Research Article

Functional-Type Caristi–Kirk Theorem on Menger Space and Application

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In this paper, we extend Caristi's fixed point theorem in metric spaces to probabilistic metric spaces, and also, we prove some common fixed point theorems for a pair of mappings satisfying a system of Caristi-type contractions in the setting of a Menger space. Two examples are given to support the main results. Furthermore, we have functional equations as an application for the main theorem.

1. Introduction and Preliminaries

The theory of probabilistic metric spaces is of fundamental importance in random functional analysis especially due to its extensive applications in random differential and random integral equations. In 1942, Menger introduced in [1] a generalization of metric space, called a statistical metric space, by using distribution functions instead of nonnegative real numbers as values of the metric. The connection between probabilistic and geometric concepts has been established in 1956 by Špaček [2]. However, the main influence upon the development of this theory is owed to Schweizer and Sklar and their coworkers (see [3–6]). Since then, a large number of fixed point theorems for single-valued and multivalued mappings in probabilistic metric spaces have been proved by many authors (see, for examples, [7–13]). Since every metric space is a probabilistic metric space, we can use many results in probabilistic metric spaces to prove some fixed point theorems in metric spaces and Banach spaces.

In this paper, first, we extend Caristi's fixed point theorem in metric spaces to probabilistic metric spaces. Secondly, we prove, in the setting of a Menger space, some common fixed point theorems for a pair of mappings

satisfying a system of Caristi-type contractions. Two examples are given to support our main results.

Let $\mathbb{R}_+$ denote the set of positive real numbers and $\mathbb{N}$ denote the set of all natural numbers. For brevity, $T(x)$ and $S(x)$ will be denoted by Tx and Sx , respectively. Let $c: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be some function. Let, for $\alpha \in \mathbb{R}_+$,

$$\begin{aligned} \liminf_{t \rightarrow \alpha^+} c(t) &= \sup_{\varepsilon > 0} \inf c([\alpha, \alpha + \varepsilon]), \\ \limsup_{t \rightarrow \alpha^+} c(t) &= \inf_{\varepsilon > 0} \sup c([\alpha, \alpha + \varepsilon]). \end{aligned} \quad (1)$$

Definition 1. We say that a function c is right lower (upper) semicontinuous at α if $\liminf_{t \rightarrow \alpha^+} c(t) = c(\alpha)$ ($\limsup_{t \rightarrow \alpha^+} c(t) = c(\alpha)$).

From the definition, it is easy to prove that the following proposition is needed in the sequel.

Proposition 1. If c is a right lower (upper) semicontinuous function at α , then it is right locally bounded below (above) at α , that is, there exists $\lambda > 0$ such that $\inf(c[\alpha, \alpha + \lambda]) > -\infty$ ($\sup(c[\alpha, \alpha + \lambda]) < +\infty$).

Definition 2. A mapping $g: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is said to be locally bounded above if it is bounded above on each $[0, a]$, ($a > 0$).

In order to set the framework needed to state our main results, we recall the following notions.

Definition 3 (distribution function). A mapping $F: \mathbb{R} \longrightarrow \mathbb{R}_+$ is called a distribution function if it is non-decreasing, left continuous with $\inf F = 0$, and $\sup F = 1$.

In what follows, we always denote by $\mathcal{D}$ the set of all distribution functions.

Example 1. A simple example of distribution function is a Heaviside function:

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \quad (2)$$

Definition 4 (probabilistic metric space). A probabilistic metric space (PM-space, for short) is an ordered pair $(X, \mathcal{F})$, where X is an abstract set of elements and $\mathcal{F}$ is a mapping of $X \times X \longrightarrow \mathcal{D}$ (we shall denote the distribution function $\mathcal{F}(p, q)$ by $F_{p,q}$, and $F_{p,q}(t)$ will represent the value of $F_{p,q}$ at $t \in \mathbb{R}$). The functions $F_{p,q}$, $p, q \in X$, are assumed to satisfy the following conditions:

- (1) $\forall t > 0$, $F_{p,q}(t) = 1$ if and only if $p = q$
- (2) $F_{p,q}(0) = 0$
- (3) $F_{p,q} = F_{q,p}$
- (4) If $F_{p,q}(t_1) = 1$ and $F_{q,r}(t_2) = 1$, then $F_{p,r}(t_1 + t_2) = 1$

Definition 5 (see [14]). Let $(X, \mathcal{F})$ be a probabilistic metric space. For $p \in X$ and $t > 0$, the strong t -neighborhood of p is the set

$$N_p(t) = \{q \in X : F_{p,q}(t) > 1 - t\}. \quad (3)$$

The strong neighborhood system at p is the collection $N_p = \{N_p(t) : t > 0\}$, and the strong neighborhood system for X is the union $N = \cup_{p \in X} N_p$.

Definition 6 (see [14]). The (ε, λ) -topology on $(X, \mathcal{F})$ is the topology introduced on X by the family of the neighborhoods $\{U_p(\varepsilon, \lambda)\}_{p \in X, \varepsilon > 0, \lambda > 0}$, where

$$U_p(\varepsilon, \lambda) = \{q \in X : F_{p,q}(\varepsilon) > 1 - \lambda\}. \quad (4)$$

Remark 1. $U_p(t, t) = N_p(t)$ for all $t > 0$.

Definition 7 (triangular norm). A triangular norm (briefly a t -norm) is a binary operation τ on the unit interval $[0, 1]$ which is associative, commutative, and nondecreasing in each of its variables and such that $\tau(x, 1) = x$ for every $x \in [0, 1]$.

Example 2. The following are the three basic t -norms:

- (i) Minimum: $\tau_M(x, y) = \min\{x, y\}$

- (ii) Product: $\tau_p(x, y) = xy$

- (iii) Lukasiewicz t -norm: $\tau_L(x, y) = \max\{x + y - 1, 0\}$

Definition 8 (Menger space). A Menger space is a triplet $(X, \mathcal{F}, \tau)$ where $(X, \mathcal{F})$ is a PM-space and τ is a t -norm satisfying the following triangle inequality:

$$F_{p,r}(t_1 + t_2) \geq \tau(F_{p,q}(t_1), F_{q,r}(t_2)), \quad \forall p, q, r \in X, t_1, t_2 \geq 0. \quad (5)$$

The following result is due to Schweizer et al. [6].

Theorem 1. If $(X, \mathcal{F}, \tau)$ is a Menger space such that $\sup_{t < 1} \tau(t, t) = 1$, then $(X, \mathcal{F}, \tau)$ is a Hausdorff space in the topology induced by the family $\{U_p(\varepsilon, \lambda)\}_{p \in X, \varepsilon > 0, \lambda > 0}$ of neighborhoods.

Definition 9. Let $(X, \mathcal{F}, \tau)$ be a Menger space. A sequence $(p_n)_{n \in \mathbb{N}}$ in X is said to be

- (i) τ -convergent to $p \in X$ if for any $\varepsilon > 0$ and any $\lambda > 0$, there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{p_n, p}(\varepsilon) > 1 - \lambda$, whenever $n \geq N$
- (ii) τ -Cauchy sequence if for any $\varepsilon > 0$ and any $\lambda > 0$, there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{p_n, p_m}(\varepsilon) > 1 - \lambda$, whenever $n, m \geq N$

Definition 10. A Menger space $(X, \mathcal{F}, \tau)$ is said to be τ -complete if each τ -Cauchy sequence in X is τ -convergent to some point in X .

Definition 11. Let $(X, \mathcal{F}, \tau)$ be a Menger space such that $\sup_{t < 1} \tau(t, t) = 1$, and let S be a self-mapping on $(X, \mathcal{F}, \tau)$. S is said to be a τ -continuous mapping if for each sequence $(p_n)_{n \in \mathbb{N}}$ in X which is τ -convergent to a point $p \in X$, the sequence $(Sp_n)_{n \in \mathbb{N}}$ is τ -convergent to Sp .

The following result was established by Schweizer and Sklar in [5].

Theorem 2. Let $(X, \mathcal{F}, \tau)$ be a Menger space such that $\sup_{t < 1} \tau(t, t) = 1$. Then, a sequence $(p_n)_{n \in \mathbb{N}}$ in X is τ -convergent to $p \in X$ if and only if for each $t \in \mathbb{R}$,

$$\lim_{n \longrightarrow \infty} F_{p_n, p}(t) = H(t). \quad (6)$$

The following theorem establishes a connection between metric spaces and Menger spaces.

Theorem 3 (see [13]). Let (X, d) be a metric space. Let $\mathcal{F}: X \times X \longrightarrow \mathcal{D}$ be the mapping defined by

$$F_{p,q}(t) = H(t - d(p, q)), \quad p, q \in X, t \in \mathbb{R}. \quad (7)$$

Then, $(X, \mathcal{F}, \tau_M)$ is a Menger space. It is complete if the metric d is complete.

We say that a t -norm τ satisfies the condition $(\mathcal{P})$ if for all $y \in [0, 1]$,

$$\lim_{x \rightarrow 1^-} \tau(x, y) = y. \quad (8)$$

Remark 2. It is easy to see that if a t -norm τ satisfies the condition $(\mathcal{P})$, then

$$\sup_{t < 1} \tau(t, t) = 1. \quad (9)$$

Theorem 4 (see [15]). Let $(X, \mathcal{F}, \tau)$ be a Menger space with τ satisfying the condition $(\mathcal{P})$. Let $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$ be two sequences in X such that $(p_n)_{n \in \mathbb{N}}$ is τ -convergent to $p \in X$ and $(q_n)_{n \in \mathbb{N}}$ is τ -convergent to $q \in X$. Then,

(1) For any given $t \in \mathbb{R}$, we have

$$\liminf_{n \rightarrow \infty} F_{p_n, q_n}(t) \geq F_{p, q}(t). \quad (10)$$

(2) If $t \in \mathbb{R}$ is any continuous point of $F_{p, q}$, then

$$\lim_{n \rightarrow \infty} F_{p_n, q_n}(t) = F_{p, q}(t). \quad (11)$$

Lemma 1. (see [15], Proposition 2). If $(X, \mathcal{F}, \tau)$ is a Menger space with τ satisfying the condition $(\mathcal{P})$, then it is metrizable. In addition, if $(X, \mathcal{F}, \tau)$ is sequentially complete, then it must be net-complete.

2. Main Results

Theorem 5. Let $(X, \mathcal{F}, \tau)$ be a complete Menger space with τ satisfying the condition $(\mathcal{P})$. Let $T, S: X \rightarrow X$ be two mappings such that, for all $p \in X$ and for all $t > 0$,

$$\begin{cases} F_{p, Sp}(t) \geq H(t - h(c(\phi(p)), c(\phi(Tp))) (\phi(p) - \phi(Tp))), \\ F_{p, Tp}(t) \geq H(t - h(c(\phi(p)), c(\phi(Sp))) (\phi(p) - \phi(Sp))), \end{cases} \quad (12)$$

where $\phi: X \rightarrow \mathbb{R}_+$ is a lower semicontinuous function, $c: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a right locally bounded from above, and h is a locally bounded function from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}_+$.

Then, there exists an element $p^* \in X$ such that $Sp^* = p^* = Tp^*$.

Proof.

Step 1: let $\alpha = \inf(\phi(X))$. Since c is locally bounded from the above, there exists $\lambda > 0$ such that $\mu = \sup c([\alpha, \alpha + \lambda]) < \infty$. For this μ , there exists $\nu > 0$ such that $h(t, s) \leq \nu$ for all $s, t \in [0, \mu]$.

For some $p_0 \in X$ such that $\alpha \leq \phi(p_0) \leq \alpha + \lambda$, we consider the following set:

$$X_0 = \{p \in X : \phi(p) \leq \phi(p_0)\}. \quad (13)$$

One can see that X_0 is nonempty and closed since ϕ is lower semicontinuous. For all $p \in X_0$, $\phi(Tp) \leq \phi(p)$. If not, then there exists some $p_1 \in X_0$ such that $\phi(p_1) < \phi(Tp_1)$. By applying (12), we obtain

$F_{p_1, Sp_1}(t) = 1$, for all $t > 0$; this implies that $Sp_1 = p_1$ and consequently $Tp_1 = p_1$, which is contradiction. Hence, $\phi(Tp) \leq \phi(p)$, for all $p \in X_0$. So,

$$\begin{aligned} \phi(Tp) &\leq \phi(p) \leq \phi(p_0), \\ \phi(Sp) &\leq \phi(p) \leq \phi(p_0), \end{aligned} \quad (14)$$

for all $p \in X_0$.

Thus, $TX_0 \subset X_0$ and $SX_0 \subset X_0$. And since $\phi(p)$, $\phi(Tp)$, $\phi(Sp) \in [\alpha, \alpha + \lambda]$, for all $p \in X_0$, we obtain

$$\begin{cases} \max\{c(\phi(p)), c(\phi(Tp)), c(\phi(Sp))\} \leq \mu, \\ \max\{h(c(\phi(p)), c(\phi(Tp))), h(c(\phi(p)), c(\phi(Sp)))\} \leq \nu. \end{cases} \quad (15)$$

Step 2: we define a partial order “ $\leq$ ” on X_0 as follows:

$$p \leq q \iff F_{p, q}(t) \geq H(t - \nu(\phi(p) - \phi(q))), \quad (16)$$

$\forall t > 0$.

If $p, q \in X_0$ such that $p \leq q$, then $\phi(q) \leq \phi(p)$. On the contrary, the reflexivity and The symmetry of “ $\leq$ ” are obvious. Let us prove the transitivity.

If $p, q, r \in X_0$ such that $p \leq q$, $q \leq r$, then $\phi(r) \leq \phi(q) \leq \phi(p) \leq \phi(p_0)$

Moreover, let t be a positive real number.

If $t \leq \nu(\phi(p) - \phi(r))$, we have

$$F_{p, r}(t) \geq 0 = H(t - \nu(\phi(p) - \phi(r))). \quad (17)$$

If $t > \nu(\phi(p) - \phi(r))$, then, by considering $t_1, t_2 > 0$ such that $t_1 > \nu(\phi(p) - \phi(q))$, $t_2 > \nu(\phi(q) - \phi(r))$, and $t = t_1 + t_2$, we have

$$\begin{aligned} F_{p, r}(t) &\geq \tau(F_{p, q}(t_1), F_{q, r}(t_2)) \\ &\geq \tau(H(t_1 - \nu(\phi(p) - \phi(q))), \\ &\quad H(t_2 - \nu(\phi(q) - \phi(r)))) \\ &= \tau(1, 1) = 1 = H(t - \nu(\phi(p) - \phi(r))). \end{aligned} \quad (18)$$

Again following (16), we have $p \leq r$. This shows that “ $\leq$ ” is a partial order on X_0 .

Step 3: now, let $\{p_\alpha\}_{\alpha \in I}$ be a totally ordered subset of X_0 , where I is an index set. We stipulate that $p_\alpha \leq p_\beta \iff \alpha \leq \beta$. Hence, $(I, \leq)$ is a directed set, and $(\phi(p_\alpha))_{\alpha \in I}$ is a decreasing monotone net. Let $s \geq 0$ such that $\phi(p_\alpha) \rightarrow s$. So, for any given $\varepsilon > 0$, $\lambda > 0$, and $\varepsilon > \lambda$, there exists $\alpha_0 \in I$ such that when $\alpha_0 \leq \alpha$, we have $s \leq \phi(p_\alpha) < s + (\lambda/\nu)$. Hence, for any $\alpha, \beta \in I$ such that $\alpha_0 \leq \alpha \leq \beta$, we have $\phi(p_\alpha) - \phi(p_\beta) \leq (\lambda/\nu)$, and then

$$\begin{aligned} F_{p_\alpha, p_\beta}(\varepsilon) &\geq H(\varepsilon - \nu(\phi(p_\alpha) - \phi(p_\beta))) \\ &\geq H(\varepsilon - \lambda) = 1 \\ &> 1 - \lambda. \end{aligned} \quad (19)$$

This implies that $(\phi(p_\alpha))_{\alpha \in I}$ is a Cauchy net on X_0 . By the completeness of X_0 and according to Proposition 2 in [15], there exists some $\bar{p} \in X_0$ such that $p_\alpha \longrightarrow \bar{p}$. In view of the lower semicontinuity of ϕ , it follows that

$$\begin{aligned} \phi(\bar{p}) &\leq \liminf_{\alpha} \phi(p_\alpha) \\ &= \lim_{\alpha} \phi(p_\alpha) = s \\ &\leq \phi(p_\alpha), \quad \forall \alpha \in I. \end{aligned} \quad (20)$$

Now, we prove that $\bar{p}$ is an upper bound of $(p_\alpha)_{\alpha \in I}$. In fact, for any given $\alpha \in I$, when $t > 0$ is a continuous point of $F_{p_\alpha, \bar{p}}$, then, by applying Theorem 1 in [15], we have

$$\begin{aligned} F_{p_\alpha, \bar{p}}(t) &= \lim_{\beta} F_{p_\alpha, p_\beta}(t) \\ &\geq \liminf_{\beta} H(t - \nu(\phi(p_\alpha) - \phi(p_\beta))) \\ &\geq H(t - \nu(\phi(p_\alpha) - \phi(\bar{p}))). \end{aligned} \quad (21)$$

Using the same idea as in [15], Theorem 3, we prove that

$$F_{p_\alpha, \bar{p}}(t) \geq H(t - \nu(\phi(p_\alpha) - \phi(\bar{p}))), \quad \forall t > 0, \forall \alpha \in I. \quad (22)$$

This means that, for all $\alpha \in I$, $p_\alpha \leq \bar{p}$, i.e., $\bar{p}$ is an upper bound of $(p_\alpha)_{\alpha \in I}$. By Zorn's lemma, $(X_0, \leq)$ has a maximal element p^* . If $\phi(Sp^*) \leq \phi(Tp^*)$, then, according to (1), we have

$$F_{p^*, Sp^*}(t) \geq H(t - \nu(\phi(p^*) - \phi(Sp^*))), \quad \forall t > 0. \quad (23)$$

Hence, $p^* \leq Sp^*$, which implies that $p^* = Sp^*$ and $p^* = Tp^*$. The same conclusion holds if $\phi(Tp^*) \leq \phi(Sp^*)$.

If we take $T = S$ and $h(x, y) = 1$, for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$, we obtain the following result. $\square$

Corollary 1 (see [15]). *Let $(X, \mathcal{F}, \tau)$ be a complete Menger space with τ satisfying the condition $(\mathcal{P})$. Let $T: X \longrightarrow X$ be a mapping such that*

$$F_{p, Tp}(t) \geq H(t - (\phi(p) - \phi(Tp))), \quad \forall p \in X, \forall t > 0, \quad (24)$$

where $\phi: X \longrightarrow \mathbb{R}_+$ is a lower semicontinuous function. Then, there exists an element $p^* \in X$ such that $Tp^* = p^*$.

The following corollary is the metric version of Theorem 5.

Corollary 2 (see [16]). *Let (X, d) be a complete metric space. Let $T, S: X \longrightarrow X$ be two mappings such that*

$$\begin{cases} d(p, Sp) \leq h(c(\phi(p)), c(\phi(Tp))) (\phi(p) - \phi(Tp)), \\ d(p, Tp) \leq h(c(\phi(p)), c(\phi(Sp))) (\phi(p) - \phi(Sp)), \\ \forall p \in X, \end{cases} \quad (25)$$

where $\phi: X \longrightarrow \mathbb{R}_+$ is a lower semicontinuous function, $c: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a right locally bounded from the above, and h is a locally bounded function from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}_+$.

Then, there exists an element $p^* \in X$ such that $Tp^* = p^* = Sp^*$.

Proof. Consider the mapping $\mathcal{F}: X \times X \longrightarrow \mathcal{D}$ defined as follows:

$$\mathcal{F}(p, q) = F_{p, q}(t) = H(t - d(p, q)), \quad t \in \mathbb{R}, p, q \in X. \quad (26)$$

By Theorem 3, the space $(X, \mathcal{F}, \tau_M)$ is a complete Menger space. It is easy to see that the t -norm τ_M satisfies the condition $(\mathcal{P})$. Therefore, all conditions of Theorem 5 are satisfied. Then, the conclusion holds. $\square$

Example 3. Consider the set $X = \mathbb{R}_+$ endowed with the metric d defined by

$$d(p, q) = |p - q|, \quad \text{for all } (p, q) \in X^2, \quad (27)$$

and define the function $F: X \times X \longrightarrow \mathcal{D}$ as follows:

$$\begin{aligned} F(p, q)(t) &= F_{p, q}(t) = H(t - |p - q|), \\ &\text{for all } (p, q) \in X^2, t \in \mathbb{R}. \end{aligned} \quad (28)$$

Consider the two mappings $T, S: X \longrightarrow X$ such that

$$\begin{aligned} Sp &= \begin{cases} \frac{p}{4}, & \text{if } p \in [0, 1[; \\ \frac{p}{4} + \frac{3}{4}, & \text{if } p \in [1, +\infty[, \end{cases} \\ Tp &= \begin{cases} \frac{p}{2}, & \text{if } p \in [0, 1[; \\ \frac{p}{2} + \frac{1}{2}, & \text{if } p \in [1, +\infty[, \end{cases} \end{aligned} \quad (29)$$

and the lower semicontinuous function $\phi: X \longrightarrow [0, +\infty[$ defined by

$$\phi(p) = \begin{cases} \exp(2p) & \text{if } p \in [0, 1]; \\ \exp(2p + 1) & \text{if } p \in]1, +\infty[. \end{cases} \quad (30)$$

Let us show that, for all $p \in X$ and $t > 0$,

$$\begin{cases} F_{p, Sp}(t) \geq H(t - (\phi(p) - \phi(Tp))); \\ F_{p, Tp}(t) \geq H(t - (\phi(p) - \phi(Sp))). \end{cases} \quad (31)$$

Since the function H is nondecreasing, it suffices to show that, for all $p \in X$,

$$(S): \begin{cases} |p - Sp| \leq \phi(p) - \phi(Tp); \\ |p - Tp| \leq \phi(p) - \phi(Sp). \end{cases} \quad (32)$$

(i) If $p \in [0, 1[$, we have $|p - Sp| = 3/4p$, $|p - Tp| = 1/2p$, and

$$\begin{cases} c\phi(p) - \phi(Tp) = e^{2p} - e^p = p + \sum_{n=2}^{+\infty} \frac{1}{n!} ((2p)^n - p^n), \\ \phi(p) - \phi(Sp) = e^{2p} - e^{p/2} = \frac{3}{2}p + \sum_{n=2}^{+\infty} \frac{1}{n!} \left((2p)^n - \left(\frac{p}{2}\right)^n \right). \end{cases} \quad (33)$$

Then, we obtain $|p - Sp| \leq \phi(p) - \phi(Tp)$ and $|p - Tp| \leq \phi(p) - \phi(Sp)$.

(ii) If $p = 1$, system (S) is obvious.

(iii) If $p > 1$, we have $|p - Sp| = 3/4(p - 1)$, $|p - Tp| = 1/2(p - 1)$, and

$$\begin{cases} \phi(p) - \phi(Tp) = e^{2p+1} - e^{p+2} = p - 1 \\ + \sum_{n=2}^{+\infty} \frac{1}{n!} [(2p+1)^n - (p+2)^n], \\ \phi(p) - \phi(Sp) = e^{2p} - e^{p/2} = \frac{3}{2}p - \frac{3}{2} \\ + \sum_{n=2}^{+\infty} \frac{1}{n!} [(2p+1)^n - (\frac{p}{2} + \frac{5}{2})^n]. \end{cases} \quad (34)$$

Then, we obtain $|p - Sp| \leq \phi(p) - \phi(Tp)$ and $|p - Tp| \leq \phi(p) - \phi(Sp)$.

Hence, we have

$$\begin{cases} F_{p,Sp}(t) \geq H(t - (\phi(p) - \phi(Tp))), \\ F_{p,Tp}(t) \geq H(t - (\phi(p) - \phi(Sp))), \\ \forall p \in X, \forall t > 0, \end{cases} \quad (35)$$

and 0, 1 are two common fixed points of T and S .

Theorem 6. Let $(X, \mathcal{F}, \tau)$ be a complete Menger space with τ satisfying the condition (P). Let $\phi, \psi: X \rightarrow \mathbb{R}_+$ be two lower semicontinuous functions and $T, S: X \rightarrow X$ be two mappings such that, for all $p \in X$ and for all $t > 0$,

$$\begin{cases} F_{p,Sp}(t) \geq H(t - h(c(\phi(p) + \psi(p)), c(\phi(Tp) \\ + \psi(Tp))) (\psi(p) - \psi(Tp))), \\ F_{p,Tp}(t) \geq H(t - h(c(\phi(p) + \psi(p)), c(\phi(Sp) \\ + \psi(Sp))) (\phi(p) - \phi(Sp))), \end{cases} \quad (36)$$

where $c: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a right locally bounded function from the above and h is a locally bounded function from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}_+$.

Assume that

- (i) $\phi \circ T, \phi \circ S, \psi \circ T$, and $\psi \circ S$ are lower semicontinuous functions
- (ii) There exists $p_0 \in X$ such that $\psi(Sp_0) \leq \psi(Tp_0)$ and $\phi(Tp_0) \leq \phi(Sp_0)$

Then, there exists an element $p^* \in X$ such that $Tp^* = p^* = Sp^*$.

Proof. The set $X_0 = \{p \in X: \psi(Sp) \leq \psi(Tp) \text{ and } \phi(Tp) \leq \phi(Sp)\}$ is nonempty ($p_0 \in X_0$) and closed because the functions $\phi \circ T, \phi \circ S, \psi \circ T$, and $\psi \circ S$ are lower semicontinuous.

Step 1: let $\alpha = \inf(\phi + \psi)(X_0)$. Since the function c is locally bounded from the above, there exists $\lambda > 0$ such that $\mu = \supc([\alpha, \alpha + \lambda]) < \infty$. Also, there exists $\nu > 0$ with $h(t, s) \leq \nu$, whenever $(t, s) \in [0, \mu]$.

Let $p_1 \in X_0$ such that $\alpha \leq (\phi + \psi)(p_1) \leq \alpha + \lambda$. Consider the set X_1 defined by

$$X_1 = \{p \in X_0: (\phi + \psi)(p) \leq (\phi + \psi)(p_1)\}. \quad (37)$$

Then, it is nonempty ($p_1 \in X_1$) and closed since $\phi + \psi$ is lower semicontinuous.

Let $p \in X_1$. Since $p \in X_0$, $\psi(Sp) \leq \psi(Tp)$ and $\phi(Tp) \leq \phi(Sp)$. So,

$$\begin{aligned} (\phi + \psi)(Tp) &\leq \phi(Tp) + \psi(p) \leq \phi(Sp) + \psi(p) \\ &\leq (\phi + \psi)(p) \leq (\phi + \psi)(p_1). \end{aligned} \quad (38)$$

Then, $Tp \in X_1$. Similarly, we can show that $Sp \in X_1$, for all $p \in X_1$.

Since $(\phi + \psi)(p), (\phi + \psi)(Tp), (\phi + \psi)(Sp) \in [\alpha, \alpha + \lambda]$,

$$\max\{c((\phi + \psi)(p)), c((\phi + \psi)(Tp)), c((\phi + \psi)(Sp))\} \leq \mu. \quad (39)$$

And consequently, for all $p \in X_1$,

$$\begin{cases} h(c((\phi + \psi)(p)), c((\phi + \psi)(Tp))) \leq \nu, \\ h(c((\phi + \psi)(p)), c((\phi + \psi)(Sp))) \leq \nu. \end{cases} \quad (40)$$

Step 2: we introduce the partial order " $\preceq$ " defined on X_1 by

$$p \leq q \iff F_{p,q}(t) \geq H(t - \nu(\phi + \psi)(p) + (\phi + \psi)(q)), \quad \forall t > 0. \quad (41)$$

By the same arguments as in the proof of Theorem 5, one can show that the partial ordered set $(X_1, \preceq)$ has a maximal element p^* .

For all $t > 0$,

$$\begin{aligned} F_{p^*,Sp^*}(t) &\geq H(t - h(c(\phi(p^*) + \psi(p^*)), c(\phi(Tp^*) + \psi(Tp^*))) \\ &\quad (\psi(p^*) - \psi(Tp^*))) \\ &\geq H(t - \nu(\psi(p^*) - \psi(Tp^*))) \\ &\geq H(t - \nu(\psi(p^*) - \psi(Sp^*))). \end{aligned} \quad (42)$$

Since H is nondecreasing and $\phi(p^*) - \phi(Sp^*) \geq 0$,

$$\begin{aligned}
F_{p^*, Sp^*}(t) &\geq H(t - \nu(\psi(p^*) + \phi(p^*) - \phi(Sp^*) - \psi(Sp^*))) \\
&\geq H(t - \nu((\psi + \phi)(p^*) - (\psi + \phi)(Sp^*))).
\end{aligned} \tag{43}$$

Hence, $p^* \leq Sp^*$, which implies that $Sp^* = p^*$ and $Tp^* = p^*$. $\square$

Example 4. Consider the space $X = \mathbb{R}_+$ with the metric d defined by

$$d(p, q) = |p - q|, \quad \text{for all } (p, q) \in X^2. \tag{44}$$

Consider the two mappings $T, S: X \longrightarrow X$ such that

$$\begin{aligned}
Sp &= \begin{cases} \frac{p}{3}, & \text{if } p \in [0, 1]; \\ \frac{p}{3} + \frac{2}{3}, & \text{if } p \in]1, +\infty[, \end{cases} \\
Tp &= \begin{cases} \frac{p}{5}, & \text{if } p \in [0, 1]; \\ \frac{p}{5} + \frac{4}{5}, & \text{if } p \in]1, +\infty[. \end{cases}
\end{aligned} \tag{45}$$

Consider the two lower semicontinuous functions $\phi, \psi: X \longrightarrow [0, +\infty[$ such that

$$\begin{aligned}
\phi(p) &= \begin{cases} e^{5p}, & \text{if } p \in [0, 1]; \\ e^{5p+1}, & \text{if } p \in]1, +\infty[, \end{cases} \\
\psi(p) &= \begin{cases} e^{3p}, & \text{if } p \in [0, 1]; \\ e^{3p+1}, & \text{if } p \in]1, +\infty[. \end{cases}
\end{aligned} \tag{46}$$

One can see that

$$\begin{aligned}
\phi(Sp) &= \begin{cases} e^{5p/3}, & \text{if } p \in [0, 1]; \\ e^{(5p/3)+(13/3)}, & \text{if } p \in]1, +\infty[. \end{cases} \\
\psi(Tp) &= \begin{cases} e^{3p/5}, & \text{if } p \in [0, 1]; \\ e^{(3p/5)+(17/5)}, & \text{if } p \in]1, +\infty[. \end{cases}
\end{aligned} \tag{47}$$

Consider the function $F: X \times X \longrightarrow \mathcal{D}$ defined by

$$\begin{aligned}
F(p, q)(t) &= F_{p,q}(t) = H(t - |p - q|), \\
&\text{for all } (p, q) \in X^2, t \in \mathbb{R}.
\end{aligned} \tag{48}$$

Let $p \in X$. Let us show that

$$\begin{cases} |p - Sp| \leq \psi(p) - \psi(Tp); \\ |p - Tp| \leq \phi(p) - \phi(Sp). \end{cases} \tag{49}$$

(i) If $p \in [0, 1]$, we have $|p - Sp| = 2/3p$, $|p - Tp| = 4/5p$, and

$$\begin{cases} \psi(p) - \psi(Tp) = e^{3p} - e^{3p/5} = \frac{12p}{5} + \sum_{n=2}^{+\infty} \frac{1}{n!} \left((3p)^n - \left(\frac{3p}{5} \right)^n \right), \\ \phi(p) - \phi(Sp) = e^{5p} - e^{5p/3} = \frac{10}{3}p + \sum_{n=2}^{+\infty} \frac{1}{n!} \left((5p)^n - \left(\frac{5p}{3} \right)^n \right). \end{cases} \tag{50}$$

Then, $|p - Sp| \leq \psi(p) - \psi(Tp)$ and $|p - Tp| \leq \phi(p) - \phi(Sp)$.

(ii) If $p = 1$, the system is obvious.

(iii) If $p > 1$, we have $|p - Sp| = 2/3(p - 1)$, $|p - Tp| = 4/5(p - 1)$, and

$$\begin{cases} \psi(p) - \psi(Tp) = e^{3p+1} - e^{(3p+17)/5} = \frac{12}{5}(p - 1) \\ \quad + \sum_{n=2}^{+\infty} \frac{1}{n!} \left((3p + 1)^n - \left(\frac{3p+17}{5} \right)^n \right), \\ \phi(p) - \phi(Sp) = e^{5p+1} - e^{(5p+13)/3} = \frac{10}{3}(p - 1) \\ \quad + \sum_{n=2}^{+\infty} \frac{1}{n!} \left((5p + 1)^n - \left(\frac{5p+13}{3} \right)^n \right). \end{cases} \tag{51}$$

Then, $|p - Sp| \leq \psi(p) - \psi(Tp)$ and $|p - Tp| \leq \phi(p) - \phi(Sp)$.

In all the cases, we have

$$\begin{cases} |p - Sp| \leq \psi(p) - \psi(Tp); \\ |p - Tp| \leq \phi(p) - \phi(Sp). \end{cases} \tag{52}$$

Since H is nondecreasing,

$$\begin{cases} F_{p,Sp}(t) \geq H(t - (\psi(p) - \psi(Tp))); \\ F_{p,Tp}(t) \geq H(t - (\phi(p) - \phi(Sp))), \\ \forall p \in X, \forall t > 0. \end{cases} \tag{53}$$

Moreover, the functions $\psi \circ T$, $\psi \circ S$, $\phi \circ T$, and $\phi \circ S$ are lower semicontinuous, $\phi(T0) \leq \phi(S0)$, and $\psi(S0) \leq \psi(T0)$.

Therefore, all the assumptions of Theorem 5 are satisfied, and 0 is a common fixed point of S and T .

3. Application

Let C_1 be the space of the functions $f: \mathbb{N}^* \longrightarrow]0, +\infty]$ such that the series $\sum_{n \geq 1} 2^{-n}/f(n)$ is convergent and C_2 be the space of the functions $f: \mathbb{N}^* \longrightarrow [0, +\infty[$ such that the series $\sum_{n \geq 1} 2^{-n}f(n)$ is convergent. We consider two metrics d_1 and d_2 defined, respectively, on C_1 and C_2 by

$$d_1(f, g) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \left| \frac{1}{f(n)} - \frac{1}{g(n)} \right|, \tag{54}$$

$$d_2(f, g) = \sum_{n=1}^{+\infty} \frac{1}{2^n} |f(n) - g(n)|.$$

Since the function $m: (C_1, d_1) \longrightarrow (C_2, d_2)$ defined by $m(f) = 1/f$, for all $f \in C_1$, is a homeomorphism and (C_2, d_2) is complete, then also (C_1, d_1) is complete.

Let $\mathcal{F}$ be the family of distribution functions defined on C_1 by

$$F_{f,g}(t) = H(t - d_1(f, g)), \quad \text{for all } t \in [0, +\infty[. \quad (55)$$

Then, the space $(C_1, \mathcal{F}, \tau_M)$ is a complete Menger space.

Now, let $c \geq 1$, and three sequences $g: \mathbb{N}^* \longrightarrow [1, +\infty[$, $h: \mathbb{N}^* \longrightarrow [0, +\infty[$, and $r: \mathbb{N}^* \longrightarrow [1, +\infty[$.

Theorem 7. *The equation $f(n) = g(n)(f(n-1))^{r(n)} + h(n)$, $n \in \mathbb{N}^*$, of unknown f , has a solution in C_1 .*

Proof. Let $\mathcal{C}_{g,h,c}$ be the set of elements f of C_1 such that $f(1) = c$ and

$$f(n) \geq g(n)(f(n-1))^{r(n)} + h(n), \quad \text{for all } n \in \mathbb{N}^*. \quad (56)$$

$\mathcal{C}_{g,h,c}$ is nonempty since the sequence f , defined by $f(1) = c$ and $f(n) = +\infty$ for all $n \geq 2$, is an element of $\mathcal{C}_{g,h,c}$.

$\mathcal{C}_{g,h,c}$ is a closed subset of C_1 . Indeed, let $f \in C_1$ and $(f_p)_p \in (\mathcal{C}_{g,h,c})^{\mathbb{N}^*}$ such that $f_p \tau_M$ -convergent to f . Then, for $\varepsilon > 0$ and $0 < \lambda < 1$, there exists $N \in \mathbb{N}$ such that

$$F_{f_p, f}(\varepsilon) = H(\varepsilon - d_1(f_p, f)) > 1 - \lambda, \quad \text{for all } p \geq N, \quad (57)$$

so $\varepsilon - d_1(f_p, f) > 0$, for all $p \geq N$, which shows that $d_1(f_p, f) \longrightarrow 0$; according to [17], Lemma 1, we have $f_p(n) \longrightarrow f(n)$, for all $n \in \mathbb{N}^*$, when $p \longrightarrow +\infty$. We know that, for each $p \in \mathbb{N}^*$,

$$f_p(n) \geq g(n)(f_p(n-1))^{r(n)} + h(n), \quad \text{for all } n \in \mathbb{N}^*. \quad (58)$$

Let us fix $n \geq 2$. By letting $p \longrightarrow +\infty$ in the above inequality, we obtain

$$f(n) \geq g(n)(f(n-1))^{r(n)} + h(n). \quad (59)$$

Consider the mapping T defined on $\mathcal{C}_{g,h,c}$ by

$$\begin{cases} T(f)(1) = c, \\ T(f)(n) = g(n)(f(n-1))^{r(n)} + h(n), \quad \text{for all } n \geq 2. \end{cases} \quad (60)$$

Then, we have $f \in \mathcal{C}_{g,h,c}$.

(i) Since $r(n) \geq 1$, $h(n) \geq 0$, and $g(n) \geq 1$, for all $n \in \mathbb{N}^*$, then $f(n) \geq f(n-1)$, for all $n \in \mathbb{N} \setminus \{0, 1\}$. Hence, $(f(n))_{n \in \mathbb{N}^*}$ is an increasing sequence.

(ii) Let $n \in \mathbb{N} \setminus \{0, 1\}$. Then, we have

$$\begin{aligned} T(f)(n) &\geq g(n)(f(n-1))^{r(n)} + h(n) \\ &\geq g(n)(g(n-1)(f(n-2))^{r(n-1)} + h(n-1))^{r(n)} \\ &\quad + h(n) \\ &\geq g(n)(T(f)(n-1))^{r(n)} + h(n), \end{aligned} \quad (61)$$

and $T(f)(1) = c$, which shows that $T(\mathcal{C}_{g,h,c}) \subset \mathcal{C}_{g,h,c}$.

$$(iii) \sum_{n=1}^{+\infty} 1/2^n f(n) = (1/2c) + \sum_{n=2}^{+\infty} 1/2^n f(n) \leq (1/2c) + (1/c) \sum_{n=2}^{+\infty} 1/2^n = (1/c).$$

The function $\varphi: \mathcal{C}_{g,h,c} \longrightarrow [0, +\infty)$ defined by

$$\varphi(f) = \frac{1}{c} - \sum_{n=1}^{+\infty} \frac{1}{2^n f(n)} \quad (62)$$

is lower semicontinuous in $\mathcal{C}_{g,h,c}$.

(iv) Let $f \in \mathcal{C}_{g,h,c}$. Since $f(n) \geq T(f)(n)$, for all $n \in \mathbb{N}^*$,

$$\begin{aligned} d_1(f, \varphi(T(f))) &= \sum_{n=1}^{+\infty} \frac{1}{2^n T(f)(n)} - \sum_{n=1}^{+\infty} \frac{1}{2^n f(n)} \\ &= \varphi(f) - \varphi(T(f)). \end{aligned} \quad (63)$$

And since H is nondecreasing, we have, for all $t \geq 0$,

$$F_{f, T(f)}(t) = H(t - d(f, T(f))) \geq H(t - (\varphi(f) - \varphi(T(f)))). \quad (64)$$

By applying Corollary 2, there exists $f^* \in C_1$ such that

$$T(f^*)(n) = f^*(n), \quad \text{for all } n \in \mathbb{N}^*. \quad (65)$$

□

Example 5. The following functional equation

$$\begin{cases} f(1) = e, \\ f(n) = n^2(f(n-1)) + e^{(n-1)^2}(e^{2n-1} - n^2), \quad \text{for all } n \geq 2, \end{cases} \quad (66)$$

admits a solution in C_1 . In fact, one can apply Theorem 7 for $g(n) = n^2 \geq 1$, $h(n) = e^{(n-1)^2}(e^{2n-1} - n^2) \geq 0$, and $r(n) = 1$, for all $n \in \mathbb{N}^*$.

4. Conclusion

The results in this paper

- (1) Extend the main results of [16] from a standard metric space to a Menger space
- (2) Generalize and improve the main result of [15]

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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