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Research Article

Nearly Soft Menger Spaces

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In this paper, we define a weak type of soft Menger spaces, namely, nearly soft Menger spaces. We give their complete description using soft s-regular open covers and prove that they coincide with soft Menger spaces in the class of soft regular* spaces. Also, we study the role of enriched and soft regular spaces in preserving nearly soft Mengerness between soft topological spaces and their parametric topological spaces. Finally, we establish some properties of nearly soft Menger spaces with respect to hereditary and topological properties and product spaces.

1. Introduction

The theory of selection principles is an area of mathematics that studies the possibility of generating a mathematical object of one kind from a sequence of objects of the same or different kind. The beginnings of this theory are going back to Borel, Hurewicz, Menger, Rothberger, and Sierpiński. This theory is one of the important tools of numerous subareas of mathematics such as set theory and general topology, Ramsey theory, game theory, hyperspaces, function spaces, uniform structures, cardinal invariants, and dimension theory.

In 1924, Menger [1] studied selection property under the name Menger basis property and Hurewicz [2], in 1925, reformulated it in the present form. Menger's property is strictly between σ-compactness and Lindelöfness. The papers [3, 4] carried out a systematic study of selection principles in topology and then research in this field expanded immensely and attracted many researchers (see survey papers [5–7] and references therein). Some types of selection principles (so-called weak selection principles) have been formulated by applying the interior and closure operators in the definition of a selection property (see [8–21]) and the other types have been explored by replacing sequences of open covers by sequences of covers by some generalized open sets (see [22–24]). In this paper, we apply the ideas from selection principles theory to soft topological spaces. In fact, we are focused on a weaker form of the classical Menger covering property in the soft topology settings.

Soft sets were established by Molotdsov [25], in 1999, as a new technique to approach real-life problems which suffer vague and uncertainties. He investigated merits of soft sets compared with probability theory and fuzzy set theory. Many applications of soft sets have been recently given on the different areas such as decision-making problem, information theory, computer sciences, engineering, and medical sciences. In 2011, Shabir and Naz [26] employed soft sets that are defined over an initial universe set with a fixed set of parameters to introduce the concept of soft topological space. Then, researchers have studied several concepts of classical topological spaces through soft topological spaces. Soft compactness [27] and some weak variants of it [28–32] have been established and investigated. One of divergences between soft topological spaces and classical topological spaces was discussed in [33].

This paper is organized as follows. Section 2 provides basic definitions and results which are used in this paper. In Section 3, we establish some properties of soft semiopen and soft s-regular open sets which will help us to prove some results in the next sections. Section 4 introduces the concept of nearly soft Menger spaces and investigates its fundamental properties with the help of examples. Section 5 concludes the paper.

2. Preliminaries

In what follows, we recall the main definitions and results which shall be used throughout this work.

2.1. Selection Principles

Definition 1 (see [3]). Let $\mathscr{A}$ and $\mathscr{B}$ be given families of sets. Then, $S_{fin}(\mathscr{A},\mathscr{B})$ denotes the selection hypothesis: for each sequence $(A_n\colon n\in\mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(B_n\colon n\in\mathbb{N})$ such that B_n is a finite subset of A_n for each n and $\cup_{n\in\mathbb{N}}B_n\in\mathscr{B}$.

If $\mathscr{O}$ denotes the collection of all open covers of a topological space X, then spaces satisfying $S_{fin}(\mathscr{O},\mathscr{O})$ are said to have the Menger (covering) property (or X is a Menger space).

Definition 2 (see [22]). A topological space (X,τ) is said to be nearly Menger if for each sequence $(\mathscr{U}_n\colon n\in\mathbb{N})$ of open covers of X there is a sequence $(\mathscr{V}_n\colon n\in\mathbb{N})$ such that, for each $n\in\mathbb{N}$, $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ and $\cup_{n\in\mathbb{N}}\cup\{\mathrm{Int}(\mathrm{Cl}(V))\colon V\in\mathscr{V}_n\}=X$.

Evidently, every nearly compact space [34] (every open cover $\mathscr{U}$ has a finite subcover $\mathscr{V}$ such that the interiors of closures of members of $\mathscr{V}$ cover the space) is nearly Menger, and every nearly Menger space is nearly Lindelöf [35] (every open cover $\mathscr{U}$ has a countable subcover $\mathscr{V}$ such that the interiors of closures of members of $\mathscr{V}$ cover the space).

2.2. Soft Sets. In this paper, X will be a nonempty set (called the initial universal set), 2^X its power set, E a fixed set (called the set of parameters), and $A,B,\ldots$ subsets of E.

Definition 3 (see [25]). A pair (G,E) is said to be a soft set over X provided that G is a mapping of a set of parameters E into 2^X.

A soft set is identified with the set of ordered pairs $(G,E)=\{(e,G(e))\colon e\in E \text{ and } G(e)\in 2^X\}$. The collection of all soft sets defined over X under a parameter set E is denoted by $SS(X_E)$.

Definition 4 (see [36]). A soft set (G,E) over X is said to be finite (resp. countable) if $G(e)$ is finite (resp. countable) for each $e\in E$. Otherwise, it is infinite (resp. uncountable).

Definition 5 (see [36]). A soft set (G,E) over X is called a soft point if there exist $e\in E$ and $x\in X$ such that $G(e)=\{x\}$ and $G(b)=\varnothing$ for each $b\in E\backslash\{e\}$.

Throughout this study such soft point is briefly denoted by P_e^x.

In a similar way, we define $P_e^{\{x,y\}}$ as a soft set (G,E) such that $G(e)=\{x,y\}$ and $G(b)=\varnothing$ for each $b\in E\backslash\{e\}$.

Definition 6 (see [37]). The relative complement of a soft set (G,E), denoted by $(G,E)^c$, is given by $(G,E)^c=(G^c,E)$, where $G^c\colon E\longrightarrow 2^X$ is the mapping defined by $G^c(e)=X\backslash G(e)$ for each $e\in E$.

Definition 7 (see [38]). A soft set (G,E) over X is said to be a null soft set, denoted by $\widetilde{\varnothing}$, if $G(e)=\varnothing$ for each $e\in E$. (G,E) is said to the absolute soft set, denoted by $\widetilde{X}$, if $G(e)=X$ for each $e\in E$.

Definition 8 (see [38]). The union of two soft sets (G,A) and (F,B) over X, denoted by $(G,A)\widetilde{\cup}(F,B)$, is a soft set (H,D), where $D=A\cup B$ and the mapping $H\colon D\longrightarrow 2^X$ is given as follows:

$$H(d)=\begin{cases} G(d), & d\in A\backslash B,\\ F(d), & d\in B\backslash A,\\ G(d)\cup F(d), & d\in A\cap B.\end{cases} \tag{1}$$

If (G_i,E), $i\in I$, is an indexed family of soft sets over X, then $\widetilde{\cup}\{(G_i,E)L\colon i\in I\}=(H,E)$, where $H(e)=\cup_{i\in I}G_i(e)$.

Definition 9 (see [38]). The intersection of two soft sets (G,A) and (F,B) over X, denoted by $(G,A)\widetilde{\cap}(F,B)$, is the soft set (H,D), where $D=A\cap B\neq\varnothing$, and the mapping $H\colon D\longrightarrow 2^X$ is given by $H(d)=G(d)\cap F(d)$ for every $d\in D$.

For a family (G_i,E), $i\in I$, of soft sets over X, one defines $\widetilde{\cap}\{(G_i,E)L\colon i\in I\}=(H,E)$, where $H(e)=\cap_{i\in I}G_i(e)$ for every $e\in E$.

Definition 10 A soft set (G,A) is a soft subset of a soft set (F,B), denoted by $(G,A)\widetilde{\subseteq}(F,B)$, if $A\subseteq B$ and for all $a\in A$, we have $G(a)\subseteq F(a)$.The soft sets (G,A) and (F,B) are soft equal if each one of them is a soft subset of the other.

Definition 11 (see [39]). A soft mapping between $SS(X_A)$ and $SS(Y_B)$ is a pair (f,φ), denoted also by f_φ, of mappings such that $f\colon X\longrightarrow Y$ and $\varphi\colon A\longrightarrow B$ and the image of $(G,A)\in SS(X_A)$ and preimage of $(H,B)\in SS(Y_B)$ are defined by

(i) $f_\varphi(G,A)(b)=(f_\varphi(G),B)=f(\cup_{a\in\varphi^{-1}(b)}G(a))$, $b\in B$

(ii) $f_\varphi^{-1}(H,B)(a)=(f_\varphi^{-1}(H),A)=f_\varphi^{-1}(H(\varphi(a)))$. $a\in A$

A soft mapping $f_\varphi\colon SS(X_A)\longrightarrow SS(Y_B)$ is said to be injective (resp. surjective and bijective) if f and φ are injective (resp. surjective and bijective).

2.3. Soft Topological Spaces

Definition 12 (see [26]). A triple (X,τ,E) is said to be a soft topological space if τ is a collection of soft sets over X satisfying the following axioms:

(ST.1) $\widetilde{\varnothing}$ and $\widetilde{X}$ belong to τ

(ST.2) If $(G, E) \in \tau$ and $(F, E) \in \tau$, then $(G, E)\widetilde{\cap}$ $(F, E) \in \tau$

(ST.3) If $\{(G_i, E)\colon i \in I\}$ is any subset of τ, then $\widetilde{\cup}\{(G_i, E)\colon i \in I\} \in \tau$

Elements of τ are called soft open sets, and their relative complements are called soft closed.

If (ST.1) is replaced by (ST.1′) $(G, E) \in \tau$, whenever $G(e) = X$ or $G(e) = \varnothing$ for each $e \in E$, then τ is called the enriched soft topology (see [27]).

Throughout this paper, the ordered triple (X, τ, E) indicates a soft topological space.

Proposition 1 (see [26]). *Let (X, τ, E) be a soft topological space. Then, the collection $\tau_e = \{G(e)\colon (G, E) \in \tau\}$ defines a topology on X for each $e \in E$. This collection is called a parametric topology on X.*

The family,

$$\tau^* = \{(G, E)\colon G(e) \in \tau_e \text{ for each } e \in E\}, \tag{2}$$

is a soft topology on X finer than τ.

The soft topology τ^* given in the proposition above is called an extended soft topology.

The equivalence between the enriched and extended soft topologieswas proved in [40].

Definition 13 (see [29, 41]). Let (G, E) be a soft set in (X, τ, E). Then,

(i) The interior of (G, E), denoted by $\mathrm{Int}(G, E)$, is the union of all soft open sets in X contained in (G, E); the closure of (G, E), denoted by $\mathrm{Cl}(G, E)$, is the intersection of all soft closed sets containing (G, E)

(ii) (G, E) is soft semiopen in X if $(G, E) \widetilde{\subseteq} \mathrm{Cl}$ $(\mathrm{Int}((G, E)))$; the relative complement of a semi open set is called semiclosed

(iii) The semi interior of (G, E), denoted by $\mathrm{sInt}(G, E)$, is the union of all soft semiopen sets contained in (G, E); the semi closure of (G, E), denoted by $\mathrm{sCl}(G, E)$, is the intersection of all soft semi closed sets containing (G, E)

Definition 14 (see [26]). For a subset $Y \neq \widetilde{\varnothing}$ of (X, τ, E), the family $\tau_Y = \{\widetilde{Y} \widetilde{\cap} (G, E)\colon (G, E) \in \tau\}$ is called a soft relative topology on Y and the triple (Y, τ_Y, E) is called a soft subspace of (X, τ, E).

Definition 15 (see [26, 42]). Let (G, E) be a soft set over X and $x \in X$. We write

(i) $x \Subset (G, E)$ if $x \in G(e)$ for some $e \in E$; $y \Subset (G, E)$ if $x \notin G(e)$ for every $e \in E$.

(ii) $x \in (G, E)$ if $x \in G(e)$ for every $e \in E$; $y \notin (G, E)$ if $x \notin G(e)$ for some $e \in E$.

Definition 16. (see [26, 43]). (X, τ, E) is said to be

(i) Soft regular if for every soft closed set (H, E) and $x \in X$ such that $x \notin (H, E)$, there are disjoint soft open sets (U, E) and (V, E) containing (H, E) and x, respectively

(ii) Soft regular* if for every soft closed set (H, E) and $P_e^x \widetilde{\in} \widetilde{X}$ such that $P_e^x \notin (H, E)$, there are disjoint soft open sets (U, E) and (V, E), containing (H, E) and P_e^x, respectively.

Definition 17 (see [42]). A soft set (G, E) over X is called stable provided that there is $S \subseteq X$ such that $G(e) = S$ for each $e \in E$; (X, τ, E) is called stable provided that all proper nonnull soft open sets are stable.

A family $\{(G_i, E)\colon i \in I\}$ of soft sets in (X, τ, E) is said to be a soft cover of (X, τ, E) (or a soft cover of $\widetilde{X}$) if $\widetilde{\cup}_{i\in I}(G_i, E) = \widetilde{X}$. A soft cover $\{(G_i, E)\colon i \in I\}$ is said to be locally finite if for each soft point P_e^x has a soft neighbourhood intersecting only finitely many (G_i, E). A soft cover $\{(G_i, E)\colon i \in I\}$ is a soft refinement of a soft cover $\{(H_j, E)\colon j \in J\}$ if for each (G_i, E) there is (H_j, E) such that $(G_i, E) \ \widetilde{\subset} (H_j, E)$.

Definition 18 A soft space (X, τ, E) is said to be

(i) Soft compact (resp. soft Lindelöf) [27] provided that every soft open cover of X has a finite (resp. countable) subcover

(ii) Soft paracompact [30, 44] if every soft open cover has a soft open, locally finite refinement

Definition 19 (see [36]). A soft map $f_\varphi\colon (X, \tau, A) \longrightarrow (Y, \theta, B)$ is said to be

(i) Soft continuous if the inverse image of each soft open set is soft open

(ii) Soft open (resp. soft closed) if the image of each soft open (resp. soft closed) set is soft open (resp. soft closed)

(iii) Soft homeomorphism if it is bijective, soft continuous, and open

Definition 20 Let (G, A) and (F, B) be soft sets over X and Y, respectively. The cartesian product of (G, A) and (F, B) is a soft set $(G \times F, A \times B)$ over $X \times Y$ such that $(G \times F)(a, b) = G(a) \times F(b)$ for each $(a, b) \in A \times B$.

Theorem 1 (see [40]). *(X, τ, E) is extended if and only if $(Int(H), E) = Int(H, E)$ and $(Cl(H), E) = Cl(H, E)$ for any soft subset (H, E) of (X, τ, E).*

Theorem 2 (see [40]). *If (X, τ, E) is a soft regular space, then $(Int(H), E) = Int(H, E)$ and $(Cl(H), E) = Cl(H, E)$ for any stable soft subset (H, E) of (X, τ, E).*

3. Further Properties of Soft Semiopen Sets

This section is devoted to investigation of some properties of soft semiopen and soft s-regular open sets, which we need to prove some results in Section 4.

Proposition 2 (see [45]). *If* (G,E) *is a soft open set, then*

$$(G,E)\,\tilde{\cap}\,\mathrm{Cl}(H,E)\,\tilde{\subseteq}\,\mathrm{Cl}[(G,E)\,\tilde{\cup}\,(H,E)], \tag{3}$$

for every soft subset (H,E) *of* (X,τ,E).

Proposition 3 *If* (F,E) *is a soft closed set, then*

$$\mathrm{Int}[(F,E)\,\tilde{\cup}\,(H,E)]\,\tilde{\subseteq}\,(F,E)\,\tilde{\cup}\,\mathrm{Int}(H,E), \tag{4}$$

for every soft subset (F,E) *of* (X,τ,E).

Proof. Let $P_e^x \notin (F,E)\,\tilde{\cup}\,\mathrm{Int}(H,E)$. Then, $P_e^x \notin (F,E)$ and $P_e^x \notin \mathrm{Int}(H,E)$. Therefore, there is a soft open set (U,E) such that $P_e^x \in (U,E)\,\tilde{\subseteq}\,(F^c,E)$. Suppose that $P_e^x \in \mathrm{Int}[(F,E)\,\tilde{\cup}\,(H,E)]$. Then, there is a soft open set (V,E) such that $P_e^x \in (V,E)\,\tilde{\subseteq}\,(F,E)\,\tilde{\cup}\,(H,E)$. Now, $(U,E)\,\tilde{\cap}\,(V,E)$ is a soft open set containing P_e^x such that $(U,E)\,\tilde{\cap}\,(V,E)\,\tilde{\subseteq}\,(F^c,E)$ and $(U,E)\,\tilde{\cap}\,(V,E)\,\tilde{\subseteq}\,(F,E)\,\tilde{\cap}\,(H,E)$. This implies that $P_e^x \in \mathrm{Int}(H,E)$. This is a contradiction. Thus, $P_e^x \notin \mathrm{Int}[(H,E)\,\tilde{\cup}\,(F,E)]$, as required. □

Proposition 4 $(G,E)\,\tilde{\cup}\,Int(Cl(G,E)) = sCl(G,E)$ *for every soft subset* (G,E) *of* (X,τ,E).

Proof. Since $\mathrm{sCl}(G,E)$ is soft semiclosed, then $\mathrm{Int}(\mathrm{Cl}(\mathrm{sCl}(G,E)))\,\tilde{\subseteq}\,\mathrm{sCl}(G,E)$. Obviously,

$$\mathrm{Int}(\mathrm{Cl}(G,E))\,\tilde{\subseteq}\,\mathrm{Int}(\mathrm{Cl}(\mathrm{sCl}(G,E))),\ (G,E)\,\tilde{\subseteq}\,\mathrm{sCl}(G,E). \tag{5}$$

Therefore,

$$(G,E)\,\tilde{\cup}\,\mathrm{Int}(\mathrm{Cl}(G,E))\,\tilde{\subseteq}\,\mathrm{sCl}(G,E). \tag{6}$$

Conversely, it can be observed that (using Proposition 3)

$$\mathrm{Int}(\mathrm{Cl}[(G,E)\,\tilde{\cup}\,\mathrm{Int}(\mathrm{Cl}(G,E))] = \mathrm{Int}[\mathrm{Cl}(G,E)\,\tilde{\cup}\,\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(G,E)))]\,\tilde{\subseteq}\,\mathrm{Cl}(G,E)\,\tilde{\cup}\,\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(G,E))))\tilde{\subseteq}\,\mathrm{Cl}(G,E)\tilde{\cup}\,\mathrm{Cl}(\mathrm{Int}(G,E)) = \mathrm{Cl}(G,E). \tag{7}$$

Now, we have

$$\mathrm{Int}(\mathrm{Cl}[(G,E)\,\tilde{\cup}\,\mathrm{Int}(\mathrm{Cl}(G,E))]\tilde{\subseteq}\,\mathrm{Int}(\mathrm{Cl}(G,E))\tilde{\subseteq}\,(G,E)\tilde{\cup}\,\mathrm{Int}(\mathrm{Cl}(G,E)). \tag{8}$$

Thus, $(G,E)\tilde{\cup}\,\mathrm{Int}(\mathrm{Cl}(G,E))$ is a soft semiclosed set. Hence,

$$\mathrm{sCl}(G,E)\tilde{\subseteq}\,(G,E)\tilde{\cup}\,\mathrm{Int}(\mathrm{Cl}(G,E)), \tag{9}$$

as required. □

Corollary 1. *Int*$(Cl(G,E)) = sCl(G,E)$ *for every soft open subset* (G,E) *of* (X,τ,E).

Definition 21. A soft subset (H,E) of (X,τ,E) is said to be soft s-regular open if $(H,E) = \mathrm{Int}(\mathrm{sCl}(H,E))$.

Proposition 5. *Every soft s-regular open set is soft open and soft semiclosed.*

Proof. Let (H,E) be a soft s-regular open set. Then, $(H,E) = \mathrm{Int}(\mathrm{sCl}(H,E))$. Obviously, (H,E) is soft regular open. It follows, from Corollary 1, that $\mathrm{Int}(\mathrm{sCl}(H,E)) = \mathrm{Int}(\mathrm{Int}(\mathrm{Cl}(H,E))) = \mathrm{Int}(\mathrm{Cl}(H,E))$. Thus, $(H,E) = \mathrm{Int}(\mathrm{Cl}(H,E))$ and $\mathrm{Int}(\mathrm{Cl}(H,E)\,\tilde{\subseteq}\,(H,E)$. Hence, the desired result is proved. □

Proposition 6. *Int*$(sCl(H,E))$ *is a soft s-regular open set for every soft subset* (H,E) *of* (X,τ,E).

Proof. Since $\mathrm{Int}(\mathrm{sCl}(H,E))\tilde{\subseteq}\,\mathrm{sCl}(H,E)$, then

$$\mathrm{Int}(\mathrm{sCl}(\mathrm{Int}(\mathrm{sCl}(H,E))))\tilde{\subseteq}\,\mathrm{Int}(\mathrm{sCl}(H,E)). \tag{10}$$

Conversely, $\mathrm{Int}(\mathrm{sCl}(H,E))\tilde{\subseteq}\,\mathrm{sCl}(\mathrm{Int}(\mathrm{sCl}(H,E)))$, then

$$\mathrm{Int}(\mathrm{sCl}(H,E))\tilde{\subseteq}\,\mathrm{Int}(\mathrm{sCl}(\mathrm{Int}(\mathrm{sCl}(H,E)))). \tag{11}$$

Thus, $\mathrm{Int}(\mathrm{sCl}(H,E)) = \mathrm{Int}(\mathrm{sCl}(\mathrm{Int}(\mathrm{sCl}(H,E))))$, as required. □

4. Nearly Soft Menger Spaces

Definition 22 (see [46]). A soft topological space (X,τ,E) is said to be soft Menger if for each sequence $(\mathscr{U}_n\colon n \in \mathbb{N})$ of soft open covers of $\tilde{X}$ there is a sequence $(\mathscr{V}_n\colon n \in \mathbb{N})$ such that $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ for each n and $\cup_{n\in\mathbb{N}}\mathscr{V}_n$ is a soft open cover of $\tilde{X}$.

In this section, we formulate the concept of nearly soft Menger spaces and show its relationships with soft compact, soft Lindelöf, and soft Menger spaces with the help of examples. Also, we characterize this concept in terms of soft s-regular open covers and study under what conditions the concepts of soft Menger and nearly soft Menger spaces are equivalent.

Definition 23. A soft topological space (X,τ,E) is said to be nearly soft Menger if for each sequence $(\mathscr{U}_n\colon n \in \mathbb{N})$ of soft open covers of $\widetilde{X}$ there is a sequence $(\mathscr{V}_n: n \in \mathbb{N})$ such that, for every n, $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ and $\widetilde{\cup}_{n\in\mathbb{N}}\widetilde{\cup}\{\mathrm{Int}(\mathrm{Cl}(V,E))\colon (V,E) \in \mathscr{V}\}_n = \widetilde{X}$.

Definition 24. A soft space (X,τ,E) is said to be nearly soft compact (resp. nearly soft Lindelöf) if every soft open cover of $\widetilde{X}$ has a finite (resp. countable) subcover such that the soft interiors of soft closures of whose members cover $\widetilde{X}$.

Clearly, every nearly soft compact space and every soft Menger space are nearly soft Menger, and every nearly soft Menger space is nearly soft Lindelöf.

The three examples below show that the above implications are not reversible.

Example 1. Let $E = \{e_1, e_2\}$ be a set of parameters. Consider the soft topological space $(\mathbb{Z},\tau,E)$, where $\mathbb{Z}$ is the set of integers, and τ is the discrete soft topology on $\mathbb{Z}$. Then, $(\mathbb{Z},\tau,E)$ is a nearly soft Menger space, but it is not nearly soft compact.

Example 2. Let $E = \{e_1, e_2\}$ be a parameters set and $\tau = \{(G,E)\widetilde{\subseteq}\widetilde{\mathbb{R}}\colon 1 \in (G,E) \text{ or } (G,E) = \widetilde{\varnothing}\}$ be a soft topology on the set of real numbers $\mathbb{R}$. Then, $(\mathbb{R},\tau,E)$ is not a soft Lindelöf space, so it is not soft Menger. To show that $(\mathbb{R},\tau,E)$ is a nearly soft Menger space, let (G,E) be a nonnull soft open set. Then, $\widetilde{\mathbb{R}}$ is the only soft closed set containing (G,E). This implies that $\mathrm{Int}(\mathrm{Cl}(G,E)) = \widetilde{\mathbb{R}}$. Therefore, for any sequence $(\mathscr{U}_n: n \in \mathbb{N})$ of soft open covers of $\widetilde{\mathbb{R}}$, we choose only one element (G_n,E) of $\mathscr{U}_n$. Set $\mathscr{V}_n = \{(G_n,E)\}$, $n \in \mathbb{N}$. We have the sequence of finite subsets of $\mathscr{U}_n$, $n \in \mathbb{N}$, such that $\widetilde{\cup}_{n\in\mathbb{N}}\widetilde{\cup}\{\mathrm{Int}(\mathrm{Cl}(G_n,E))\colon (G_n,E) \in \mathscr{V}_n\} = \widetilde{\mathbb{R}}$. Hence, $(\mathbb{R},\tau,E)$ is a nearly soft Menger space.

Example 3. It is well known that a soft topological space $(\mathbb{R},\tau,E)$ is the classical topological space if $E = \{e\}$ is a singleton. Then, we can consider the Sorgenfrey line as an example of a soft Lindelöf, hence a soft nearly Lindelöf space. Since for any open subset (G,E) of $\widetilde{\mathbb{R}}$ we have $\mathrm{Int}(\mathrm{Cl}(G,E)) = (G,E)$, one concludes that it is not a Menger space.

An immediate consequence of Corollary 1 gives a characterization of nearly soft Menger spaces in terms of soft semiclosed sets.

Proposition 7. *A soft space (X,τ,E) is nearly soft Menger if for each sequence $(\mathscr{U}_n: n \in \mathbb{N})$ of soft open covers of $\widetilde{X}$ there is a sequence $(\mathscr{V}_n: n \in \mathbb{N})$ such that $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ for each n, and $\widetilde{\cup}_{n\in\mathbb{N}}\widetilde{\cup}\{sCl(V,E)\colon (V,E) \in \mathscr{V}_n\} = \widetilde{X}$.*

The following result is a consequence of Corollary 1 and Definition 23 and characterizes a nearly soft Menger space in terms of soft s-regular open covers.

Corollary 2. *A soft space (X,τ,E) is a nearly soft Menger space if and only if it satisfies $S_{fin}(s\mathscr{R}\mathscr{R}, s\mathscr{R}\mathscr{R})$, where $s\mathscr{R}\mathscr{R}$ denotes the family of all soft s-regular open covers of $\widetilde{X}$.*

We investigate in the following result under what condition the concepts of soft Menger and nearly soft Menger spaces are equivalent.

Theorem 3. *The following two properties are equivalent if (X,τ,E) is a soft regular* space:*

(i) (X,τ,E) is a soft Menger space

(ii) (X,τ,E) is a nearly soft Menger space

Proof

(i) ⇒ (ii) It follows from Proposition 7.

(ii) ⇒ (i) Let $(\mathscr{U}_n: n \in \mathbb{N})$ be a sequence of soft open covers of (X,τ,E). Then, for every $n \in \mathbb{N}$, there is a sequence $(\mathscr{V}_n: n \in \mathbb{N})$ of soft open covers such that $\mathscr{V}_n' = \{\mathrm{sCl}(V,E)\colon (V,E) \in \mathscr{V}_n\}$ forms a soft open refinement of $\mathscr{U}_n$ (because X is soft regular*). Since (X,τ,E) is a nearly soft Menger space, there is a sequence $(\mathscr{W}_n: n \in \mathbb{N})$ such that $\mathscr{W}_n$ is a finite subset of $\mathscr{V}_n'$ for each n, and $\cup_{n\in\mathbb{N}}\cup\{\mathrm{sCl}(W,E)\colon (W,E) \in \mathscr{W}_n\} = \widetilde{X}$. For every $n \in \mathbb{N}$ and every $(W,E) \in \mathscr{W}_n$, there is $(U_W,E) \in \mathscr{U}_n$ such that $(W,E)\widetilde{\subseteq}(U_W,E)$. Let $U_n' = \{(U_W,E)\colon (W,E) \in \mathscr{W}_n\}$. To prove that $\cup_{n\in\mathbb{N}}\mathscr{U}_n'$ covers $\widetilde{X}$, let $P_e^x \widetilde{\in} \widetilde{X}$. Then, there are $n \in \mathbb{N}$ and $(W,E) \in \mathscr{W}_n$ such that $P_e^x \in (W,E)$. This means that there is $(U_W,E) \in \mathscr{U}_n'$ such that $(W,E)\widetilde{\subseteq}(U_W,E)$. Thus, $P_e^x \in (U_W,E)$. Hence, $\widetilde{X} = \widetilde{\cup}_{n\in\mathbb{N}}\widetilde{\cup}\{(U_W,E)\colon (U_W,E) \in \mathscr{U}_n\}$. □

Corollary 3. *The following two properties are equivalent if (X,τ,E) is a soft paracompact:*

(i) (X,τ,E) is a soft Menger space

(ii) (X,τ,E) is a nearly soft Menger space

One of topological results states that Mengerness and near Mengerness coincide in spaces X in which $Int(Cl(A))$ is a finite set for any $A \subseteq X$. This result does not valid for soft topological spaces as the following example shows.

Example 4. Let the set of parameters E be the set of real numbers $\mathbb{R}$ and $\tau = \{(G,E)\widetilde{\subseteq}\widetilde{X}\colon 1 \in (G,E)$ or (G,E) is the null soft set $\}$ be a soft topology on $X = \{x,y\}$. Following the similar arguments given in Example 2, we infer that (X,τ,E) is a nearly soft Menger space, but it is not soft Menger.

Definition 25. A soft subset (G,E) of (X,τ,E) is said to be nearly soft Menger in X (or nearly soft Menger relative to (X,τ,E)) if for every sequence $(\mathscr{U}_n: n \in \mathbb{N})$ of covers by soft open subsets of (X,τ,E) there is a sequence $(\mathscr{V}_n: n \in \mathbb{N})$ such that $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ for each n and $\cup_{n\in\mathbb{N}}\cup\{\mathrm{Int}(\mathrm{Cl}(V,E)\colon (V,E) \in \mathscr{V}_n)\}^{(G,E)}$.

Proposition 8. *The soft union of two nearly soft Menger sets in a soft space (X,τ,E) is also nearly soft Menger in X.*

Proof. Let (F_1,E) and (F_2,E) be two nearly soft Menger sets in X and let $(\mathscr{U}_n: n \in \mathbb{N})$ be a sequence of soft open covers of $(F_1,E)\widetilde{\cup}(F_2,E)$. Then, there is a sequence $(\mathscr{V}_n: n \in \mathbb{N})$ of soft open covers of (F_1,E) and a sequence $(\mathscr{V}_n': n \in \mathbb{N})$ of soft open covers of (F_2,E) such that $\mathscr{V}_n$ and $\mathscr{V}_n'$ are finite subsets of $\mathscr{U}_n$ for each n, and $\cup_{n\in\mathbb{N}}\cup\{\mathrm{Int}(\mathrm{Cl}$

$(V,E))$: $(V,E) \in \mathscr{V}_n\}$ and $\cup_{n\in\mathbb{N}}\cup\{\mathrm{Int}(\mathrm{Cl}(V',E))$: $(V',E) \in \mathscr{V}'_n\}$ are soft covers of (F_1,E) and (F_2,E), respectively. Thus, the sequence $(\mathscr{W}_n = \mathscr{V}_n\cup\mathscr{V}'_n$: $n \in \mathbb{N})$ witnesses for $(\mathscr{U}_n$: $n \in \mathbb{N})$ that $(F_1,E)\tilde{\cup}(F_2,E)$ is nearly soft Menger in X. □

Proposition 9. *The property of being a nearly soft Menger is hereditary by soft clopen subspaces.*

Proof. Let (Y,τ_Y,E) be a soft clopen subspace of a nearly soft Menger space (X,τ,E). Suppose that $(\mathscr{U}_n$: $n \in \mathbb{N})$ is a sequence of soft open covers of (Y,τ_Y,E). Then, for each n and each $(U,E) \in \mathscr{U}_n$, there is a soft open subset (H_U,E) of (X,τ,E) such that $(U,E) = \tilde{Y}\tilde{\cap}(H_U,E)$. Set $\mathscr{H}_n = \{(H_U,E)$: $(U,E) \in U_n\}\cup\{\tilde{Y}^c\}$. Now, $(\mathscr{H}_n$: $n \in \mathbb{N})$ is a sequence of soft open covers of (X,τ,E). By hypothesis, there is a sequence $(\mathscr{W}_n$: $n \in \mathbb{N})$ such that $\mathscr{W}_n$ is a finite subset of $\mathscr{H}_n$ for each n and $\cup_{n\in\mathbb{N}}\cup\{\mathrm{Int}(\mathrm{Cl}(W,E))$: $(W,E) \in \mathscr{W}_n\}$ is a soft cover of $\tilde{X}$. By taking $\mathscr{V}_n = \{(U,E)$: $(H_U,E) \in \mathscr{W}_n\}$ for each n, we find that $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ for each n, and $\cup_{n\in\mathbb{N}}\cup\{\mathrm{Int}(\mathrm{Cl}(U,E))$: $(U,E) \in \mathscr{V}_n\}$ is a soft cover of $\tilde{Y}$. Hence, (Y,τ_Y,E) is a nearly soft Menger space.

Example 2 illustrates that the nearly soft Menger property is not soft closed hereditary. The following example illustrates that the nearly soft Menger property is not soft open hereditary as well. □

Example 5. Let $E = \{e_1,e_2\}$ be a set of parameters and $\tau = \{(G,E)\subseteq\tilde{\mathbb{R}}$: $1\Subset(G,E)\}$ be a soft topology on the set of real numbers $\mathbb{R}$. Since $(\mathbb{R},\tau,E)$ is soft compact, it is a nearly soft Menger space. On the contrary, let $Y = \mathbb{R}\backslash\{1\}$. Then, $\tilde{Y}$ is an uncountable soft open set and τ_Y is the discrete soft topology. Hence, (Y,τ_Y,E) is not a nearly soft Menger space.

Proposition 10. *If (X,τ,E) is an enriched nearly soft Menger space, then E is a countable set.*

Proof. Let (X,τ,E) be enriched. Then, $\mathscr{U} = \{(U,E)$: $U(e) = X$ and $U(e\prime) = \varnothing$ for each $e\prime \neq e\}$ is a soft open cover of $\tilde{X}$. We construct a sequence of soft open covers of $\tilde{X}$ as follows: $\mathscr{U}_n = \mathscr{U}$ for each $n \in \mathbb{N}$. By nearly soft Mengerness, there is a sequence $(\mathscr{V}_n$: $n \in \mathbb{N})$ such that $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ for each n and $\cup_{n\in\mathbb{N}}\cup\{\mathrm{Int}(\mathrm{Cl}(U,E))$: $(U,E) \in \mathscr{V}_n\}$ is a soft cover of $\tilde{X}$. Since (U,E) is a soft clopen set, then $\cup_{n\in\mathbb{N}}\cup\{(U,E)$: $(U,E) \in \mathscr{V}_n\}$ is also a soft cover of $\tilde{X}$. Hence, E must be countable.

Example below shows that the converse of the abovementioned proposition fails. □

Example 6. Let τ be the discrete soft topology on the real numbers set $\mathbb{R}$ and let $E = \{e_1,e_2,e_3\}$ be a set of parameters. Then, (X,τ,E) is enriched and E is countable. However, it is not a nearly soft Menger space.

The following results investigate nearly soft Mengerness between a soft topological space and its parametric topological spaces.

Proposition 11. *If (X,τ,E) is an enriched nearly soft Menger space, then (X,τ_e) is a nearly Menger space for each $e \in E$.*

Proof. Let $(\mathscr{U}_n^{(e)}$: $n \in \mathbb{N})$ be a sequence of open covers of (X,τ_e), and let for each n, $\mathscr{U}_n^{(e)} = \{U_i^{(e)}$: $i \in I\}$. Set $\mathscr{V} = \{(V_i,E)$: $i \in I\}$, where $V_i(e) = U_i^{(e)}$ and $V_i(e\prime) = X$ for each $e\prime \neq e$, and consider the sequence $(\mathscr{V}_n = \mathscr{V}$: $n \in \mathbb{N})$ of soft open covers of (X,τ,E). By near soft Mengerness of (X,τ,E), there is a sequence $(\mathscr{W}_n$: $n \in \mathbb{N})$ such that, for each n, $\mathscr{W}_n$ is a finite subset of $\mathscr{V}_n$ and $\cup_{n\in\mathbb{N}}\cup(\cup\{\mathrm{Int}(\mathrm{Cl}(W,E))$: $(W,E) \in \mathscr{W}_n\}$ is a soft cover of $\tilde{X}$. It follows from Theorem 1 that $\cup_{n\in\mathbb{N}}\cup\{(\mathrm{Int}(\mathrm{Cl}(W)),E)$: $(W,E) \in \mathscr{W}_n\}$ is also a soft cover of $\tilde{X}$. Denote $\mathscr{H}_n = \{\mathrm{Int}(\mathrm{Cl}(W_n(e))) = \mathrm{Int}(\mathrm{Cl}(U_n(e)))$: $(W_n,E) \in \mathscr{W}_n\}$. Then, the sequence $(\mathscr{H}_n$: $n \in \mathbb{N})$ witnesses for $(\mathscr{U}_n^{(e)}$: $n \in \mathbb{N})$ that (X,τ_e) is a nearly Menger space. □

Proposition 12. *If all (X,τ_e), $e \in E$, are nearly Menger spaces, then (X,τ,E) is a nearly soft Menger space provided that (X,τ,E) is enriched and E is countable.*

Proof. Let $(\mathscr{U}_n$: $n \in \mathbb{N})$ be a sequence of soft open covers of (X,τ,E), where $\mathscr{U}_n = \{(U_i,E)$: $i \in I_n\}$. Fix e and let $\mathscr{V}_n^{(e)} = \{U_i(e)$: $i \in I_n\}$. Then, $(V_n^{(e)}$: $n \in \mathbb{N})$ is a sequence of open covers of (X,τ_e). By near Mengerness of (X,τ_e), there is a sequence $(\mathscr{W}_n^{(e)}$: $n \in \mathbb{N})$ such that, for each n, $\mathscr{W}_n^{(e)}$ is a finite subset of $\mathscr{V}_n^{(e)}$ and $(\cup_{n\in\mathbb{N}}\{\mathrm{Int}(\mathrm{Cl}(U_i(e)))$: $U_i(e) \in \mathscr{W}_n^{(e)}\}$ is an open cover of X. Taking a sequence $(\mathscr{H}_n$: $n \in \mathbb{N})$ such that $\mathscr{H}_n = \{(\mathrm{Int}(\mathrm{Cl}(W)),E)$: $W(e) \in W_n^{(e)}\}$. It follows from Theorem 1 that $\mathscr{H}_n = \{\mathrm{Int}(\mathrm{Cl}(W,E))$: $W(e) \in W_n^{(e)}\}$. Since $W_n^{(e)}$ is finite, then $\mathscr{H}_n$ is a finite set and since E is countable, then the sequence $(\mathscr{H}_n$: $n \in \mathbb{N})$ witnesses for $(\mathscr{U}_n$: $n \in \mathbb{N})$ that (X,τ,E) is a nearly soft Menger space.

The following example elucidates that the condition of countability of E given in the abovementioned proposition is essential. □

Example 7. Let $\tau = \{(G,E)\tilde{\subseteq}\tilde{X}$: $G(e) = X$ or $\varnothing\}$ be a soft topology on $X = \{1,2\}$, and E be an uncountable set of parameters. Then, all (X,τ_e) are the indiscrete topologies on X. Therefore, all (X,τ_e) are Menger spaces. On the contrary, it follows from Proposition 10 that (X,τ,E) is not a soft Menger space.

Proposition 13. *Let (X,τ,E) be a soft regular space. Then, it is nearly soft Menger if and only if (X,τ_e) is nearly Menger for all $e \in E$.*

Proof (necessity). Let $(\mathscr{U}_n^{(e)}$: $n \in \mathbb{N})$ be a sequence of open covers of (X,τ_e) such that $\mathscr{U}_n^{(e)} = \{U_i^{(e)}$: $i \in I_n\}$. Then, $\mathscr{V}_n = \{(V_i,E)$: $i \in I\}$ such that $V_i(e) = U_i^{(e)}$ for each $e \in E\}$ is a soft cover of (X,τ,E). Since (X,τ,E) is soft regular, then every soft open set is stable. Therefore, $(\mathscr{V}_n$: $n \in \mathbb{N})$ is a sequence of soft open covers of (X,τ,E). By nearly soft Mengerness of (X,τ,E), there is a sequence $(\mathscr{W}_n$: $n \in \mathbb{N})$ such that for each n, $\mathscr{W}_n$ is a finite subset of $\mathscr{V}_n$ and $\cup_{n\in\mathbb{N}}\cup\{(\mathrm{Int}(\mathrm{Cl}(V_i)),E)$: $(V_i,E) \in \mathscr{W}_n\}$ is a soft open cover of $\widetilde{X}$. From Theorem 2, it follows that $\cup_{n\in\mathbb{N}}\{\mathrm{Int}(\mathrm{Cl}(V_i), E)\colon (V_i, E) \in \mathscr{W}_n\}$ is also a soft open cover of $\widetilde{X}$. Set $\mathscr{H}_n = \{\mathrm{Int}(\mathrm{Cl}(W_n(e))) = \mathrm{Int}(\mathrm{Cl}(U_n(e)))\colon (W_n, E) \in \mathscr{W}_n\}$. Then, the sequence $(\mathscr{H}_n\colon n \in \mathbb{N})$ testifies for $(\mathscr{U}_n^{(e)}\colon n \in \mathbb{N})$ that (X, τ_e) is a nearly Menger space.

Sufficiency: let $(\mathscr{U}_n\colon n \in \mathbb{N})$ be a sequence of soft open covers of (X, τ, E) such that $\mathscr{U}_n = \{(U_i, E)\colon i \in I_n\}$. Since (U_i, E) is stable, then $\mathscr{V}_n = \{U_i(e)\colon i \in I_n\}$ is an open covers of (X, τ_e) for each $e \in E$ and each $n \in \mathbb{N}$. By the nearly Menger property of (X, τ_e), there is a sequence $(\mathscr{W}_n\colon n \in \mathbb{N})$ such that, for each n, $\mathscr{W}_n$ is a finite subset of $\mathscr{V}_n$ and $\cup_{n\in\mathbb{N}}\{\mathrm{Int}(\mathrm{Cl}(U_i(e)))\colon U_i(e) \in \mathscr{W}_n\}$ is an open cover of X. Consider the sequence $(\mathscr{H}_n\colon n \in \mathbb{N})$ such that $\mathscr{H}_n = \{(\mathrm{Int}(\mathrm{Cl}(W)), E)\colon W(e) \in \mathscr{W}_n \text{ for each } e \in E\}$, of soft open covers of (X, τ, E). Since $\mathscr{H}_n = \{\mathrm{Int}(\mathrm{Cl}(W, E))\colon W(e) \in \mathscr{W}_n \text{ for each } e \in E\}$, we conclude that (X, τ, E) is a nearly soft Menger space.

In what follows, we examined some features of a nearly soft Menger space under some types of soft mappings. □

Proposition 14. *The soft continuous open surjective image of a nearly soft Menger space is a nearly soft Menger space.*

Proof. Let g_φ be a soft continuous open mapping of a nearly soft Menger space (X, τ, A) onto (Y, θ, B) and $(\mathscr{U}_n\colon n \in \mathbb{N})$ be a sequence of soft open covers of (Y, θ, B). From the soft continuity of g_φ, we obtain that $\mathscr{V}_n = \{g_\varphi^{-1}(U_i, B)\colon (U_i, B) \in \mathscr{U}_n\}$ is a sequence of soft open covers of (X, τ, A). Therefore, there is a sequence $(\mathscr{H}_n\colon n \in \mathbb{N})$ such that, for each n, $\mathscr{H}_n$ is a finite subset of $\mathscr{V}_n$ and $\widetilde{\cup}_{n\in\mathbb{N}}\widetilde{\cup}\{\mathrm{Int}(\mathrm{Cl}(g_\varphi^{-1}(U_i, B)))\colon g_\varphi^{-1}(U_i, B) \in \mathscr{H}_n\} = \widetilde{X}$. Since g_φ is surjective, then $\widetilde{Y} = g_\varphi(\widetilde{X}) = g_\varphi(\widetilde{\cup}_{n\in\mathbb{N}}\widetilde{\cup}\{\mathrm{Int}(g_\varphi^{-1}(\mathrm{Cl}(U_i, B)))\colon g_\varphi^{-1}(U_i, B) \in \mathscr{H}_n\})$ and since g_φ is soft open, then $\widetilde{Y} = \widetilde{\cup}_{n\in\mathbb{N}}\widetilde{\cup}\{\mathrm{Int}(g_\varphi(g_\varphi^{-}1(\mathrm{Cl}(U_i, B))))\colon g_\varphi^{-1}(U_i, B) \in \mathscr{H}_n\} = \widetilde{\cup}_{n\in\mathbb{N}}\widetilde{\cup}\{\mathrm{Int}(\mathrm{Cl}(U_i, B))\colon g_\varphi^{-1}(U_i, B) \in \mathscr{H}_n\}$. Thus, $\{\mathrm{Int}(\mathrm{Cl}(U_i, B))\colon g_\varphi^{-1}(U_i, B) \in \mathscr{H}_n\}$ is a soft cover of (Y, θ, B). Hence, the sequence $(\mathscr{H}_n\colon n \in \mathbb{N})$ witnesses for $(\mathscr{U}_n\colon n \in \mathbb{N})$ that (Y, θ, B) is a nearly soft Menger space. □

Corollary 4. *The property of being a nearly soft Menger is a soft topological property.*

Example 2.15 in [9] shows that the product of two nearly Menger spaces is not always a nearly Menger space. Since a classical topological space is a special case of a soft topological space when a parameters set is a singleton, then this result is still valid on soft topological spaces.

In the rest of this section, we prove that the product of a nearly soft Menger space and a nearly soft compact space is a nearly soft Menger.

Lemma 1. *Let (G, A) and (H, B) be two subsets of (X, τ, A) and (Y, θ, B), respectively. Then,*

(i) $sCl(G, A) \times sCl(H, B) = sCl((G, A) \times (H, B))$

(ii) $sInt(G, A) \times sInt(H, B) = sInt((G, A) \times (H, B))$

Proof

(i) Suppose that $P_{(a,b)}^{(x,y)} \notin \mathrm{sCl}((G, A) \times (H, B))$. Then, there exists a soft semiopen subset $(U, A) \times (V, B)$ of $(X \times Y, \tau \times \theta, A \times B)$ containing $P_{(a,b)}^{(x,y)}$ such that $((U, A) \times (V, B))\widetilde{\cap}((G, A) \times (H, B)) = \widetilde{\varnothing}$. So. $(U, A)\widetilde{\cap}(G, A) = \widetilde{\varnothing}$ or $(V, B)\widetilde{\cap}(H, B) = \widetilde{\varnothing}$. This means that $P_a^x \notin \mathrm{sCl}(G, A)$ or $P_b^y \notin \mathrm{sCl}(H, B)$. Thus, $P_{(a,b)}^{(x,y)} \notin \mathrm{sCl}(G, A) \times \mathrm{sCl}(H, B)$. Hence, $\mathrm{sCl}(G, A) \times \mathrm{sCl}(H, B) \widetilde{\subseteq} \mathrm{sCl}((G, A) \times (H, B))$.

Suppose now that $P_a^x \notin \mathrm{sCl}(G, A)$ or $P_b^y \notin \mathrm{sCl}(H, B)$. Without loss of generality, let $P_a^x \notin \mathrm{sCl}(G, A)$. Then, there exists a soft semiopen subset (U, A) of (X, τ, A) containing P_a^x such that $(U, A)\widetilde{\cap}(G, A) = \widetilde{\varnothing}$. Now, $(G, A) \times \widetilde{Y}$ is a soft semiopen subset of $(X \times Y, \tau \times \theta, A \times B)$ containing $P_{(a,b)}^{(x,y)}$ satisfies that $((G, A) \times \widetilde{Y})\widetilde{\cap}((G, A) \times (H, B)) = \widetilde{\varnothing}$. This means $P_{(a,b)}^{(x,y)} \notin \mathrm{sCl}((G, A) \times (H, B))$. Hence, $\mathrm{sCl}((G, A) \times (H, B)) \widetilde{\subseteq} \mathrm{sCl}(G, A) \times \mathrm{sCl}(H, B)$.

(ii) By using a similar argumentation, one can prove item (ii). □

Theorem 4. *The product of a nearly soft Menger space and a nearly soft compact space is a nearly soft Menger.*

Proof. Let (X, τ, A) be a nearly soft Menger space and (Y, θ, B) be a nearly soft compact space, and let $(\mathscr{W}_n\colon n \in \mathbb{N})$ is a sequence of soft open covers of $(X \times Y, \tau \times \theta, A \times B)$. One may assume that, for every $n \in \mathbb{N}$, $\mathscr{W}_n = \mathscr{U}_n \times \mathscr{V}_n$, where all $\mathscr{U}_n$ are soft open covers of (X, τ, A) and all $\mathscr{V}_n$ are soft open covers of (Y, θ, B). Since (Y, θ, B) is nearly soft compact there is a sequence $(\mathscr{H}_n\colon n \in \mathbb{N})$ such that, for each n, $\mathscr{H}_n$ is a finite subset of $\mathscr{V}_n$ and $\widetilde{Y} = \widetilde{\cup}\{\mathrm{sCl}(H, B)\colon (H, B) \in \mathscr{H}_n\}$. As (X, τ, A) is nearly soft Menger there are finite sets $\mathscr{G}_n \subset \mathscr{U}_n$ such that $\widetilde{\cup}_{n\in\mathbb{N}}\widetilde{\cup}\{\mathrm{sCl}(G, A)\colon (G, A) \in \mathscr{G}_n\} = \widetilde{X}$. Let $\mathscr{F}_n = \mathscr{G}_n \times \mathscr{H}_n$ for each $n \in \mathbb{N}$. Then, for every n, $\mathscr{F}_n$ is a finite subset of $\mathscr{W}_n$. By using the previous lemma, it is easy now to see that $\widetilde{\cup}_{n\in\mathbb{N}}\widetilde{\cup}\{\mathrm{sCl}(F, A \times B)\colon (F, A \times B) \in \mathscr{F}_n\} = \widetilde{X \times Y}$ which means that $(X \times Y, \tau \times \theta, A \times B)$ is nearly soft Menger. □

5. Conclusion

This study is devoted to introducing and investigating the concept of nearly soft Menger spaces. We provide several examples to discuss its relationships with soft Menger and soft Lindelöf spaces, and to show the interchangeability of nearly Mengerness between a soft space and its parametric spaces. In general, we have initiated many properties of it parallel to their corresponding properties from classical topology. Our hope is that the introduced concepts will be beneficial for the researchers to further promote and advance the study selection principles in soft topologies.

Data Availability

The data used to support the findings of this study are cited at relevant places within the text as references. They are also available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interests regarding the publication of this article.

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