Research Article

Bounds for the Remainder in Simpson's Inequality via n -Polynomial Convex Functions of Higher Order Using Katugampola Fractional Integrals

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The goal of this paper is to derive some new variants of Simpson's inequality using the class of n -polynomial convex functions of higher order. To obtain the main results of the paper, we first derive a new generalized fractional integral identity utilizing the concepts of Katugampola fractional integrals. This new fractional integral identity will serve as an auxiliary result in the development of the main results of this paper.

1. Introduction and Preliminaries

The following inequality known in the literature as Simpson's inequality [1].

$$\left| \int_a^b \Lambda(t) dt - \frac{b-a}{6} \left[\Lambda(a) + 4\Lambda\left(\frac{a+b}{2}\right) + \Lambda(b) \right] \right| \leq \frac{(b-a)^4}{2880} \|\Lambda^{(4)}\|_{\infty}, \quad (1)$$

where $\Lambda: [a, b] \rightarrow \mathbb{R}$ is a four times continuously differentiable function on (a, b) and

$$\|\Lambda^{(4)}\|_{\infty} = \sup_{t \in (a, b)} |\Lambda^{(4)}(t)| < \infty. \quad (2)$$

Simpson's inequality plays a significant role in analysis [2–4]. Over the years, it has been extended and generalized in different directions using novel and innovative approaches. The survey by Dragomir et al. [5] is very informative regarding the developments of Simpson's inequality and its applications.

In recent years, the fractional calculus [6–10] is often known as noninteger calculus which has become a powerful tool in mathematics because it provides a good tool to describe physical memory. Fractional calculus has wide

applications in real life through its help in solving different physical problems [11–20]. The classic definition of Riemann–Liouville fractional integrals is one of the most basic concepts in fractional calculus which is defined as:

Definition 1. Let $\Lambda \in L[a, b]$. Then Riemann–Liouville integrals $J_{a^+}^{\alpha} \Lambda$ and $J_{b^-}^{\alpha} \Lambda$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$\begin{aligned} J_{a^+}^{\alpha} \Lambda(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \Lambda(t) dt, & x > a, \\ J_{b^-}^{\alpha} \Lambda(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \Lambda(t) dt, & x < b, \end{aligned} \quad (3)$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx, \quad (4)$$

is a well known gamma function.

In past few decades, several successful attempts have been made in generalizing the classical concepts of fractional calculus. Erdelyi–Kober operator is a significant generalization of fractional integrals introduced and was studied by

Arthur Erdelyi and Hermann Kober. But there is a drawback that one cannot get the Hadamard version of the derivatives and integrals from Erdelyi–Kober operators. Katugampola [21] gave a well-defined concept of fractional integrals as:

Definition 2. Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then the left and right sides of Katugampola fractional integrals of order $\alpha > 0$ of $\Lambda \in X_c^\alpha(a, b)$ are defined by

$$\begin{aligned} {}^{\rho}I_{a^+}^\alpha \Lambda(x) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} \Lambda(t) dt, \\ {}^{\rho}I_{a^-}^\alpha \Lambda(x) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} \Lambda(t) dt, \end{aligned} \quad (5)$$

if the integral exists.

If we take $\rho = 1$, then we can recapture Riemann–Liouville fractional integrals from the Katugampola fractional integrals. It worth to mention here that Erdelyi–Kober operators and Katugampola fractional integrals are not equivalent to each other.

Sarikaya et al. [22] are the first authors to utilize the concepts of Riemann–Liouville fractional integrals in obtaining the fractional analogues of Hermite–Hadamard’s inequality. This idea inspired several inequalities expert, and resultantly huge number of articles have been written on the fractional analogues of classical inequalities. For example, Hu et al. [23] obtained some new fractional analogues of integral inequalities using Katugampola fractional integrals. Nie et al. [24] obtained k -fractional analogues of Simpson’s inequality. Peng et al. [25] also obtained some new fractional analogues of Simpson’s inequality. Set [26] obtained fractional analogues of Ostrowski’s inequality. Wu et al. [27] obtained fractional analogues of inequalities using k -th order differentiable functions. Kermausuor [28] obtained new Simpson type inequalities involving Katugampola fractional integrals essentially using the class of Breckner type s -convex function.

Recently, Toplu et al. [29] introduced the notion of n -polynomial convex functions as follows.

Definition 3. Let $n \in \mathbb{N}$. Then a nonnegative function $\Lambda: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be a n -polynomial convex function if the inequality

$$\Lambda(tx + (1-t)y) \leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \Lambda(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \Lambda(y), \quad (6)$$

holds for every $x, y \in I$ and $t \in [0, 1]$.

Many researchers have also derived several new Hermite–Hadamard’s like inequalities [30–36] using the concept of n -polynomial convex functions. We would like to point out here that the class of n -polynomial convex functions generalize the class of convex functions if we take $n = 1$, then we have the class of 1-polynomial convex functions which is just the classical convex functions. Also we can get other type of convexities: for example, for $n = 2$, we have 2-polynomial convexity. Another point of pondering here is that every n -polynomial convex function is an h -convex function with the function $h(t) = 1/n \sum_{s=1}^n [1 - (1-t)^s]$. So more generally, every non-negative convex function is also an n -polynomial convex function.

The idea behind the study of this article is to extend the notion of n -polynomial convex functions with the introduction of higher order n -polynomial convex functions. We derive a new fractional integral identity using the concepts of Katugampola fractional integrals. This new identity will serve as an auxiliary result in the development of some new fractional analogues of Simpson’s inequalities using the concept of n -polynomial convex functions of higher order.

Before we move to our main results, we would like to introduce the notion of n -polynomial convex functions of higher order.

Definition 4. Let $n \in \mathbb{N}$. Then a nonnegative function $\Lambda: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be a higher order n -polynomial convex function if the inequality

$$\Lambda(tx + (1-t)y) \leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \Lambda(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \Lambda(y) - \mu(t^\sigma(1-t) + t(1-t)^\sigma) \|y - x\|^\sigma, \quad (7)$$

holds for every $x, y \in I$, $\sigma > 0$, and $t \in [0, 1]$.

Remark 1. Note that if $\sigma = 0$ in (7), then the class of n -polynomial convex functions of higher order reduces to the class of n -polynomial convex functions. If $\sigma = 2$, then we have a new class of strongly n -polynomial convex functions. If $n = 1$, then we have the class of higher-order convex functions [37]. And along with $n = 1$, if we have $\sigma = 2$, then the class of n -polynomial convex functions of higher order reduces to the class of strongly convex functions [38]. From this, it is evident that the class of

n -polynomial convex functions of higher order is quite unifying one as it relates several other unrelated classes of convexity [39–45].

2. Main Results

In this section, we discuss our main results.

2.1. A Key Lemma. We now derive the main auxiliary result of the paper.

Lemma 1. Let $\alpha, \varrho > 0$, and let $\Lambda: [a^e, b^e] \longrightarrow \mathbb{R}$ be a differentiable function on (a^e, b^e) , with $0 \leq \alpha < b$ such that $\Lambda' \in L_1([a^e, b^e])$, then

$$\begin{aligned} & \frac{1}{(i+1)(i+2)} \left[\Lambda(a^e) + (i+1) \left(\Lambda\left(\frac{ia^e + b^e}{i+1}\right) + \Lambda\left(\frac{a^e + ib^e}{i+1}\right) \right) + \Lambda(b^e) \right] \\ & - \frac{(i+1)^{\alpha-1} \varrho^\alpha \Gamma(\alpha+1)}{(b^e - a^e)^\alpha} \left[{}^e I_{a^+}^\alpha \Lambda\left(\frac{ia^e + b^e}{i+1}\right) + {}^e I_{b^-}^\alpha \Lambda\left(\frac{a^e + ib^e}{i+1}\right) \right] \\ & = \frac{\varrho(b^e - a^e)}{i+1} \left[\int_0^1 \left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right) t^{\varrho-1} \Lambda' \left(\frac{i+t^e}{i+1} a^e + \frac{1-t^e}{i+1} b^e \right) dt - \int_0^1 \left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right) t^{\varrho-1} \Lambda' \left(\frac{1-t^e}{i+1} a^e + \frac{i+t^e}{i+1} b^e \right) dt \right]. \end{aligned} \quad (8)$$

Proof. Consider

$$\begin{aligned} I &= \frac{\varrho(b^e - a^e)}{i+1} \left[\int_0^1 \left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right) t^{\varrho-1} \Lambda' \left(\frac{i+t^e}{i+1} a^e + \frac{1-t^e}{i+1} b^e \right) dt - \int_0^1 \left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right) t^{\varrho-1} \Lambda' \left(\frac{1-t^e}{i+1} a^e + \frac{i+t^e}{i+1} b^e \right) dt \right] \\ I &= \frac{\varrho(b^e - a^e)}{i+1} [I_1 - I_2]. \end{aligned} \quad (9)$$

Integrating by parts I_1 , we have

$$\begin{aligned} I_1 &= \int_0^1 \left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right) t^{\varrho-1} \Lambda' \left(\frac{i+t^e}{i+1} a^e + \frac{1-t^e}{i+1} b^e \right) dt \\ &= -\frac{i+1}{\varrho(b^e - a^e)} \left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right) \Lambda \left(\frac{i+t^e}{i+1} a^e + \frac{1-t^e}{i+1} b^e \right) \Big|_0^1 - \frac{\alpha}{b^e - a^e} \int_0^1 t^{\alpha\varrho-1} \Lambda \left(\frac{i+t^e}{i+1} a^e + \frac{1-t^e}{i+1} b^e \right) dt, \\ &= \frac{\Lambda(a^e)}{\varrho(b^e - a^e)(i+2)} + \frac{i+1}{\varrho(i+2)(b^e - a^e)} \Lambda \left(\frac{ia^e + b^e}{i+1} \right) - \frac{(i+1)^\alpha \varrho^{\alpha-1} \Gamma(\alpha+1)^\varrho}{(b^e - a^e)^{\alpha+1}} {}^e I_{a^+}^\alpha \Lambda \left(\frac{ia^e + b^e}{i+1} \right). \end{aligned} \quad (10)$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^1 \left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right) t^{\varrho-1} \Lambda' \left(\frac{1-t^e}{i+1} a^e + \frac{i+t^e}{i+1} b^e \right) dt \\ &= -\frac{\Lambda(b^e)}{\varrho(b^e - a^e)(i+2)} - \frac{i+1}{\varrho(i+2)(b^e - a^e)} \Lambda \left(\frac{a^e + ib^e}{i+1} \right) + \frac{(i+1)^\alpha \varrho^{\alpha-1} \Gamma(\alpha+1)^\varrho}{(b^e - a^e)^{\alpha+1}} {}^e I_{b^-}^\alpha \Lambda \left(\frac{a^e + ib^e}{i+1} \right). \end{aligned} \quad (11)$$

By substituting the values of I_1 and I_2 in I , we get the required result.

This completes the proof. $\square$

Theorem 1. Let $\alpha, \varrho > 0$ and let $\Lambda: [a^e, b^e] \rightarrow \mathbb{R}$ be a differentiable function on (a^e, b^e) , with $0 \leq \alpha < b$ such that $\Lambda' \in L_1([a^e, b^e])$. If $|\Lambda'|$ is higher-order n -polynomial convex function of order $\sigma > 0$, then

2.2. Results and Discussions. We now derive the main results using Lemma 1.

$$\begin{aligned} & \left| \frac{1}{(i+1)(i+2)} \left[\Lambda(a^e) + \Lambda(b^e) + (i+1) \left(\Lambda\left(\frac{ia^e + b^e}{i+1}\right) + \Lambda\left(\frac{a^e + ib^e}{i+1}\right) \right) \right] - \frac{(i+1)^{\alpha-1} \varrho^\alpha \Gamma(\alpha+1)}{(b^e - a^e)^\alpha} \left[{}^e I_{a^+}^\alpha \Lambda\left(\frac{ia^e + b^e}{i+1}\right) \right. \right. \\ & \quad \left. \left. + {}^e I_{b^-}^\alpha \Lambda\left(\frac{a^e + ib^e}{i+1}\right) \right] \right| \\ & \leq \frac{(b^e - a^e)}{(i+1)(i+2)} \left[\left(\frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s (2s-i+1) + i^{s+1} - 1}{(i+1)^s (s+1)} (|\Lambda'(a^e)| + |\Lambda'(b^e)|) \right) - \frac{4\mu}{(\sigma+1)(\sigma+2)} \|b^e - a^e\|^\sigma \right]. \end{aligned} \quad (12)$$

Proof. By using Lemma 1 and because $|\Lambda'|$ is higher-order n -polynomial convex function of order $\sigma > 0$, we have

$$\begin{aligned} & \left| \frac{1}{(i+1)(i+2)} \left[\Lambda(a^e) + \Lambda(b^e) + (i+1) \left(\Lambda\left(\frac{ia^e + b^e}{i+1}\right) + \Lambda\left(\frac{a^e + ib^e}{i+1}\right) \right) \right] - \frac{(i+1)^{\alpha-1} \varrho^\alpha \Gamma(\alpha+1)}{(b^e - a^e)^\alpha} \left[{}^e I_{a^+}^\alpha \Lambda\left(\frac{ia^e + b^e}{i+1}\right) \right. \right. \\ & \quad \left. \left. + {}^e I_{b^-}^\alpha \Lambda\left(\frac{a^e + ib^e}{i+1}\right) \right] \right| \\ & \leq \frac{\varrho(b^e - a^e)}{i+1} \left[\int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda'\left(\frac{i+t^e}{i+1} a^e + \frac{1-t^e}{i+1} b^e\right) \right| dt + \int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda'\left(\frac{1-t^e}{i+1} a^e + \frac{i+t^e}{i+1} b^e\right) \right| dt \right] \\ & = \frac{\varrho(b^e - a^e)}{i+1} [I_1 + I_2]. \end{aligned} \quad (13)$$

Note that

$$\begin{aligned} I_1 &= \int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda'\left(\frac{i+t^e}{i+1} a^e + \frac{1-t^e}{i+1} b^e\right) \right| dt \\ &= \frac{1}{\varrho} \int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right| \left| \Lambda'\left(\frac{i+x}{i+1} a^e + \frac{1-x}{i+1} b^e\right) \right| dx \\ &\leq \frac{1}{\varrho} \left[\left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right| dx \right) \left(\int_0^1 \left| \Lambda'\left(\frac{i+x}{i+1} a^e + \frac{1-x}{i+1} b^e\right) \right| dx \right) \right] \\ &\leq \frac{1}{\varrho} \left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right| dx \right) \left(\int_0^1 \left(\frac{1}{n} \sum_{s=1}^n \left(1 - \left(1 - \left(\frac{i+x}{i+1} \right)^s \right) \right) |\Lambda'(a^e)| \right. \right. \\ & \quad \left. \left. + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{i+x}{i+1} \right)^s \right) |\Lambda'(b^e)| - \mu(x^\sigma(1-x) + x(1-x)^\sigma) \|b^e - a^e\|^\sigma dx \right) \right) \\ &= \frac{1}{\varrho(i+2)} \left[\left(\frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s (1+s) - 1}{(i+1)^s (1+s)} |\Lambda'(a^e)| + \frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s (s-i) + i^{s+1}}{(i+1)^s (1+s)} |\Lambda'(b^e)| - \frac{2\mu}{(\sigma+1)(\sigma+2)} \|b^e - a^e\|^\sigma \right)^{1/q} \right]. \end{aligned} \quad (14)$$

Similarly,

$$\begin{aligned}
 I_2 &= \int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda' \left(\frac{1-t^{\varrho}}{i+1} a^{\varrho} + \frac{i+t^{\varrho}}{i+1} b^{\varrho} \right) \right| dt \\
 &= \frac{1}{\varrho} \int_0^1 \left| \frac{1}{i+2} - \frac{x^{\alpha}}{i+1} \right| \left| \Lambda' \left(\frac{1-x}{i+1} a^{\varrho} + \frac{i+x}{i+1} b^{\varrho} \right) \right| dx \\
 &\leq \frac{1}{\varrho} \left[\left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^{\alpha}}{i+1} \right| dx \right) \left(\int_0^1 \left| \Lambda' \left(\frac{1-x}{i+1} a^{\varrho} + \frac{i+x}{i+1} b^{\varrho} \right) \right| dx \right) \right] \\
 &\leq \frac{1}{\varrho} \left[\left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^{\alpha}}{i+1} \right| dx \right) \left(\int_0^1 \left(\frac{1}{n} \sum_{s=1}^n \left(1 - \left(1 - \left(\frac{1-x}{i+1} \right)^s \right) \right) |\Lambda'(a^{\varrho})| \right. \right. \right. \\
 &\quad \left. \left. \left. + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{1-x}{i+1} \right)^s \right) |\Lambda'(b^{\varrho})| - \mu(x^{\sigma}(1-x) + x(1-x)^{\sigma}) \|b^{\varrho} - a^{\varrho}\|^{\sigma} dx \right) \right) \right] \\
 &= \frac{1}{\varrho(i+2)} \left[\left(\frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s(s-i) + i^{s+1}}{(i+1)^s(1+s)} |\Lambda'(a^{\varrho})| + \frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)} |\Lambda'(b^{\varrho})| - \frac{2\mu}{(\sigma+1)(\sigma+2)} \|b^{\varrho} - a^{\varrho}\|^{\sigma} \right) \right],
 \end{aligned} \tag{15}$$

where we have used the fact that $|1/i+2 - x^{\alpha}/i+1| \leq 1/i+2$ for all $x \in [0, 1]$. Using I_1 and I_2 in (13), we get the required result. $\square$

Theorem 2. Let $\alpha, \varrho > 0$, and let $\Lambda: [a^{\varrho}, b^{\varrho}] \rightarrow \mathbb{R}$ be a differentiable function on $(a^{\varrho}, b^{\varrho})$, with $0 \leq \alpha < b$ such that $\Lambda' \in L_1([a^{\varrho}, b^{\varrho}])$. If $|\Lambda'|^q$ is higher-order n -polynomial convex function of order $\sigma > 0$, where $r^{-1} + q^{-1} = 1$ and $q > 1$, then

$$\begin{aligned}
 &\left| \frac{1}{(i+1)(i+2)} \left[\Lambda(a^{\varrho}) + \Lambda(b^{\varrho}) + (i+1) \left(\Lambda\left(\frac{ia^{\varrho} + b^{\varrho}}{i+1}\right) + \Lambda\left(\frac{a^{\varrho} + ib^{\varrho}}{i+1}\right) \right) \right] \right. \\
 &\quad \left. - \frac{(i+1)^{\alpha-1} \varrho^{\alpha} \Gamma(\alpha+1)}{(b^{\varrho} - a^{\varrho})^{\alpha}} \left[{}^{\varrho}I_{a^+}^{\alpha} \Lambda\left(\frac{ia^{\varrho} + b^{\varrho}}{i+1}\right) + {}^{\varrho}I_{b^-}^{\alpha} \Lambda\left(\frac{a^{\varrho} + ib^{\varrho}}{i+1}\right) \right] \right| \\
 &\leq \frac{(b^{\varrho} - a^{\varrho})}{(i+1)(i+2)} \left[\left(\frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)} |\Lambda'(a^{\varrho})|^q + \frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s(s-i) + i^{s+1}}{(i+1)^s(1+s)} |\Lambda'(b^{\varrho})|^q - \frac{2\mu}{(\sigma+1)(\sigma+2)} \|b^{\varrho} - a^{\varrho}\|^{\sigma} \right)^{1/q} \right. \\
 &\quad \left. + \left(\frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s(s-i) + i^{s+1}}{(i+1)^s(1+s)} |\Lambda'(a^{\varrho})|^q + \frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)} |\Lambda'(b^{\varrho})|^q - \frac{2\mu}{(\sigma+1)(\sigma+2)} \|b^{\varrho} - a^{\varrho}\|^{\sigma} \right)^{1/q} \right].
 \end{aligned} \tag{16}$$

Proof. Using Lemma 1, Holder's inequality and because $|\Lambda'|^q$ is a higher-order n -polynomial convex function of order $\sigma > 0$, we have

$$\begin{aligned}
 &\left| \frac{1}{(i+1)(i+2)} \left[\Lambda(a^{\varrho}) + \Lambda(b^{\varrho}) + (i+1) \left(\Lambda\left(\frac{ia^{\varrho} + b^{\varrho}}{i+1}\right) + \Lambda\left(\frac{a^{\varrho} + ib^{\varrho}}{i+1}\right) \right) \right] \right. \\
 &\quad \left. - \frac{(i+1)^{\alpha-1} \varrho^{\alpha} \Gamma(\alpha+1)}{(b^{\varrho} - a^{\varrho})^{\alpha}} \left[{}^{\varrho}I_{a^+}^{\alpha} \Lambda\left(\frac{ia^{\varrho} + b^{\varrho}}{i+1}\right) + {}^{\varrho}I_{b^-}^{\alpha} \Lambda\left(\frac{a^{\varrho} + ib^{\varrho}}{i+1}\right) \right] \right| \\
 &\leq \frac{\varrho(b^{\varrho} - a^{\varrho})}{i+1} \left[\int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda' \left(\frac{i+t^{\varrho}}{i+1} a^{\varrho} + \frac{1-t^{\varrho}}{i+1} b^{\varrho} \right) \right| dt + \int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda' \left(\frac{1-t^{\varrho}}{i+1} a^{\varrho} + \frac{i+t^{\varrho}}{i+1} b^{\varrho} \right) \right| dt \right] \\
 &= \frac{\varrho(b^{\varrho} - a^{\varrho})}{i+1} [I_3 + I_4].
 \end{aligned} \tag{17}$$

Note that

$$\begin{aligned}
I_3 &= \int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda' \left(\frac{i+t^\varrho}{i+1} a^\varrho + \frac{1-t^\varrho}{i+1} b^\varrho \right) \right| dt \\
&= \frac{1}{\varrho} \int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right| \left| \Lambda' \left(\frac{i+x}{i+1} a^\varrho + \frac{1-x}{i+1} b^\varrho \right) \right| dx \\
&\leq \frac{1}{\varrho} \left[\left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right|^r dx \right)^{1/r} \left(\int_0^1 \left| \Lambda' \left(\frac{i+x}{i+1} a^\varrho + \frac{1-x}{i+1} b^\varrho \right) \right|^q dx \right)^{1/q} \right] \\
&\leq \frac{1}{\varrho} \left[\left[\left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right|^r dx \right)^{1/r} \left(\left(\int_0^1 \left(\frac{1}{n} \sum_{s=1}^n \left(1 - \left(1 - \left(\frac{i+x}{i+1} \right)^s \right) \right) |\Lambda'(a^\varrho)|^q + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{i+x}{i+1} \right)^s \right) |\Lambda'(b^\varrho)|^q \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \mu(x^\sigma(1-x) + x(1-x)^\sigma) \|b^\varrho - a^\varrho\|^\sigma dx \right)^{1/q} \right) \right] \right] \\
&= \frac{1}{\varrho(i+2)} \left[\left(\frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)} |\Lambda'(a^\varrho)|^q + \frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s(s-i)+i^{s+1}}{(i+1)^s(1+s)} |\Lambda'(b^\varrho)|^q - \frac{2\mu}{(\sigma+1)(\sigma+2)} \|b^\varrho - a^\varrho\|^\sigma \right)^{1/q} \right]. \tag{18}
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_4 &= \int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda' \left(\frac{1-t^\varrho}{i+1} a^\varrho + \frac{i+t^\varrho}{i+1} b^\varrho \right) \right| dt \\
&= \frac{1}{\varrho} \int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right| \left| \Lambda' \left(\frac{1-x}{i+1} a^\varrho + \frac{i+x}{i+1} b^\varrho \right) \right| dx \\
&\leq \frac{1}{\varrho} \left[\left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right|^r dx \right)^{1/r} \left(\int_0^1 \left| \Lambda' \left(\frac{1-x}{i+1} a^\varrho + \frac{i+x}{i+1} b^\varrho \right) \right|^q dx \right)^{1/q} \right] \\
&\leq \frac{1}{\varrho} \left[\left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right|^r dx \right)^{1/r} \left(\int_0^1 \left(\frac{1}{n} \sum_{s=1}^n \left(1 - \left(1 - \left(\frac{1-x}{i+1} \right)^s \right) \right) |\Lambda'(a^\varrho)|^q + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{1-x}{i+1} \right)^s \right) |\Lambda'(b^\varrho)|^q - \mu(x^\sigma(1-x) + x(1-x)^\sigma) \|b^\varrho - a^\varrho\|^\sigma dx \right)^{1/q} \right] \\
&= \frac{1}{\varrho(i+2)} \left[\left(\frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s(s-i)+i^{s+1}}{(i+1)^s(1+s)} |\Lambda'(a^\varrho)|^q + \frac{1}{n} \sum_{s=1}^n \frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)} |\Lambda'(b^\varrho)|^q - \frac{2\mu}{(\sigma+1)(\sigma+2)} \|b^\varrho - a^\varrho\|^\sigma \right)^{1/q} \right], \tag{19}
\end{aligned}$$

where $|1/i+2 - x^\alpha/i+1| \leq 1/i+2$ for all $x \in [0, 1]$. Using I_3 and I_4 in (17), we get the required result. $\square$

Theorem 3. Let $\alpha, \varrho > 0$, and let $\Lambda: [a^\varrho, b^\varrho] \rightarrow \mathbb{R}$ be a differentiable function on (a^ϱ, b^ϱ) , with $0 \leq \alpha < b$ such that $\Lambda' \in L_1([a^\varrho, b^\varrho])$. If $|\Lambda'|^q$ is a higher-order n -polynomial convex function of order $\sigma > 0$ where $q > 1$, then

$$\begin{aligned}
&\left| \frac{1}{(i+1)(i+2)} \left[\Lambda(a^\varrho) + \Lambda(b^\varrho) + (i+1) \left(\Lambda \left(\frac{ia^\varrho + b^\varrho}{i+1} \right) + \Lambda \left(\frac{a^\varrho + ib^\varrho}{i+1} \right) \right) \right] \right. \\
&\quad \left. - \frac{(i+1)^{\alpha-1} \varrho^\alpha \Gamma(\alpha+1)}{(b^\varrho - a^\varrho)^\alpha} \left[{}^\varrho I_{a^\varrho}^\alpha \Lambda \left(\frac{ia^\varrho + b^\varrho}{i+1} \right) + {}^\varrho I_{b^\varrho}^\alpha \Lambda \left(\frac{a^\varrho + ib^\varrho}{i+1} \right) \right] \right| \\
&\leq \frac{(b^\varrho - a^\varrho)}{(i+1)(i+2)^{1/r}} \left[\left(\frac{1}{i(i+2)} \sum_{s=1}^n \frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)} |\Lambda'(a^\varrho)|^q + \frac{1}{n(i+2)} \sum_{s=1}^n \frac{(i+1)^s(s-i)+i^{s+1}}{(i+1)^s(1+s)} |\Lambda'(b^\varrho)|^q - \frac{2\mu}{(i+2)(\sigma+1)(\sigma+2)} \|b^\varrho - a^\varrho\|^\sigma \right)^{1/q} \right. \\
&\quad \left. + \left(\frac{1}{n(i+2)} \sum_{s=1}^n \frac{(i+1)^s(s-i)+i^{s+1}}{(i+1)^s(1+s)} |\Lambda'(a^\varrho)|^q + \frac{1}{n(i+2)} \sum_{s=1}^n \frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)} |\Lambda'(b^\varrho)|^q - \frac{2\mu}{(i+2)(\sigma+1)(\sigma+2)} \|b^\varrho - a^\varrho\|^\sigma \right)^{1/q} \right]. \tag{20}
\end{aligned}$$

Proof. Using Lemma 1, Power mean inequality, and because $|\Lambda'|^q$ is a higher-order n -polynomial convex function of order $\sigma > 0$, we have

$$\begin{aligned}
 & \left| \frac{1}{(i+1)(i+2)} \left[\Lambda(a^e) + \Lambda(b^e) + (i+1) \left(\Lambda\left(\frac{ia^e + b^e}{i+1}\right) + \Lambda\left(\frac{a^e + ib^e}{i+1}\right) \right) \right] \right. \\
 & \quad \left. - \frac{(i+1)^{\alpha-1} \varrho^\alpha \Gamma(\alpha+1)}{(b^e - a^e)^\alpha} \left[{}^e I_{a^+}^\alpha \Lambda\left(\frac{ia^e + b^e}{i+1}\right) + {}^e I_{b^-}^\alpha \Lambda\left(\frac{a^e + ib^e}{i+1}\right) \right] \right| \\
 & \leq \frac{\varrho(b^e - a^e)}{i+1} \left[\int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda'\left(\frac{i+t^e}{i+1} a^e + \frac{1-t^e}{i+1} b^e\right) \right| dt + \int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda'\left(\frac{1-t^e}{i+1} a^e + \frac{i+t^e}{i+1} b^e\right) \right| dt \right] \\
 & = \frac{\varrho(b^e - a^e)}{i+1} [I_5 + I_6].
 \end{aligned} \tag{21}$$

Note that

$$\begin{aligned}
 I_5 &= \int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda'\left(\frac{i+t^e}{i+1} a^e + \frac{1-t^e}{i+1} b^e\right) \right| dt \\
 &= \frac{1}{\varrho} \int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right| \left| \Lambda'\left(\frac{i+x}{i+1} a^e + \frac{1-x}{i+1} b^e\right) \right| dx \\
 &\leq \frac{1}{\varrho} \left[\left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right| dx \right)^{1/r} \left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right| \left| \Lambda'\left(\frac{i+x}{i+1} a^e + \frac{1-x}{i+1} b^e\right) \right|^q dx \right)^{1/q} \right] \\
 &\leq \frac{1}{\varrho} \left[\left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right| dx \right)^{1/r} \left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^\alpha}{i+1} \right| \left(\frac{1}{n} \sum_{s=1}^n \left(1 - \left(1 - \left(\frac{i+x}{i+1} \right)^s \right) \right) |\Lambda'(a^e)|^q + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{i+x}{i+1} \right)^s \right) |\Lambda'(b^e)|^q \right. \right. \right. \\
 &\quad \left. \left. \left. - \mu(x^\sigma(1-x) + x(1-x)^\sigma) \|b^e - a^e\|^\sigma dx \right)^{1/q} \right] \right] \\
 &= \frac{1}{\varrho(i+2)^{1/r}} \left[\left(\frac{1}{n(i+2)} \sum_{s=1}^n \frac{(i+1)^s(1+s) - 1}{(i+1)^s(1+s)} |\Lambda'(a^e)|^q + \frac{1}{n(i+2)} \sum_{s=1}^n \frac{(i+1)^s(s-i) + i^{s+1}}{(i+1)^s(1+s)} |\Lambda'(b^e)|^q \right. \right. \\
 &\quad \left. \left. - \frac{2\mu}{(i+2)(\sigma+1)(\sigma+2)} \|b^e - a^e\|^\sigma \right)^{1/q} \right].
 \end{aligned} \tag{22}$$

Similarly,

$$\begin{aligned}
 I_6 &= \int_0^1 \left| \frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1} \right| t^{\varrho-1} \left| \Lambda' \left(\frac{1-t^{\varrho}}{i+1} a^{\varrho} + \frac{i+t^{\varrho}}{i+1} b^{\varrho} \right) \right| dt \\
 &= \frac{1}{\varrho} \int_0^1 \left| \frac{1}{i+2} - \frac{x^{\alpha}}{i+1} \right| \left| \Lambda' \left(\frac{1-x}{i+1} a^{\varrho} + \frac{i+x}{i+1} b^{\varrho} \right) \right| dx \\
 &\leq \frac{1}{\varrho} \left[\left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^{\alpha}}{i+1} \right| dx \right)^{1/r} \left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^{\alpha}}{i+1} \right| \left| \Lambda' \left(\frac{1-x}{i+1} a^{\varrho} + \frac{i+x}{i+1} b^{\varrho} \right) \right|^q dx \right)^{1/q} \right] \\
 &\leq \frac{1}{\varrho} \left[\left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^{\alpha}}{i+1} \right| dx \right)^{1/r} \left(\int_0^1 \left| \frac{1}{i+2} - \frac{x^{\alpha}}{i+1} \right| \left(\frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{1-x}{i+1} \right)^s \right) \right) |\Lambda'(a^{\varrho})|^q + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{1-x}{i+1} \right)^s \right) |\Lambda'(b^{\varrho})|^q \right. \right. \\
 &\quad \left. \left. - \mu (x^{\sigma}(1-x) + x(1-x)^{\sigma}) \|b^{\varrho} - a^{\varrho}\|^{\sigma} dx \right) \right)^{1/q} \right] \\
 &= \frac{1}{\varrho(i+2)^{1/r}} \left[\left(\frac{1}{i(i+2)} \sum_{s=1}^n \frac{(i+1)^s (s-i) + i^{s+1}}{(i+1)^s (1+s)} |\Lambda'(a^{\varrho})|^q + \frac{1}{n(i+2)} \sum_{s=1}^n \frac{(i+1)^s (1+s) - 1}{(i+1)^s (1+s)} |\Lambda'(b^{\varrho})|^q \right. \right. \\
 &\quad \left. \left. - \frac{2\mu}{(i+2)(\sigma+1)(\sigma+2)} \|b^{\varrho} - a^{\varrho}\|^{\sigma} \right) \right]^{1/q},
 \end{aligned} \tag{23}$$

where we have used the fact that $|1/i+2 - x^{\alpha}/i+1| \leq 1/i+2$ for all $x \in [0, 1]$. Using I_5 and I_6 in (21), we get the required result. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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