Research Article

Hermite-Hadamard- and Jensen-Type Inequalities for Interval (h_1, h_2) Nonconvex Function

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In the present study, we will introduce the definition of interval (h_1, h_2) nonconvex function. We will investigate some properties of interval (h_1, h_2) nonconvex function. Moreover, we will develop Hermite-Hadamard- and Jensen-type inequalities for interval (h_1, h_2) nonconvex function.

1. Introduction

Since its inception five decades ago, the theory of fuzzy sets has advanced in a variety of ways. Application of the theory of fuzzy sets covered many areas like artificial intelligence, decision theory, computer science, logic operational research, and robotics [1–3]. Initially good books like probability theory by Dubois and Prade in 1988, Behavioral and social science by Smithsons in 1987, and Fuzzy Control by sugeno 1985 and pedrycz 1989 and others have been published. We refer [4, 5] recent developments in this field.

For some other results and application of interval analysis theory, we refer the readers [4, 6–13]. Due to vast application of fuzzy sets, many integral inequalities have been derived by different authors [1–3, 14–17].

Costa [18] there is a new fuzzy version of Jensen-type integral inequality for fuzzy interval valued function. Also in [19], Zhao et al. develop new Hermite-Hadamard-type inequality for h -convex interval valued function.

For more about Hermite-Hadamard inequalities, refer [20–24]. We will introduce the interval (h_1, h_2) non-convex function. The second objective of this article is to develop

Hermite-Hadamard- and Jensen-type inequality for above said generalization.

2. Preliminaries

In this section, we define some basic definitions, properties, results, and notations on interval analysis, which are used throughout the paper [17, 25]. Here, $\mathbb{R}_I$ and $\mathbb{R}_I^+$ denote the family of all intervals and positive interval and it is equipped with the algebraic operations “+” and “.” given, respectively, by $[x_1, x_2] + [y_1, y_2] = [x_1 + y_1, x_2 + y_2]$ and

$$\alpha \cdot [x_1, x_2] = \begin{cases} [\alpha x_1, \alpha x_2], & \text{if } 0 \leq \alpha, \\ [\alpha x_2, \alpha x_1], & \text{if } \alpha < 0. \end{cases} \quad (1)$$

A function $F: V \subseteq \mathbb{R} \longrightarrow \mathbb{R}_I$ with $F(x) = [f_1(x), f_2(x)]$, where $f_1, f_2: V \longrightarrow \mathbb{R}$ are real functions with $f_1(x) \leq f_2(x)$ for all $x \in V$, it is called an interval-valued function.

For intervals $[\underline{x}, \bar{x}]$ and $[\underline{y}, \bar{y}]$, the Hausdorff distance is defined by $d([\underline{x}, \bar{x}], [\underline{y}, \bar{y}]) = \max\{|\underline{x} - \underline{y}|, |\bar{x} - \bar{y}|\}$.

Then $(\mathbb{R}_I, d)$ is complete.

A set of numbers $f\{x_{i-1}, y_i, x_i\}_{i=1}^m$ is said to be a tagged partition P of $[x, y]$ if

$$x = x_0 < x_1 < \cdots < x_m = y, \quad (2)$$

and if $x_{i-1} \leq y_i \leq x_i$ for all $i = 1, 2, \dots, m$. Moreover, if we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta x_i \leq \delta$ if for each i , then we say that the partition is δ -fine. The family of all δ -fine partitions of $[x, y]$ is denoted by $P(\delta, [x, y])$.

Given $X \in P(\delta, [x, y])$, we define an integral sum of $f: [x, y] \rightarrow \mathbb{R}_I$ as follows:

$$S(f, X, \delta, [x, y]) = \sum_{i=1}^m f(y_i)(x_i - x_{i-1}). \quad (3)$$

Throughout the paper, IR -integrable means interval Riemann integrable. The concept of IR -integrable is given in [19], Definition 2.2, is equivalent to IR integral given in [10], Definition 9.1.

Definition 1. Let $f: [x, y] \rightarrow \mathbb{R}_I$. f is called IR -integrable on $[x, y]$ with IR -integral $A = (IR) \int_x^y f(t)dt$, if there exists an $A \in RI$ such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d(S(f, X, \delta, [x, y]), A) < \epsilon, \quad (4)$$

for each $X \in P(\delta, [x, y])$. Let $IR([x, y])$ denote the set of all IR -integrable functions on $[x, y]$.

Definition 2. (See p -convex set, [26]).

The real interval I is known as p -convex set if for all $x, y \in I$ and $\alpha \in [0, 1]$, implies that

$$[\alpha x^p + (1 - \alpha)y^p]^{(1/p)} \in I, \quad (5)$$

where $p = 2k + 1$ or $p = (n_1/n_2)$, $n_1 = 2r + 1$, $n_2 = 2t + 1$, and $k, r, t \in \mathbb{N}$.

Definition 3. (See p -convex function, [26]). For a p -convex set I , the mapping $f: I \rightarrow \mathbb{R}$:

$$f([\lambda^p + (1 - \lambda)y^p]^{(1/p)}) \leq \lambda f(x) + (1 - \lambda)f(y), \quad (6)$$

is called p -convex function, for all $x, y \in I$, and $\lambda \in [0, 1]$.

Definition 4. (See h -convex function, [27]). The nonnegative function $f: [a, b] \rightarrow \mathbb{R}$ is h -convex if

$$f(\lambda x + (1 - \lambda)y) \leq h(\lambda)f(x) + h(1 - \lambda)f(y). \quad (7)$$

$\forall x, y \in [a, b]$ and $\lambda \in (0, 1)$ and $h \neq 0$, h is nonnegative real-valued function or f belongs to the class $SX(h; [a, b])$.

If the inequality (7) is reversed, then f is said to be h -concave, i.e., $f \in SV(h, I)$.

Definition 5. (h_1, h_2) -convex function [28]. Let $h_1, h_2: [0, 1] \subseteq J \rightarrow \mathbb{R}^+$ be a nonnegative function, $h_1, h_2 \neq 0$. The nonnegative function $f: J \rightarrow \mathbb{R}$ is an (h_1, h_2) -convex if

$$f(\lambda x + (1 - \lambda)y) \leq h_1(\lambda)h_2(1 - \lambda)f(x) + h_1(1 - \lambda)h_2(\lambda)f(y), \quad (8)$$

for all $x, y \in J$ and $\lambda \in (0, 1)$ or that f belongs to the class $SX((h_1, h_2); J, \mathbb{R})$. If the inequality (8) is reversed, then f is said to be (h_1, h_2) -concave, i.e., $f \in SV((h_1, h_2), J, \mathbb{R}_I^+)$.

Definition 6. Interval (h_1, h_2) -convex function [29]. Let $h_1, h_2: [0, 1] \subseteq J \rightarrow \mathbb{R}^+$ be a nonnegative function, $h_1, h_2 \neq 0$. We say that $f: J \rightarrow \mathbb{R}_I^+$ is an interval (h_1, h_2) -convex function or that f belongs to the class $SX((h_1, h_2); J, \mathbb{R}_I^+)$, if f is nonnegative and for all $x, y \in J$ and $\lambda \in (0, 1)$, we have

$$\begin{aligned} & h_1(\lambda)h_2(1 - \lambda)f(x) + h_1(1 - \lambda)h_2(\lambda)f(y) \\ & \subseteq f(\lambda x + (1 - \lambda)y). \end{aligned} \quad (9)$$

If the inequality (9) is reversed, then f is said to be interval (h_1, h_2) -concave, i.e., $f \in SV((h_1, h_2), J, \mathbb{R})$.

Remark 1

- (1) If $h_2 = 1$, then Definition 6 becomes interval h -convex in [19].
- (2) If $h_1 = h_2 = 1$, then Definition 6 becomes P -function in [30].
- (3) If $h_1(t) = t^s$, $h_2 = 1$, then Definition 6 becomes to s -convex fuzzy process in [31].

We wind up the current section by introducing the new concept of interval (h_1, h_2) nonconvexity. This idea is inspired by An et al. [29]. Throughout the paper, for interval $[\underline{x}, \bar{x}]$ and $[\underline{y}, \bar{y}]$,

$$[\underline{x}, \bar{x}] \subseteq [\underline{y}, \bar{y}] \Rightarrow \underline{y} \leq \underline{x} \text{ and } \bar{x} \leq \bar{y}. \quad (10)$$

Definition 7. Interval (h_1, h_2) nonconvex function. Let $h_1, h_2: [0, 1] \subseteq J \rightarrow \mathbb{R}^+$ be a nonnegative function, $h_1, h_2 \neq 0$. For a nonconvex function, $f: J \rightarrow \mathbb{R}_I^+$ is an interval (h_1, h_2) nonconvex function if

$$\begin{aligned} & h_1(\lambda)h_2(1 - \lambda)f(x) + h_1(1 - \lambda)h_2(\lambda)f(y) \\ & \subseteq f(\lambda x^p + (1 - \lambda)y^p)^{(1/p)}. \end{aligned} \quad (11)$$

$\forall x, y \in J$, and $\lambda \in (0, 1)$

For convenience, in this paper, the class of interval nonconvex function is denoted by $SX((h_1, h_2, p); J, \mathbb{R}_I^+)$. If the inequality (11) is reversed, then f is said to be interval (h_1, h_2) nonconcave, i.e., $f \in SV((h_1, h_2, p), J, \mathbb{R})$.

Throughout the paper $H(x, y) = h_1(x)h_2(y)$ for $x, y \in [0, 1]$.

Theorem 1. Let $f: [r, s] \rightarrow \mathbb{R}_I^+$ be an interval-valued function such that $f(\lambda) = [f(\lambda), \bar{f}(\lambda)]$. Then $f \in SX((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$ if and only if $\underline{f} \in SX((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$ and $\bar{f} \in SV((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$.

Proof. Let f be an interval (h_1, h_2) non-convex function and suppose that $x, y \in [r, s]$, $\lambda \in (0, 1)$, then

$$\begin{aligned} & h_1(\lambda)h_2(1 - \lambda)f(x) + h_1(1 - \lambda)h_2(\lambda)f(y) \\ & \subseteq f(\lambda x^p + (1 - \lambda)y^p)^{(1/p)}, \end{aligned} \quad (12)$$

that is,

$$\begin{aligned} & [h_1(\lambda)h_2(1-\lambda)\underline{f}(x) + h_1(1-\lambda)h_2(\lambda)\underline{f}(y) \\ & \quad \times h_1(\lambda)h_2(1-\lambda)\overline{f}(x) + h_1(1-\lambda)h_2(\lambda)\overline{f}(y)] \\ & \subseteq \left[\underline{f}(\lambda x^p + (1-\lambda)y^p)^{(1/p)}, \overline{f}(\lambda x^p + (1-\lambda)y^p)^{(1/p)} \right]. \end{aligned} \quad (13)$$

It follows that

$$\begin{aligned} & h_1(\lambda)h_2(1-\lambda)\underline{f}(x) + h_1(1-\lambda)h_2(\lambda)\underline{f}(y) \\ & \geq \underline{f}(\lambda x^p + (1-\lambda)y^p)^{(1/p)} h_1(\lambda)h_2(1-\lambda)\overline{f}(x) \\ & \quad + h_1(1-\lambda)h_2(\lambda)\overline{f}(y) \\ & \leq \overline{f}(\lambda x^p + (1-\lambda)y^p)^{(1/p)}. \end{aligned} \quad (14)$$

This shows that $f \in SX((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$ and $\overline{f} \in SV((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$.

Conversely suppose that

$\underline{f} \in SX((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$ and $\overline{f} \in SV((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$ then from above definition and inclusion set (13), it follows that $f \in SX((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$.

This completes the proof. $\square$

Theorem 2. Let $f: [r, s] \rightarrow \mathbb{R}_I^+$ be an interval-valued function such that $f(\lambda) = [\underline{f}(\lambda), \overline{f}(\lambda)]$. Then $f \in SV((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$ if $\underline{f}$ and $\overline{f}$ only if $\underline{f} \in SV((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$ and $\overline{f} \in SX((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$.

Proof. The proof is similar to that of Theorem 1. $\square$

3. Main Result

Theorem 3. Let $f: [r, s] \rightarrow \mathbb{R}_I^+$, $h_1, h_2: [0, 1] \rightarrow \mathbb{R}^+$ and $H((1/2), (1/2)) \neq 0$ if $f \in SX((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$ and $f \in IR_{[r, s]}$ then

$$\begin{aligned} & \frac{1}{2H((1/2), (1/2))} f\left(\frac{r^p + s^p}{2}\right)^{(1/p)} \supseteq \frac{p}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\ & \supseteq [f(r) + f(s)] \int_0^1 H(x, 1-x) dx. \end{aligned} \quad (15)$$

Proof. By assumption, we have

$$\begin{aligned} & H\left(\frac{1}{2}, \frac{1}{2}\right) \underline{f}(xr^p + (1-x)s^p)^{(1/p)} + H\left(\frac{1}{2}, \frac{1}{2}\right) \\ & \quad \underline{f}((1-x)r^p + xs^p)^{(1/p)} \\ & \geq \underline{f}\left(\frac{r^p + s^p}{2}\right)^{(1/p)} H\left(\frac{1}{2}, \frac{1}{2}\right) \overline{f}(xr^p + (1-x)s^p)^{(1/p)} \\ & \quad + H\left(\frac{1}{2}, \frac{1}{2}\right) \overline{f}((1-x)r^p + xs^p)^{(1/p)} \leq \overline{f}\left(\frac{r^p + s^p}{2}\right)^{(1/p)}. \end{aligned} \quad (16)$$

Integrating above w.r.t. “ x ” on $[0, 1]$, we get

$$\begin{aligned} & \int_0^1 \underline{f}(xr^p + (1-x)s^p)^{(1/p)} dx \\ & \quad + \int_0^1 \underline{f}((1-x)r^p + xs^p)^{(1/p)} dx \\ & \geq \frac{1}{H((1/2), (1/2))} \underline{f}\left(\frac{r^p + s^p}{2}\right)^{(1/p)}, \\ & \int_0^1 \overline{f}(xr^p + (1-x)s^p)^{(1/p)} dx \\ & \quad + \int_0^1 \overline{f}((1-x)r^p + xs^p)^{(1/p)} dx \\ & \leq \frac{1}{H((1/2), (1/2))} \overline{f}\left(\frac{r^p + s^p}{2}\right)^{(1/p)}, \end{aligned} \quad (17)$$

it follows that

$$\begin{aligned} & \frac{2p}{s^p - r^p} \int_0^1 \lambda^{p-1} \underline{f}(\lambda) d\lambda \geq \frac{1}{H((1/2), (1/2))} \underline{f}\left(\frac{r^p + s^p}{2}\right)^{(1/p)} \\ & \frac{2p}{s^p - r^p} \int_0^1 \lambda^{p-1} \overline{f}(\lambda) d\lambda \leq \frac{1}{H((1/2), (1/2))} \overline{f}\left(\frac{r^p + s^p}{2}\right)^{(1/p)}. \end{aligned} \quad (18)$$

This implies that

$$\frac{p}{s^p - r^p} \int_0^1 \lambda^{p-1} f(\lambda) d\lambda \subseteq \frac{1}{2H((1/2), (1/2))} f\left(\frac{r^p + s^p}{2}\right)^{(1/p)}. \quad (19)$$

Now by def. of interval nonconvex function, we have

$$\begin{aligned} & h_1(x)h_2(1-x)f(r) + h_1(1-x)h_2(x)f(s) \\ & \subseteq f(xr^p + (1-x)s^p)^{(1/p)}, \end{aligned} \quad (20)$$

integrated with respect to “ x ” on $[0, 1]$, we get

$$\begin{aligned}
& f(r) \int_0^1 h_1(x) h_2(1-x) dx + f(s) \int_0^1 h_1(1-x) h_2(x) dx \\
& \subseteq \int_0^1 f(xr^p + (1-x)s^p)^{(1/p)} dx [f(r) + f(s)] \\
& \quad \times \int_0^1 h_1(x) h_2(1-x) dx \\
& \subseteq \int_0^1 f(xr^p + (1-x)s^p)^{(1/p)},
\end{aligned} \tag{21}$$

it follows that

$$\begin{aligned}
& [f(r) + f(s)] \cdot \int_0^1 h_1(x) h_2(1-x) dx \\
& \subseteq \frac{p}{s^p - r^p} \int_0^1 \lambda^{p-1} f(\lambda) d\lambda.
\end{aligned} \tag{22}$$

Combining (19) and (22) we get (27). $\square$

Remark 2

- (1) If $H(x, y) = h_1(x)$ and $f = \bar{f}$ then Theorem 3 becomes ([32], Theorem 5).
- (2) If $H(x, y) = h_1(x)$ and $p = 1$ then Theorem 3 becomes ([19], Theorem 4.1).
- (3) If $h_1(x) = x^s$, $h_2(x) = 1$ and $p = 1$ then Theorem 3 becomes ([31], Theorem 4).
- (4) If $\bar{f} = \bar{f}$ then Theorem 3 becomes Hermite-Hadamard inequality for (h_1, h_2) -convex function.

Example 1. Consider $h_1(t) = t$, $h_2(t) = 1$ for $t \in [0, 1]$, $[r, s] = [-1, 1]$ and $f: [r, s] \rightarrow \mathbb{R}_I^+$ be defined by $f(\lambda) = [\lambda^p, 4 - e^{\lambda^p}]$ and p be an odd number, then we have

$$\frac{1}{2H((1/2), (1/2))} f\left(\frac{r^p + s^p}{2}\right)^{(1/p)} = f(0) = [0, 3], \tag{23}$$

and

$$\begin{aligned}
& \frac{p}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\
& = \frac{p}{2} \left[\int_{-1}^1 \lambda^{p-1} \lambda^p d\lambda, \int_{-1}^1 \lambda^{p-1} (4 - e^{\lambda^p}) d\lambda \right] \\
& = \frac{p}{2} \left[\int_{-1}^1 \lambda^{2p-1} d\lambda, \frac{8}{p} - \int_{-1}^1 \lambda^{p-1} e^{\lambda^p} d\lambda \right] \\
& = \left[0, 4 - \frac{e - e^{-1}}{2} \right].
\end{aligned} \tag{24}$$

Putting $z = \lambda^p$ in (24) and simplifying, we get

$$[f(r) + f(s)] \int_0^1 H(x, 1-x) dx = \left[0, 4 - \frac{e + e^{-1}}{2} \right]. \tag{25}$$

Combining (23)–(25), we get

$$[0, 3] \supseteq \left[0, 4 - \frac{e - e^{-1}}{2} \right] \supseteq \left[0, 4 - \frac{e + e^{-1}}{2} \right]. \tag{26}$$

Consequently, verify the Theorem 3.

Theorem 4. Let $f: [r, s] \rightarrow \mathbb{R}_I^+$, $h_1, h_2: [0, 1] \rightarrow \mathbb{R}_I^+$, and $H((1/2), (1/2)) \neq 0$ if $f \in SX((h_1, h_2, p), [r, s], \mathbb{R}_I^+)$ and $f \in IR_{[r,s]}$ then

$$\begin{aligned}
& \frac{1}{4[H((1/2), (1/2))]^2} f\left(\frac{r^p + s^p}{2}\right)^{(1/p)} \\
& \supseteq \Delta_1 \supseteq \frac{p}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\
& \supseteq \Delta_2 \supseteq [f(r) + f(s)] \left[\frac{1}{2} + H\left(\left(\frac{1}{2}\right), \left(\frac{1}{2}\right)\right) \right] \\
& \quad \times \int_0^1 H(x, 1-x) dx,
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
\Delta_1 &= \frac{1}{4H((1/2), (1/2))} \left[f\left(\frac{3r^p + s^p}{4}\right)^{(1/p)} + f\left(\frac{r^p + 3s^p}{4}\right)^{(1/p)} \right], \\
\Delta_2 &= \frac{1}{2} \left[f(r) + f(s) + 2f\left(\frac{r^p + s^p}{2}\right)^{(1/p)} \right] \int_0^1 H(x, 1-x) dx.
\end{aligned} \tag{28}$$

Proof. For $[r^p, (r^p + s^p)/2]$, one has

$$\begin{aligned}
& H\left(\left(\frac{1}{2}\right), \left(\frac{1}{2}\right)\right) f\left(xr^p + (1-x)\frac{r^p + s^p}{2}\right)^{(1/p)} \\
& + H\left(\left(\frac{1}{2}\right), \left(\frac{1}{2}\right)\right) f\left((1-x)r^p + x\frac{r^p + s^p}{2}\right)^{(1/p)} \\
& \subseteq f\left(\frac{3r^p + s^p}{4}\right)^{(1/p)}.
\end{aligned} \tag{29}$$

Consequently, we get

$$\begin{aligned}
& \frac{4p}{s^p - r^p} \int_r^{((r^p + s^p)/2)} \lambda^{p-1} f(\lambda) d\lambda \\
& \subseteq \frac{1}{H((1/2), (1/2))} f\left(\frac{3r^p + s^p}{4}\right)^{(1/p)}.
\end{aligned} \tag{30}$$

Similarly for $[r^p + s^p/2, s^p]$

$$\begin{aligned}
& \frac{4p}{s^p - r^p} \int_{(r^p + s^p/2)}^s \lambda^{p-1} f(\lambda) d\lambda \\
& \subseteq \frac{1}{H((1/2), (1/2))} f\left(\frac{r^p + 3s^p}{4}\right)^{(1/p)}.
\end{aligned} \tag{31}$$

Adding (30) and (31), we get

$$\begin{aligned}
& \frac{P}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\
& \leq \frac{1}{4H((1/2), (1/2))} \left[f\left(\frac{3r^p + s^p}{4}\right)^{(1/p)} + f\left(\frac{r^p + 3s^p}{4}\right)^{(1/p)} \right] \\
& \quad \cdot \frac{P}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\
& \leq \Delta_1,
\end{aligned} \tag{32}$$

and from Theorem 3, we have

$$\begin{aligned}
& \frac{1}{4[H((1/2), (1/2))]^2} f\left(\frac{r^p + s^p}{2}\right)^{(1/p)} \\
& = \frac{1}{4[H((1/2), (1/2))]^2} f\left(\frac{1}{2} \frac{3r^p + s^p}{4} + \frac{1}{2} \frac{r^p + 3s^p}{4}\right)^{(1/p)} \\
& \geq \frac{1}{4H((1/2), (1/2))} f\left(\frac{3r^p + s^p}{4} + \frac{r^p + 3s^p}{4}\right) \\
& \geq \Delta_1 \\
& \geq \frac{P}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\
& \geq \frac{1}{2} \left[f(r) + f(s) + 2f\left(\frac{r^p + s^p}{2}\right)^{(1/p)} \right] \int_0^1 H(x, 1-x) dx \\
& = \Delta_2 \\
& \geq [f(r) + f(s)] \left[\frac{1}{2} + H\left(\left(\frac{1}{2}\right), \left(\frac{1}{2}\right)\right) \right] \int_0^1 H(x, 1-x) dx,
\end{aligned} \tag{33}$$

and the result follows. $\square$

Theorem 5 (Jensen-type inequality). *Let $w_1, w_2, \dots, w_n \in \mathbb{R}^+$ with $n \geq 2$. If h_1, h_2 is nonnegative super multiplicative function and if $f \in SX((h_1, h_2, p), [u, v], \mathbb{R}_I^+)$ and $x_1, x_2, \dots, x_n \in I$. Then, we have following inequality:*

$$f\left(\left[\frac{1}{W_n} \sum_{i=1}^n w_i x_i^p\right]^{(1/p)}\right) \geq \sum_{i=1}^n H\left(\frac{w_i}{W_n}, \frac{W_{n-1}}{W_n}\right) f(x_i), \tag{34}$$

where $W_n = \sum_{i=1}^n w_i$

Proof. When $n=2$ inequality (33) is true. Consider the inequality (33) is true for $n-1$, then

$$\begin{aligned}
& f\left(\left[\frac{1}{W_n} \sum_{i=1}^n w_i x_i^p\right]^{(1/p)}\right) \\
& = f\left(\left[\frac{w_n}{W_n} x_n^p + \sum_{i=1}^{n-1} \frac{w_i}{W_n} x_i^p\right]^{(1/p)}\right) \\
& \geq h_1\left(\frac{w_n}{W_n}\right) h_2\left(\frac{W_{n-1}}{W_n}\right) f(x_n) \\
& \quad + h_1\left(\frac{W_{n-1}}{W_n}\right) h_2\left(\frac{w_n}{W_n}\right) f\left(\sum_{i=1}^{n-1} \frac{w_i}{W_n} x_i^p\right) \\
& \geq h_1\left(\frac{w_n}{W_n}\right) h_2\left(\frac{W_{n-1}}{W_n}\right) f(x_n) \\
& \quad + h_1\left(\frac{W_{n-1}}{W_n}\right) h_2\left(\frac{w_n}{W_n}\right) \left(\sum_{i=1}^{n-1} H\left(\frac{w_i}{W_n}, \frac{W_{n-2}}{W_{n-1}}\right) f(x_i)\right) \\
& \geq \sum_{i=1}^n H\left(\frac{w_i}{W_n}, \frac{W_{n-1}}{W_n}\right) f(x_i).
\end{aligned} \tag{35}$$

Thus, by mathematical induction the result is proved. $\square$

4. Conclusions

Hermite-Hadamard- and Jensen-type inequalities are true for this new concept of interval value convexity. Moreover by suitable substitution, the obtained results reduce to the results of [19, 31, 32].

Data Availability

All data required for this paper is included within this paper.

Conflicts of Interest

The authors do not have any conflicts of interest.

Authors' Contributions

All authors contributed equally in this paper.

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