delta_q f}{f}\right) + S(r, f). \end{aligned} \tag{62}$$

And with a view of $T(r, \Delta_q f/f) \le 2T(r, f) + S(r, f)$, we thus conclude that $\Delta_q f/f$ is transcendental and of order zero, and $d(z)$ is small with respect to $g_3(z)$.

Replacing z by qz in (18), it follows

$$\left[f\left(q^2 z\right) + f(qz)\right][f(qz) + f(z)] = d(qz)f(qz). \tag{63}$$

By combining with (18), we have

$$\frac{f\left(q^2 z\right) + f(qz)}{f(z) + f(z/q)} = \frac{d(qz)f(qz)}{d(z)f(z)}. \tag{64}$$

Since $g_3(z) = f(qz)/f(z)$, then it yields

$$\begin{aligned} f(qz) &= g_3(z)f(z), \\ f\left(q^2 z\right) &= g_3(qz)f(qz) \\ &= g_3(z)g_3(qz)f(z), \\ f\left(\frac{z}{q}\right) &= \frac{f(z)}{g_3(z/q)}. \end{aligned} \tag{65}$$

Substituting (65) into (64), we obtain

$$\frac{g_3(z)g_3(qz)f(z) + g_3(z)f(z)}{f(z) + \left(f(z)/g_3(z/q)\right)} = \frac{d(qz)}{d(z)} g_3(z), \tag{66}$$

that is,

$$g_3\left(\frac{z}{q}\right)\left(g_3(qz) + 1\right) = \frac{d(qz)}{d(z)}\left(g\left(\frac{z}{q}\right) + 1\right). \tag{67}$$

By applying Lemma 4 for (67), it follows $m(r, g_3(z/q)) = S(r, g_3)$. This leads to

$$N\left(r, g_3\left(\frac{z}{q}\right)\right) = T(r, g_3) + S(r, g_3). \tag{68}$$

Thus, in view of Lemma 2, it yields

$$\begin{aligned} (1 + o(1))N(r, g_3) &= N\left(r, g_3\left(\frac{z}{q}\right)\right) \\ &= T\left(r, g_3\left(\frac{z}{q}\right)\right) + S(r, g_3) \\ &= (1 + o(1))T(r, g_3) + S(r, g_3). \end{aligned} \tag{69}$$

This shows

$$\begin{aligned} N\left(r, \frac{\Delta_q f}{f}\right) &= N(r, g_3) \\ &= T(r, g_3) + S(r, g_3) \\ &= T\left(r, \frac{\Delta_q f}{f}\right) + S\left(r, \frac{\Delta_q f}{f}\right), \end{aligned} \tag{70}$$

which implies $\delta(\infty, \Delta_q f/f) = 0$.

Set

$$\begin{aligned} P_5(z, g_3) &:= g_3\left(\frac{z}{q}\right)\left(g_3(qz) + 1\right) \\ &\quad - \frac{d(qz)}{d(z)}\left(g\left(\frac{z}{q}\right) + 1\right) \equiv 0. \end{aligned} \tag{71}$$

For any constant $\varrho \in C/\{-2\}$, we have

$$
\begin{aligned}
P_5(z,\varrho+1) &= (\varrho+1)(\varrho+1+1) - \frac{d(qz)}{d(z)}(\varrho+1+1)\\
&= (\varrho+2)\left(\varrho+1-\frac{d(qz)}{d(z)}\right).
\end{aligned} \tag{72}
$$

Since $d(qz)/d(z)$ is not a constant, then $P_5(z,\varrho+1)\not\equiv 0$. By Lemma 1, it follows

$$
\begin{aligned}
m\left(r,\frac{1}{(\Delta_q f/f)-\varrho}\right) &= m\left(r,\frac{1}{g_3-\varrho-1}\right)\\
&= S(r,g_3) = S\left(r,\frac{\Delta_q f}{f}\right).
\end{aligned} \tag{73}
$$

This means

$$
N\left(r,\frac{1}{(\Delta_q f/f)-\varrho}\right) = T\left(r,\frac{\Delta_q f}{f}\right) + S\left(r,\frac{\Delta_q f}{f}\right), \tag{74}
$$

which implies that $\delta(\varrho,\Delta_q f/f)=0$ for any constant $\varrho\in C/\{-2\}$.

In view of (18) and Lemma 3, we have

$$
\begin{aligned}
m\left(r,\frac{1}{f(qz)+f(z)}\right) &= m\left(r,\frac{f(z)+f(z/q)}{d(z)f(z)}\right)\\
&\le m\left(r,\frac{1}{d(z)}\right) + m\left(r,\frac{f(z)+f(z/q)}{f(z)}\right)\\
&\quad + O(1)\\
&\le S(r,f).
\end{aligned} \tag{75}
$$

Thus, we can conclude from (18), (75), and Lemma 2 that

$$
\begin{aligned}
N\left(r,\frac{1}{f(qz)+f(z)}\right) &= T\left(r,\frac{1}{f(qz)+f(z)}\right) + S(r,f)\\
&= T\left(r,\frac{1}{d(z)}\,\frac{f(z)+f(z/q)}{f(z)}\right) + S(r,f)\\
&= T\left(r,\frac{f(z/q)}{f(z)}\right) + O(\log r) + S(r,f)\\
&= T\left(r,\frac{f(qz)}{f(z)}\right) + S(r,f)\\
&= T\left(r,\frac{\Delta_q f}{f}\right) + S(r,f).
\end{aligned} \tag{76}
$$

On the contrary, we can see that the zero of $(\Delta_q f/f)+2$ is the zero of $f(qz)+f(z)$, and the zero of $f(qz)+f(z)$ is also the zero of $(\Delta_q f/f)+2$. Indeed, if z_0 is a zero of $(\Delta_q f/f)+2$, that is, $((f(qz_0)-f(z_0))/f(z_0))+2=0$, then it follows $f(qz_0)+f(z_0)=0$, and this shows that z_0 is a zero of $f(qz)+f(z)$; if z_1 is a zero of $f(qz)+f(z)$, that is, $f(qz_1)+f(z_1)=0$, then it follows $(\Delta_q f(qz_1)/f(z_1))+2=0$, and this shows that z_1 is a zero of $(\Delta_q f/f)+2$. Thus, by combining with (76), we conclude

$$
\begin{aligned}
N\left(r,\frac{1}{(\Delta_q f/f)+2}\right) &= N\left(r,\frac{1}{f(qz)+f(z)}\right)\\
&= T\left(r,\frac{\Delta_q f}{f}\right) + S\left(r,\frac{\Delta_q f}{f}\right).
\end{aligned} \tag{77}
$$

This shows that $\delta(-2,\Delta_q f/f)=0$; thus, by combining with $\delta(\infty,\Delta_q f/f)=0$ and $\delta(\varrho,\Delta_q f/f)=0$ for any $\varrho\in C/\{-2\}$, we have $\delta(\varrho,\Delta_q f/f)=0$ for any $\varrho\in C$. So, $\Delta_q f/f$ has no Nevanlinna exceptional value.

Next, we prove that $f(z)$ has no Nevanlinna exceptional value.

Firstly, in view of (61) and Lemma 3, we have

$$
\begin{aligned}
m\left(r,\frac{1}{f}\right) &\le m\left(r,\frac{1}{d(z)}\right) + m\left(r,\frac{f(qz)}{f(z)}\right) + m\left(r,\frac{f(z/q)}{f(z)}\right)\\
&= S(r,f),
\end{aligned} \tag{78}
$$

which implies

$$
N\left(r,\frac{1}{f}\right) = T(r,f) + S(r,f). \tag{79}
$$

This means $\delta(0,f)=0$.

Secondly, in view of (18), we denote

$$
P_6(z,f) := [f(qz)+f(z)]\left[f(z)+f\left(\frac{z}{q}\right)\right] - d(z)f(z) \equiv 0. \tag{80}
$$

Since $d(z)$ is a nonconstant function, then for any constant $\varrho\in C/\{0\}$, it yields

$$
\begin{aligned}
P_6(z,\varrho) &= 4\varrho^2 - \varrho d(z)\\
&= \varrho(4\varrho - d(z)) \not\equiv 0.
\end{aligned} \tag{81}
$$

Thus, from Lemma 1, we conclude $m(r,1/f-\varrho) = S(r,f)$, which implies

$$
N\left(r,\frac{1}{f-\varrho}\right) = T(r,f) + S(r,f). \tag{82}
$$

This means that $\delta(\varrho,f)=0$ for any constant $\varrho\in C/\{0\}$.

Finally, in view of (77), we have

$$
\begin{aligned}
m\left(r,\frac{1}{(\Delta_q f/f)+2}\right) &= S\left(r,\frac{\Delta_q f}{f}\right)\\
&= S(r,f).
\end{aligned} \tag{83}
$$

Thus, from (18) and (83), it yields

$$m\left(r, f(z)+f\left(\frac{z}{q}\right)\right) = m\left(r, \frac{d(z)f(z)}{f(qz)+f(z)}\right)$$
$$= m\left(r, \frac{1}{(f(qz)/f(z))+1}\right) + O(\log r)$$
$$= m\left(r, \frac{1}{\left(\Delta_q f/f\right)+2}\right) + O(\log r)$$
$$= S(r, f). \tag{84}$$

Hence, by the above equality and in view of (18) and Lemma 3, it leads to

$$m(r, f) \le m\left(r, f(z)+f\left(\frac{z}{q}\right)\right) + m(r, f(qz)+f(z))$$
$$+ O(\log r)$$
$$\le 2m\left(r, f(z)+f\left(\frac{z}{q}\right)\right) + m\left(r, \frac{f(qz)+f(z)}{f(z)+f(z/q)}\right)$$
$$+ O(\log r)$$
$$= S(r, f), \tag{85}$$

which implies

$$N(r, f) = T(r, f) + S(r, f). \tag{86}$$

Hence, $\delta(\infty, f) = 0$. Together with $\delta(0, f) = 0$ and $\delta(\varrho, f) = 0$ for any $\varrho \in C/\{0\}$, we obtain that $f(z)$ has no Nevanlinna exceptional value.

(ii) Since $f(qz) = \Delta_q f(z) + f(z)$ and $f(z/q) = f(z) - \Delta_q f(z/q)$, then by substituting these into (18), it follows

$$\left[2f(z) + \Delta_q f(z)\right]\left[2f(z) - \Delta_q f\left(\frac{z}{q}\right)\right] = d(z)f(z). \tag{87}$$

If z_0 is a zero of $f(z)$ and is not a pole of $d(z)$, then in view of (87), we conclude that z_0 is a zero of $\Delta_q f(z)$ or a zero of $\Delta_q f(z/q)$. Thus, it follows from Lemma 2 that

$$N\left(r, \frac{1}{f}\right) \le N\left(r, \frac{1}{\Delta_q f(z)}\right) + N\left(r, \frac{1}{\Delta_q f(z/q)}\right) + N(r, d)$$
$$= 2N\left(r, \frac{1}{\Delta_q f}\right) + S(r, f). \tag{88}$$

Thus, by combining with (79), it yields that $\Delta_q f(z)$ has infinitely many zeros and $\lambda(\Delta_q f) = \sigma(f)$.

In view of (85) and Lemma 3, we can deduce

$$m\left(r, \Delta_q f\right) \le m(r, f) + m\left(r, \frac{\Delta_q f}{f}\right) = S(r, f)$$
$$= S\left(r, \Delta_q f\right), \tag{89}$$

which implies

$$N\left(r, \Delta_q f\right) = T\left(r, \Delta_q f\right) + S\left(r, \Delta_q f\right)$$
$$= T\left(r, \frac{1}{\Delta_q f}\right) + S\left(r, \Delta_q f\right). \tag{90}$$

Thus, by combining with (88) and (79), we have that $\Delta_q f(z)$ has infinitely many poles and $\lambda(1/\Delta_q f) = \sigma(f)$. This proves the conclusions of Theorem 8 (ii).

(iii) In view of (71), it follows

$$P_5(z, z+1) = \left(\frac{z}{q}+1\right)(qz+1+1) - \frac{d(qz)}{d(z)}\left(\frac{z}{q}+1+1\right)$$
$$= \frac{(z+q)(qz+2)}{q} - \frac{d(qz)}{d(z)}\frac{z+2q}{q}. \tag{91}$$

Since $d(z)$ is a nonconstant rational function, then let $z \longrightarrow \infty$, and we have $d(qz)/d(z) \longrightarrow q^{\kappa}$, where $\kappa = \deg_z d$. But, for $q \ne 0$, it follows $((z+q)(qz+2)/z+2q) \longrightarrow \infty$ as $z \longrightarrow \infty$. Thus, we can deduce $P_5(z, z+1) \equiv 0$. By Lemma 1, we conclude that

$$m\left(r\frac{1}{\left(\Delta_q f/f\right)-z}\right) = m\left(r, \frac{1}{g_3(z)-1-z}\right)$$
$$= S(r, g_3)$$
$$= S\left(r, \frac{\Delta_q f}{f}\right). \tag{92}$$

This leads to

$$N\left(r\frac{1}{\left(\Delta_q f/f\right)-z}\right) = T\left(r, \frac{\Delta_q f}{f}\right) + S\left(r, \frac{\Delta_q f}{f}\right), \tag{93}$$

which implies that $\Delta_q f/f$ has infinitely many fixed points and $\tau(\Delta_q f/f) = \sigma(f)$.

Therefore, this completes the proof of Theorem 8.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

H. Y. Xu and X. M. Zheng conceptualized the study. H. Y. Xu and R. Ying wrote the original draft. H. Y. Xu, B. S. Li, and X. M. Zheng reviewed and edited the manuscript. H. Y. Xu and B. S. Li obtained funding acquisition.

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