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Research Article

On the (Consecutively) Super Edge-Magic Deficiency of Subdivision of Double Stars

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Let G be a finite, simple, and undirected graph with vertex set $V(G)$ and edge set $E(G)$. A super edge-magic labeling of G is a bijection $f: V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$ and $f(u) + f(uv) + f(v)$ is a constant for every edge $uv \in E(G)$. The super edge-magic labeling f of G is called consecutively super edge-magic if G is a bipartite graph with partite sets A and B such that $f(A) = \{1, 2, \ldots, |A|\}$ and $f(B) = \{|A| + 1, |A| + 2, \ldots, |V(G)|\}$. A graph that admits (consecutively) super edge-magic labeling is called a (consecutively) super edge-magic graph. The super edge-magic deficiency of G, denoted by $\mu_s(G)$, is either the minimum nonnegative integer n such that $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such n. The consecutively super edge-magic deficiency of a graph G is defined by a similar way. In this paper, we investigate the (consecutively) super edge-magic deficiency of subdivision of double stars. We show that, some of them have zero (consecutively) super edge-magic deficiency.

1. Introduction

Besides having an interesting theoretical study, graph labelings can be applied in various fields of computer science such as coding theory, cryptography, circuit design, database management system, design of algorithms, and communication networks [1–5].

All graphs considered here are finite, simple, and undirected graphs. We denote the vertex and edge sets of a graph G by $V(G)$ and $E(G)$, respectively, where $p = |V(G)|$ and $q = |E(G)|$. For most terminology and notation in graph theory used in this paper, we follow Chartrand et al. [6].

Kotzig and Rosa [7] in 1970 introduced the concept of an edge-magic graph. A graph G is called *edge-magic* if there is a bijection $f: V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, p + q\}$ such that $f(x) + f(xy) + f(y) = k$ is a constant for every edge $xy \in E(G)$. In such a case, f is called an *edge-magic labeling* of G and k is called the *magic constant* of f. Meanwhile, Enomoto et al. [8] in 1998 introduced the terminology of a super edge-magic graph. An edge-magic graph G with an edge-magic labeling f is called *super edge-magic (SEM)* if $f(V(G)) = \{1, 2, \ldots, p\}$. In this case, f is called a *SEM labeling* of G. The following lemma, proved by Figueroa-Centeno et al. [9], provides sufficient and necessary conditions for a SEM graph.

Lemma 1 (see [9]). *A graph G is SEM if and only if there exists a bijection $f: V(G) \longrightarrow \{1, 2, \ldots, p\}$ such that the set of all edge-sums $S = \{f(x) + f(y): xy \in E(G)\}$ consists of q consecutive integers. In this case, f extends to be a SEM labeling of G with magic constant $k = p + q + \min(S)$.*

Enomoto et al. [8] provided a sufficient condition for nonexistence of a SEM labeling of a graph.

Lemma 2 (see [8]). *If G is a SEM graph, then $q \leq 2p - 3$.*

They also proposed the following conjecture.

Conjecture 1 (see [8]). *Every tree is SEM.*

A SEM labeling can be considered as a particular case of an (a, d)-super edge antimagic labeling for $d = 0$. An (a, d)-super edge antimagic labeling of a graph G is a bijection $f: V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, p + q\}$ such that $\{f(x) + f(xy) + f(y): xy \in E(G)\} = \{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$, and $f(V(G)) = \{1, 2, \ldots, p\}$. The recent results of this labeling can be seen in [10, 11]. In [11], Liu et al. computed the bounds of a of super edge-antimagic labelings of subdivided caterpillars. They also presented a partial support of Conjecture 1 by proving that some classes of subdivided caterpillars are SEM.

As same as SEM graph, Muntaner-Batle [12] in 2001 introduced the concept of a special SEM bipartite graph. In 2007, Oshima [13] called such a graph as a *consecutively SEM graph*. A bipartite graph G with partite sets A and B is called *consecutively SEM* if there exists a SEM labeling f of G with the property that $f(A) = \{1, 2, \ldots, |A|\}$ and $f(B) = \{|A| + 1, |A| + 2, \ldots, p\}$.

Kotzig and Rosa [7] proved that, for every graph G there exists a nonnegative integer n such that $G \cup nK_1$ is an edge-magic graph. This fact encourages the concept of edge-magic deficiency of a graph. The *edge-magic deficiency* of a graph G, $\mu(G)$, is defined as the minimum nonnegative integer n such that $G \cup nK_1$ is an edge-magic graph. This concept motivated Figueroa-Centeno et al. [14] to introduce the concept of super edge-magic deficiency of a graph. The *super edge-magic deficiency (SEMD)* of a graph G, $\mu_s(G)$, is defined as either the minimum nonnegative n such that $G \cup nK_1$ is a SEM graph or $+\infty$ if there exists no such n. Moreover, Ichishima et al. [15] defined a similar notion for consecutively SEM labeling. The *consecutively SEMD* of a graph G, $\mu_c(G)$, is defined to be either the smallest nonnegative integer n with the property that $G \cup nK_1$ is consecutively SEM or $+\infty$ if there exists no such n. Hence, a (consecutively) SEM graph is a graph has zero (consecutively) SEMD. As an immediate consequence of these definitions, $\mu(G) \leq \mu_s(G) \leq \mu_c(G)$ holds for every graph G.

Motivated by Conjecture 1, many researchers have investigated the SEM labeling of some families of trees. The SEM labeling of subdivision of stars $K_{1,3}$ was studied by Ngurah et al. [16]. Hussain et al. [17] studied the SEM labeling of banana trees. Ahmad et al. [18] investigated the existence of SEM labeling of subdivision of banana trees. Javaid et al. [19] found SEM labeling of subdivision of stars $K_{1,4}$ and w-trees. In [20], Ali et al. investigated the SEM labeling on w-trees. Next, Ali et al. [21] studied the SEM labeling of subdivision of stars $K_{1,n}$ for $n \geq 5$. However, Conjecture 1 is still open. Meanwhile, in [15], Ichishima et al. presented some results on consecutively SEMD of forets with two components, where its components are (non) isomorphic stars and union of paths and stars. In this paper, we study the (consecutively) SEMD of subdivision of double stars. We find the upper bound of (consecutively) SEMD of particular subdivision of double stars. We also prove that subdivision of double stars with a large order has zero (consecutively) SEMD.

2. The Results

A double star $\mathrm{DS}(m, n)$ is a tree obtained from two disjoint stars $K_{1,m}$ and $K_{1,n}$, by joining the center vertices through an edge. For $r_i, t_j \geq 1$, $1 \leq i \leq m$, $1 \leq j \leq n$, and $s \geq 0$, a *subdivision of a double star* $\mathrm{DT}(r_1, r_2, \ldots, r_m; s; t_1, t_2, \ldots, t_n)$ is a graph obtained from a double star $\mathrm{DS}(m, n)$ by inserting $r_i - 1$ vertices to each edge of $K_{1,m}$, s vertices to the edge which link the centre vertex of two stars, and $t_j - 1$ vertices to each edge of $K_{1,n}$. Thus, $\mathrm{DT}(r_1, r_2, \ldots, r_m; s; t_1, t_2, \ldots, t_n) \cong G_{(m;s;n)}$ is a graph of order $2 + s + \sum_{i=1}^{m} r_i + \sum_{j=1}^{n} t_j$.

Let vertex and edge sets of $G_{(m;s;n)}$ be defined as follows:

$$
\begin{aligned}
V\left(G_{(m;s;n)}\right) &= \left\{x, x_{i,a}: 1 \leq i \leq m,\ 1 \leq a \leq r_i\right\} \cup \left\{y_l: 1 \leq l \leq s\right\} \cup \\
&\quad \cdot \left\{z, z_{j,b}: 1 \leq j \leq n,\ 1 \leq b \leq t_j\right\}, \\
E\left(G_{(m;s;n)}\right) &= \left\{xx_{i,1}, x_{i,a}x_{i,a+1}: 1 \leq i \leq m,\ 1 \leq a \leq r_i - 1\right\} \cup \left\{xy_1\right\} \cup \\
&\quad \cdot \left\{y_l y_{l+1}: 1 \leq l \leq s - 1\right\} \cup \left\{y_s z\right\} \cup \left\{zz_{j,1}, z_{j,b}z_{j,b+1}: 1 \leq j \leq n,\ 1 \leq b \leq t_j - 1\right\}.
\end{aligned}
\tag{1}
$$

In this paper, we assume that $s \geq 0$ is odd. We denote the partite sets of $G_{(m;s;n)}$ as follows:

$$
\begin{aligned}
A &= \begin{cases}
\left\{x_{i,a}: 1 \leq i \leq m, a \equiv 1 \,(\mathrm{mod}\, 2)\right\} \cup \\
\left\{z, z_{j,b}: 1 \leq j \leq n, b \equiv 0 \,(\mathrm{mod}\, 2)\right\}, & \text{for } s = 0, \\
\left\{x_{i,a}: 1 \leq i \leq m, a \equiv 1 \,(\mathrm{mod}\, 2)\right\} \cup \\
\left\{y_l: l \equiv 1 \,(\mathrm{mod}\, 2)\right\} \cup \left\{z_{j,b}: 1 \leq j \leq n, b \equiv 1 \,(\mathrm{mod}\, 2)\right\}, & \text{for odd } s \geq 1,
\end{cases} \\
B &= \begin{cases}
\left\{x,\ x_{i,a}: 1 \leq i \leq m, a \equiv 0 \,(\mathrm{mod}\, 2)\right\} \cup \\
\left\{z_{j,b}: 1 \leq j \leq n, b \equiv 1 \,(\mathrm{mod}\, 2)\right\}, & \text{for } s = 0, \\
\left\{x,\ x_{i,a}: 1 \leq i \leq m, a \equiv 0 \,(\mathrm{mod}\, 2)\right\} \cup \\
\left\{y_l: l \equiv 0 \,((\mathrm{mod}\, 2))\right\} \cup \left\{z,\ z_{j,b}: 1 \leq j \leq n, b \equiv 0 \,(\mathrm{mod}\, 2)\right\}, & \text{for odd } s \geq 1.
\end{cases}
\end{aligned}
\tag{2}
$$

First, we investigate the upper bound of (consecutively) SEMD of $G_{(m;0;3)}$.

Theorem 1. *Let $\gamma \geq 2$ be an even integer. If $r_1 = r_2 = t_1 = \gamma$, $t_2 = t_3 = \gamma - 1$, and $r_i = 2^{i-3}(\gamma + 1)$, $3 \leq i \leq m$, then $\mu_s(G_{(m;0;3)}) = \mu_c(G_{(m;0;3)}) \leq (2^{m-3} - 1)(\gamma + 1) + 1$ for any $m \geq 3$.*

Proof. Let $G \cong G_{(m;0;3)} \cup [(2^{m-3} - 1)(\gamma + 1) + 1]K_1$ be a graph with the vertex set $V(G) = V(G_{(m;0;3)}) \cup \{w_h\colon 1 \leq h \leq (2^{m-3} - 1)(\gamma + 1) + 1\}$ and edge set $E(G) = E(G_{(m;0;3)})$.

Define a labeling $f\colon V(G) \longrightarrow \{1, 2, \ldots, p\}$, where $p = (2^{m-3} \cdot 3 + 3)\gamma + 2^{m-3} \cdot 3 - 1$, as follows:

$$f(x) = \frac{1}{2}\left[\left(2^{m-2} + 5\right)\gamma + 2^{m-2}\right],$$
$$f(z) = \frac{1}{2}\left[\left(2^{m-1} + 7\right)\gamma + 2^{m-1} - 2\right], \tag{3}$$

for $1 \leq h \leq (2^{m-3} - 1)(\gamma + 1) + 1$,

$$f(w_h) = \left(2^{m-2} + 4\right)\gamma + h + 2^{m-2} - 2,$$

$$f(u) = \begin{cases} \frac{1}{2}(\gamma - a + 1), & \text{if } u = x_{1,a} \text{ and } a \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-2} + 5\right)\gamma - a + 2^{m-2}\right], & \text{if } u = x_{1,a} \text{ and } a \neq \gamma \text{ is even},\\ \left(2^{m-3} \cdot 3 + 3\right)\gamma + 2^{m-3} \cdot 3 - 1, & \text{if } u = x_{1,\gamma},\\ \frac{1}{2}(\gamma + a + 1) & \text{if } u = x_{2,a} \text{ and } a \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-2} + 5\right)\gamma + a + 2^{m-2}\right], & \text{if } u = x_{2,a} \text{ and } a \text{ is even},\\ \frac{1}{2}\left[\left(2^{i-2} + 1\right)(\gamma + 1) - a\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-2} + 2^{i-2} + 5\right)\gamma - a + 2^{m-2} + 2^{i-2}\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is even},\\ \frac{1}{2}\left[\left(2^{m-2} + 1\right)(\gamma + 1) + b\right], & \text{if } u = z_{1,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-1} + 5\right)\gamma + b + 2^{m-1} - 2\right], & \text{if } u = z_{1,b} \text{ and } b \text{ is even},\\ \frac{1}{2}\left[\left(2^{m-2} + 3\right)\gamma - b + 2^{m-2} + 1\right], & \text{if } u = z_{2,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-1} + 7\right)\gamma - b + 2^{m-1} - 2\right], & \text{if } u = z_{2,b} \text{ and } b \text{ is even},\\ \frac{1}{2}\left[\left(2^{m-2} + 3\right)\gamma + b + 2^{m-2} + 1\right], & \text{if } u = z_{3,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-1} + 7\right)\gamma + b + 2^{m-1} - 2\right], & \text{if } u = z_{3,b} \text{ and } b \text{ is even}. \end{cases} \tag{4}$$

It can be checked that the set of all edge-sums is $S = \{c, c + 1, \ldots, c + q - 1\}$, where $c = (2^{m-3} + 2)(\gamma + 1)$ and $q = (2^{m-2} + 4)\gamma + 2^{m-2} - 2$. Hence, by Lemma 1, f extends to a SEM labeling of G with magic constant $k = (2^{m-2} \cdot 3 + 9)\gamma + 2^{m-2} \cdot 3 - 1$. Therefore, for any $m \geq 3$, $\mu_s(G_{(m;0;3)}) \leq (2^{m-3} - 1)(\gamma + 1) + 1$.

Furthermore, let A' and B' be partite sets of G, where $A' = A$ and $B' = B \cup \{w_h\colon 1 \leq h \leq (2^{m-3} - 1)(\gamma + 1) + 1\}$. Since $f(A') = \{1, 2, \ldots, d\}$ and $f(B') = \{d + 1, d + 2, \ldots, p\}$ where $d = (2^{m-3} + 2)\gamma + 2^{m-3}$, G is a consecutively SEM graph. Hence, $\mu_c(G_{(m;0;3)}) \leq (2^{m-3} - 1)(\gamma + 1) + 1$, for any $m \geq 3$.

As an illustration of the proof of Theorem 1, see Figure 1. □

Theorem 2. *Let $\gamma \geq 3$ be an odd integer. If $r_1 = r_2 = t_1 = t_2 = t_3 = \gamma$ and $r_i = 2^{i-3}\gamma$, $3 \leq i \leq m$, then $\mu_s(G_{(m;0;3)}) = \mu_c(G_{(m;0;3)}) \leq (2^{m-3} - 1)\gamma + 2$ for any $m \geq 3$.*

Proof. Let $H \cong G_{(m;0;3)} \cup [(2^{m-3} - 1)\gamma + 2]K_1$ be a graph with the vertex set $V(H) = V(G_{(m;0;3)}) \cup \{w_h\colon 1 \leq h \leq (2^{m-3} - 1)\gamma + 2\}$ and edge set $E(H) = E(G_{(m;0;3)})$.

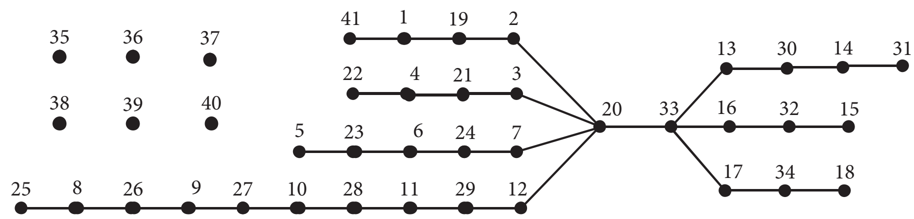

Figure 1: The (consecutively) SEM labeling of $DT(4,4,5,10;0;4,3,3)\cup 6K_1$.

Define a labeling $f\colon V(H)\longrightarrow\{1,2,\ldots,p\}$, where $p=(2^{m-3}\cdot 3+3)\gamma+4$, as follows:

The label of isolated vertices is

$$f(w_1)=\left(2^{m-3}+2\right)\gamma+4,$$
$$f(w_2)=\frac{1}{2}\left[\left(2^{m-1}+5\right)\gamma+7\right],$$
$$f(w_h)=\left(2^{m-2}+4\right)\gamma+h+1,\quad \text{for } 3\le h\le\left(2^{m-3}-1\right)\gamma+2, \tag{5}$$

and the label of the remaining vertices is as follows:

$$f(x)=\frac{1}{2}\left[\left(2^{m-2}+5\right)\gamma+7\right],$$
$$f(z)=\frac{1}{2}\left[\left(2^{m-1}+7\right)\gamma+7\right],$$

$$f(u)=\begin{cases}\frac{1}{2}(\gamma-a+2), & \text{if } u=x_{1,a} \text{ and } a \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-2}+5\right)\gamma-a+7\right], & \text{if } u=x_{1,a} \text{ and } a\neq\gamma-1 \text{ is even},\\ \left(2^{m-3}\cdot 3+3\right)\gamma+4, & \text{if } u=x_{1,\gamma-1},\\ \frac{1}{2}(\gamma+a+2), & \text{if } u=x_{2,a} \text{ and } a \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-2}+5\right)\gamma+a+7\right], & \text{if } u=x_{2,a} \text{ and } a \text{ is even},\\ \frac{1}{2}\left[\left(2^{i-2}+1\right)\gamma-a+4\right], & \text{if } u=x_{i,a},\ 3\le i\le m \text{ and } a \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-2}+2^{i-2}+5\right)\gamma-a+7\right], & \text{if } u=x_{i,a},\ 3\le i\le m \text{ and } a \text{ is even},\\ \frac{1}{2}\left[\left(2^{m-2}+1\right)\gamma+b+4\right], & \text{if } u=z_{1,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-1}+5\right)\gamma+b+7\right], & \text{if } u=z_{1,b} \text{ and } b \text{ is even},\\ \frac{1}{2}\left[\left(2^{m-2}+3\right)\gamma-b+6\right], & \text{if } u=z_{2,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-1}+7\right)\gamma-b+7\right], & \text{if } u=z_{2,b} \text{ and } b \text{ is even},\\ \frac{1}{2}\left[\left(2^{m-2}+3\right)\gamma+b+6\right], & \text{if } u=z_{3,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{m-1}+7\right)\gamma+b+7\right], & \text{if } u=z_{3,b} \text{ and } b \text{ is even}.\end{cases} \tag{6}$$

It can be verified that the set of all edge-sums is $S = \{c, c+1, \ldots, c+q-1\}$, where $c = (2^{m-3}+2)\gamma + 7$ and $q = (2^{m-2}+4)\gamma + 1$. Hence, by Lemma 1, f extends to a SEM labeling of H with magic constant $k = (2^{m-2}\cdot 3 + 9)\gamma + 12$. Therefore, for any $m \geq 3$, $\mu_s(G_{(m;0;3)}) \leq (2^{m-3}-1)\gamma + 2$.

Moreover, let A' and B' be partite sets of H, where $A' = A$ and $B' = B \cup \{w_h\colon 1 \leq h \leq (2^{m-3}-1)\gamma + 2\}$. Since $f(A') = \{1, 2, \ldots, d\}$ and $f(B') = \{d+1, d+2, \ldots, p\}$, where $d = (2^{m-3}+2)\gamma + 3$, H is a consecutively SEM graph. Hence, $\mu_c(G_{(m;0;3)}) \leq (2^{m-3}-1)\gamma + 2$ for any $m \geq 3$. □

Based on Theorems 1 and 2, we propose the following open problem.

Open Problem 1. Find an upper bound of (consecutively) SEMD of $G_{(m;0;3)}$ for the remaining cases. Furthermore, find a better upper bound of (consecutively) SEMD of $G_{(m;0;3)}$ for the same cases as in Theorems 1 and 2.

Next, we show that the graphs $G_{(2;s;n)}$ have zero (consecutively) SEMD.

Theorem 3. *Let $\gamma \geq 1$ be an odd integer. If $r_1 = r_2 = s = t_1 = t_2 = \gamma$ and $t_j = 2^{j-3}(\gamma+1)$, $3 \leq j \leq n$, then $\mu_s(G_{(2;s;n)}) = \mu_c(G_{(2;s;n)}) = 0$ for every $n \geq 1$ and odd $s \geq 1$.*

Proof. Based on the sufficient conditions, $G_{(2;s;n)}$ has order

$$p = \left\{ \begin{array}{ll} 4\gamma + 2, & \text{for } n = 1, \\ \left(2^{n-2}+4\right)\gamma + 2^{n-2} + 1, & \text{for } n \geq 2. \end{array} \right\} \quad (7)$$

We consider the proof into two cases depending on the value of n.

(i) Case 1: $n = 1$.

Define a labeling $f\colon V(G_{(2;s;1)}) \longrightarrow \{1, 2, \ldots, p\}$ as follows:

$$f(x) = \frac{1}{2}(5\gamma + 5),$$

$$f(z) = 4\gamma + 2,$$

$$f(u) = \begin{cases} \frac{1}{2}(\gamma - a + 2), & \text{if } u = x_{1,a} \text{ and } a \text{ is odd}, \\ \frac{1}{2}(5\gamma - a + 5), & \text{if } u = x_{1,a} \text{ and } a \text{ is even}, \\ \frac{1}{2}(\gamma + a + 2), & \text{if } u = x_{2,a} \text{ and } a \text{ is odd}, \\ \frac{1}{2}(5\gamma + a + 5), & \text{if } u = x_{2,a} \text{ and } a \text{ is even}, \\ \frac{1}{2}(3\gamma - l + 4), & \text{if } u = y_l \text{ and } l \text{ is odd}, \\ \frac{1}{2}(7\gamma - l + 5), & \text{if } u = y_l \text{ and } l \text{ is even}, \\ \frac{1}{2}(4\gamma - b + 5), & \text{if } u = z_{1,b} \text{ and } b \text{ is odd}, \\ \frac{1}{2}(8\gamma - b + 4), & \text{if } u = z_{1,b} \text{ and } b \text{ is even}. \end{cases} \quad (8)$$

It can be examined that the set of all edge sums is $S = \{2\gamma+4, 2\gamma+5, \ldots, 6\gamma+4\}$. Hence, by Lemma 1, f extends to a SEM labeling of $G_{(2;s;1)}$ with magic constant $k = 10\gamma + 7$. Therefore, for every odd $s \geq 1$, $\mu_s(G_{(2;s;1)}) = 0$.

(ii) Case 2: $n \geq 2$.

Define a labeling $f\colon V(G_{(2;s;1)}) \longrightarrow \{1,2,\ldots,p\}$ as follows:

$$f(x)=\frac{1}{2}\left[\left(2^{n-2}+5\right)(\gamma+1)\right],$$
$$f(z)=\left(2^{n-3}+4\right)\gamma+2^{n-3}+2,$$
$$f(u)=\begin{cases}\frac{1}{2}(\gamma-a+2), & \text{if } u=x_{1,a} \text{ and } a \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-2}+5\right)(\gamma+1)-a\right], & \text{if } u=x_{1,a} \text{ and } a \text{ is even},\\ \frac{1}{2}(\gamma+a+2), & \text{if } u=x_{2,a} \text{ and } a \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-2}+5\right)(\gamma+1)+a\right], & \text{if } u=x_{2,a} \text{ and } a \text{ is even},\\ \frac{1}{2}(3\gamma-l+4), & \text{if } u=y_l \text{ and } l \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-2}+7\right)\gamma-l+2^{n-2}+5\right], & \text{if } u=y_l \text{ and } l \text{ is even},\\ \frac{1}{2}(4\gamma-b+5), & \text{if } u=z_{1,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-2}+8\right)\gamma-b+2^{n-2}+4\right], & \text{if } u=z_{1,b} \text{ and } b \text{ is even},\\ \frac{1}{2}(4\gamma+b+5), & \text{if } u=z_{2,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-2}+8\right)\gamma+b+2^{n-2}+4\right], & \text{if } u=z_{2,b} \text{ and } b \text{ is even},\\ \frac{1}{2}\left[\left(2^{j-2}+4\right)\gamma-b+2^{j-2}+5\right], & \text{if } u=z_{j,b},\ 3\le j\le n \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-2}+2^{j-2}+8\right)\gamma-b+2^{n-2}+2^{j-2}+4\right], & \text{if } u=z_{j,b},\ 3\le j\le n \text{ and } b \text{ is even}.\end{cases} \tag{9}$$

We can examine that the set of all edge-sums is $S=\{c,c+1,\ldots,c+p-2\}$, where $c=(2^{n-3}+2)\gamma+2^{n-3}+4$. Hence, by Lemma 1, f extends to a SEM labeling of $G_{(2;s;n)}$ with magic constant $k=(2^{n-3}\cdot 5+10)\gamma+2^{n-3}\cdot 5+5$. Therefore, for every $n\ge 2$ and odd $s\ge 1$, $\mu_s(G_{(2;s;n)})=0$.

Furthermore, since for any $n\ge 1$, $f(A)=\{1,2,\ldots,d\}$ and $f(B)=\{d+1,d+2,\ldots,p\}$, where

$$d=\begin{cases}2\gamma+2, & \text{if } n=1,\\ \frac{1}{2}(5\gamma+5), & \text{if } n=2,\\ \left(2^{n-3}+2\right)\gamma+2^{n-3}+2, & \text{if } n\ge 3,\end{cases} \tag{10}$$

then $G_{(2;s;n)}$ is a consecutively SEM graph. Hence, $\mu_c(G_{(2;s;n)})=0$ for every $n\ge 1$ and odd $s\ge 1$. □

Figure 2 shows the vertex labelings of the graph $G_{(2;5;1)}$ and $G_{(2;3;4)}$, given in the proof of Theorem 3.

Theorem 4. *Let $\gamma\ge 2$ be an even integer. If $r_1=t_1=\gamma$, $r_2=s=t_2=\gamma-1$, $t_3=\gamma+2$, and $t_j=2^{j-4}(\gamma+4)$, $4\le j\le n$, then $\mu_s(G_{(2;s;n)})=\mu_c(G_{(2;s;n)})=0$ for any $n\ge 2$ and odd $s\ge 1$.*

Proof. Based on the properties of the theorem, the order of $G_{(2;s;n)}$ is

$$p=\begin{cases}5\gamma-1, & \text{for } n=2,\\ \left(2^{n-3}+5\right)\gamma+2^{n-1}-3, & \text{for } n\ge 3.\end{cases} \tag{11}$$

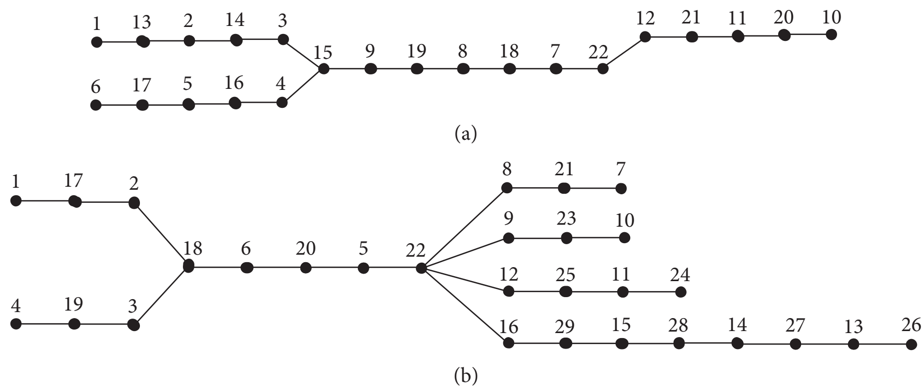

Figure 2: The (consecutively) SEM labelings of DT (5, 5; 5; 5) and DT (3, 3; 3; 3, 3, 4, 8).

Thus, there are two following cases to prove the theorem depending on the value of n.

(i) Case 1: $n = 2$.

Define a labeling $f\colon V(G_{(2;s;n)}) \longrightarrow \{1, 2, \ldots, p\}$ as follows:

$$f(x) = \frac{1}{2}(7\gamma),$$

$$f(z) = 5\gamma - 1,$$

$$f(u) = \begin{cases} \frac{1}{2}(a+1), & \text{if } u = x_{1,a} \text{ and } a \text{ is odd},\\ \frac{1}{2}(5\gamma + a), & \text{if } u = x_{1,a} \text{ and } a \text{ is even},\\ \frac{1}{2}(2\gamma + a + 1), & \text{if } u = x_{2,a} \text{ and } a \text{ is odd},\\ \frac{1}{2}(7\gamma + a), & \text{if } u = x_{2,a} \text{ and } a \text{ is even},\\ \frac{1}{2}(2\gamma - l + 1), & \text{if } u = y_l \text{ and } l \text{ is odd},\\ \frac{1}{2}(7\gamma - l), & \text{if } u = y_l \text{ and } l \text{ is even},\\ \frac{1}{2}(3\gamma + b + 1), & \text{if } u = z_{1,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}(8\gamma + b - 2), & \text{if } u = z_{1,b} \text{ and } b \text{ is even}.\\ \frac{1}{2}(5\gamma - b + 1), & \text{if } u = z_{2,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}(10\gamma - b - 2), & \text{if } u = z_{2,b} \text{ and } b \text{ is even}. \end{cases} \tag{12}$$

It can be checked that the set of all edge-sums is $S = \{(1/2)(5\gamma + 4), (1/2)(5\gamma + 6), \ldots, 6\gamma + 4\}$. Hence, by Lemma 1, f extends to a SEM labeling of $G_{(2;s;2)}$ with magic constant $k = (1/2)(25\gamma - 2)$. Therefore, for any odd $s \geq 1$, $\mu_s(G_{(2;s;2)}) = 0$.

(ii) Case 2: $n \geq 3$.

Define a labeling $f\colon V(G_{(2;s;2)}) \longrightarrow \{1, 2, \ldots, p\}$ as follows:

$$
\begin{aligned}
f(x) &= \frac{1}{2}\left[\left(2^{n-3}+7\right)\gamma + 2^{n-1} - 2\right],\\
f(z) &= \frac{1}{2}\left[\left(2^{n-3}+10\right)\gamma + 2^{n-1} - 4\right],
\end{aligned}
$$

$$
f(u) = \begin{cases}
\frac{1}{2}(a+1), & \text{if } u = x_{1,a} \text{ and } a \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-3}+5\right)\gamma + a + 2^{n-1} - 2\right], & \text{if } u = x_{1,a} \text{ and } a \text{ is even},\\
\frac{1}{2}(2\gamma + a + 1), & \text{if } u = x_{2,a} \text{ and } a \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-3}+7\right)\gamma + a + 2^{n-1} - 2\right], & \text{if } u = x_{2,a} \text{ and } a \text{ is even},\\
\frac{1}{2}(2\gamma - l + 1), & \text{if } u = y_l \text{ and } l \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-3}+7\right)\gamma - l + 2^{n-1} - 2\right], & \text{if } u = y_l \text{ and } l \text{ is even},\\
\frac{1}{2}(3\gamma + b + 1), & \text{if } u = z_{1,b} \text{ and } b \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-3}+8\right)\gamma + b + 2^{n-1} - 4\right], & \text{if } u = z_{1,b} \text{ and } b \text{ is even},\\
\frac{1}{2}(5\gamma - b + 1), & \text{if } u = z_{2,b} \text{ and } b \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-3}+10\right)\gamma - b + 2^{n-1} - 4\right], & \text{if } u = z_{2,b} \text{ and } b \text{ is even},\\
\frac{1}{2}(5\gamma + b + 1), & \text{if } u = z_{3,b} \text{ and } b \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-3}+10\right)\gamma + b + 2^{n-1} - 4\right], & \text{if } u = z_{3,b} \text{ and } b \text{ is even},\\
\frac{1}{2}\left[\left(2^{j-3}+5\right)\gamma - b + 2^{j-1} - 1\right], & \text{if } u = z_{j,b},\ 4 \le j \le n \text{ and } b \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-3}+2^{j-3}+10\right)\gamma - b + 2^{n-1} + 2^{j-1} - 4\right], & \text{if } u = z_{j,b},\ 4 \le j \le n \text{ and } b \text{ is even}.
\end{cases} \tag{13}
$$

We can check that the set of all edge-sums is $S = \{c, c+1, \ldots, c+p-2\}$, where $c = (1/2)[(2^{n-3}+5)\gamma + 2^{n-1}+2]$. Hence, by Lemma 1, f extends to a SEM labeling of $G_{(2;s;n)}$ with magic constant $k = 2p + c - 1$. Therefore, for any $n \ge 3$ and odd $s \ge 1$, $\mu_s(G_{(2;s;n)}) = 0$.

Moreover, since for any $n \ge 2$, $f(A) = \{1, 2, \ldots, d\}$, and $f(B) = \{d+1, d+2, \ldots, p\}$, where

$$
d = \begin{cases}
\frac{1}{2}(5\gamma), & \text{if } n = 2,\\
3\gamma + 1, & \text{if } n = 3,\\
\frac{1}{2}\left[\left(2^{n-3}+5\right)\gamma + 2^{n-1} - 2\right], & \text{if } n \ge 4,
\end{cases} \tag{14}
$$

then $G_{(2;s;n)}$ is a consecutively SEM graph. Hence, $\mu_c(G_{(2;s;n)}) = 0$ for any $n \geq 2$ and odd $s \geq 1$. □

The open problem related to Theorems 3 and 4 is as follows.

Open Problem 2. Decide if the graphs $G_{(2;s;n)}$ have zero (consecutively) SEMD for the remaining cases.

Lastly, we show that subdivision of double stars $G_{(m;s;n)}$ has zero (consecutively) SEMD, as the following theorems.

Theorem 5. *Let $\gamma \geq 1$ be an odd integer. If $r_1 = r_2 = s = \gamma$, $r_i = 2^{i-2}\gamma$, $t_1 = t_2 = s + \sum_{i=3}^{m} r_i$, and $t_j = 2^{j-3}(t_1 + 1)$, $3 \leq j \leq n$, then $\mu_s(G_{(m;s;n)}) = \mu_c(G_{(m;s;n)}) = 0$ for arbitrary $m \geq 3$, $n \geq 1$, and odd $s \geq 1$.*

Proof

(i) Case 1: $n = 1$.

Define a labeling $f\colon V(G_{(m;s;1)}) \longrightarrow \{1, 2, \ldots, p\}$, where $p = 2^m\gamma + 2$ as follows:

$$
\begin{aligned}
f(x) &= \frac{1}{2}\left[(2^m + 1)\gamma + 5\right],\\
f(z) &= 2^m\gamma + 2,\\
f(u) &= \begin{cases}
\frac{1}{2}(\gamma - a + 2), & \text{if } u = x_{1,a} \text{ and } a \text{ is odd},\\
\frac{1}{2}\left[(2^m + 1)\gamma - a + 5\right], & \text{if } u = x_{1,a} \text{ and } a \text{ is even},\\
\frac{1}{2}(\gamma + a + 2), & \text{if } u = x_{2,a} \text{ and } a \text{ is odd},\\
\frac{1}{2}\left[(2^m + 1)\gamma + a + 5\right], & \text{if } u = x_{2,a} \text{ and } a \text{ is even},\\
\frac{1}{2}\left[\left(2^{i-1} + 1\right)\gamma - a + 4\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is odd},\\
\frac{1}{2}\left[\left(2^m + 2^{i-1} + 1\right)\gamma - a + 5\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is even},\\
\frac{1}{2}(3\gamma - l + 4), & \text{if } u = y_l \text{ and } l \text{ is odd},\\
\frac{1}{2}\left[(2^m + 3)\gamma - l + 5\right], & \text{if } u = y_l \text{ and } l \text{ is even},\\
\frac{1}{2}(2^m\gamma - b + 5), & \text{if } u = z_{1,b} \text{ and } b \text{ is odd},\\
2^m\gamma - b + 3, & \text{if } u = z_{1,b} \text{ and } b \text{ is even}.
\end{cases}
\end{aligned}
\tag{15}
$$

(ii) Case 2: $n \geq 2$.

Define a labeling $f\colon V(G_{(m;s;n)}) \longrightarrow \{1, 2, \ldots, p\}$, where $p = (2^{n-2}(2^{m-1} - 1) + 2^m)\gamma + 2^{n-2} + 1$, as follows:

$$f(x) = \frac{1}{2}\left[\left(2^{n-2}\left(2^{m-1} - 1\right) + 2^m + 1\right)\gamma + 2^{n-2} + 5\right],$$

$$f(z) = \frac{1}{2}\left[\left(2^{n-2}\left(2^{m-1} - 1\right) + 2^{m+1}\right)\gamma + 2^{n-2} + 4\right],$$

$$f(u) = \begin{cases} \frac{1}{2}(\gamma - a + 2), & \text{if } u = x_{1,a} \text{ and } a \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{n-2}\left(2^{m-1} - 1\right) + 2^m + 1\right)\gamma - a + 2^{n-2} + 5\right], & \text{if } u = x_{1,a} \text{ and } a \text{ is even}, \\ \frac{1}{2}(\gamma + a + 2), & \text{if } u = x_{2,a} \text{ and } a \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{n-2}\left(2^{m-1} - 1\right) + 2^m + 1\right)\gamma + a + 2^{n-2} + 5\right], & \text{if } u = x_{2,a} \text{ and } a \text{ is even}, \\ \frac{1}{2}\left[\left(2^{i-1} + 1\right)\gamma - a + 4\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{n-2}\left(2^{m-1} - 1\right) + 2^m + 2^{i-1} + 1\right)\gamma - a + 2^{n-2} + 5\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is even}, \\ \frac{1}{2}(3\gamma - l + 4), & \text{if } u = y_l \text{ and } l \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{n-2}\left(2^{m-1} - 1\right) + 2^m + 3\right)\gamma - l + 2^{n-2} + 5\right], & \text{if } u = y_l \text{ and } l \text{ is odd}, \\ \frac{1}{2}\left[2^m\gamma - b + 5\right], & \text{if } u = z_{1,b} \text{ and } b \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{n-2}\left(2^{m-1} - 1\right) + 2^{m+1}\right)\gamma - b + 2^{n-2} + 4\right], & \text{if } u = z_{1,b} \text{ and } b \text{ is even}. \\ \frac{1}{2}\left[2^m\gamma + b + 5\right], & \text{if } u = z_{2,b} \text{ and } b \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{n-2}\left(2^{m-1} - 1\right) + 2^{m+1}\right)\gamma + b + 2^{n-2} + 4\right], & \text{if } u = z_{2,b} \text{ and } b \text{ is even}, \end{cases} \tag{16}$$

and for $3 \leq j \leq n$,

$$f\left(z_{j,b}\right) = \begin{cases} \frac{1}{2}\left[\left(2^m\left(2^{j-3} + 1\right) - 2^{j-2}\right)\gamma - b + 2^{j-2} + 5\right], & \text{if } b \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{n-2}\left(2^{m-1} - 1\right) + 2^m\left(2^{j-3} + 2\right) - 2^{j-2}\right)\gamma - b + 2^{n-2} + 2^{j-2} + 4\right], & \text{if } b \text{ is even}. \end{cases} \tag{17}$$

It can be verified that, for $n \geq 1$, the set of all edge-sums is $S = \{c, c+1, \ldots, c+p-2\}$, where

$$c = \begin{cases} 2^{m-1}\gamma + 4, & \text{for } n = 1, \\ \left[2^{n-3}\left(2^{m-1}-1\right) + 2^{m-1}\right]\gamma + 2^{n-3} + 4, & \text{for } n \geq 2. \end{cases} \tag{18}$$

Hence, by Lemma 1, f extends to a SEM labeling of $G_{(m;s;n)}$ with magic constants $k = 2p + c - 1$. Therefore, for arbitrary $m \geq 3$, $n \geq 1$, and odd $s \geq 1$, $\mu_s(G_{(m;s;n)}) = 0$.

Moreover, since for any $n \geq 1$, $f(A) = \{1, 2, \ldots, d\}$ and $f(B) = \{d+1, d+2, \ldots, p\}$, where

$$d = \begin{cases} 2^{m-1}\gamma + 2, & \text{if } n = 1, \\ \dfrac{1}{2}\left[\left(2^{m-1}\cdot 3 - 1\right)\gamma + 5\right], & \text{if } n = 2, \\ \left[2^{m-1}\left(2^{n-3}+1\right) - 2^{n-3}\right]\gamma + 2^{n-3} + 2, & \text{if } n \geq 3, \end{cases} \tag{19}$$

then $G_{(m;s;n)}$ is a consecutively SEM graph. Hence, $\mu_c(G_{(m;s;n)}) = 0$ for arbitrary $m \geq 3$, $n \geq 1$, and odd $s \geq 1$. □

Theorem 6. *Let $\gamma \equiv 2 \pmod 4$. If $r_1 = \gamma$, $r_2 = s = \gamma - 1$, $r_i = 2^{i-3}\gamma$, $3 \leq i \leq m$, $t_1 = (1/2)(2^{m-2}+1)\gamma$, $t_2 = t_1 - 1$, $t_3 = t_2 + 2$, and $t_j = 2^{j-4}(t_3 + 2)$, $4 \leq j \leq n$, then $\mu_s(G_{(m;s;n)}) = \mu_c(G_{(m;s;n)}) = 0$ for any $m \geq 3$, $n \geq 2$, and $s \equiv 1 \pmod 4$.*

Proof

(i) Case 1: $n = 2$.

Define a labeling $f\colon V(G_{(m;s;2)}) \longrightarrow \{1, 2, \ldots, p\}$, where $p = (2^{m-1}+3)\gamma - 1$, as follows:

$$\begin{aligned}
f(x) &= \frac{1}{2}\left(2^{m-1}+5\right)\gamma, \\
f(z) &= \left(2^{m-1}+3\right)\gamma - 1, \\
f(u) &= \begin{cases}
\dfrac{1}{2}(a+1), & \text{if } u = x_{1,a} \text{ and } a \text{ is odd}, \\
\dfrac{1}{2}\left[\left(2^{m-1}+3\right)\gamma + a\right], & \text{if } u = x_{1,a} \text{ and } a \text{ is even}, \\
\dfrac{1}{2}(2\gamma + a + 1), & \text{if } u = x_{2,a} \text{ and } a \text{ is odd}, \\
\dfrac{1}{2}\left[\left(2^{m-1}+5\right)\gamma + a\right], & \text{if } u = x_{2,a} \text{ and } a \text{ is even}, \\
\dfrac{1}{2}\left[\left(2^{i-2}+2\right)\gamma - a + 1\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is odd}, \\
\dfrac{1}{2}\left[\left(2^{m-1}+2^{i-2}+5\right)\gamma - a\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is even}, \\
\dfrac{1}{2}(2\gamma - l + 1), & \text{if } u = y_l \text{ and } l \text{ is odd}, \\
\dfrac{1}{2}\left[\left(2^{m-1}+5\right)\gamma - l\right], & \text{if } u = y_l \text{ and } l \text{ is even}, \\
\dfrac{1}{2}\left[\left(2^{m-2}+2\right)\gamma + b + 1\right], & \text{if } u = z_{1,b} \text{ and } b \text{ is odd}, \\
\dfrac{1}{2}\left[\left(2^{m-2}\cdot 3 + 5\right)\gamma + b - 2\right], & \text{if } u = z_{1,b} \text{ and } b \text{ is even}, \\
\dfrac{1}{2}\left[\left(2^{m-1}+3\right)\gamma - b + 1\right], & \text{if } u = z_{2,b} \text{ and } b \text{ is odd}, \\
\dfrac{1}{2}\left[\left(2^{m}+6\right)\gamma - b - 2\right], & \text{if } u = z_{2,b} \text{ and } b \text{ is even}.
\end{cases}
\end{aligned} \tag{20}$$

(ii) Case 2: $n \geq 3$.

Define a labeling $f\colon V(G_{(m;s;n)}) \longrightarrow \{1, 2, \ldots, p\}$, where $p = (2^{n-4}(2^{m-2}+1) + 2^{m-1} + 3)\gamma + 2^{n-3}\cdot 3 - 3$, as follows:

$$
\begin{aligned}
f(x) &= \frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m-1}+5\right)\gamma+2^{n-3}\cdot 3-2\right],\\
f(z) &= \frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m}+6\right)\gamma+2^{n-3}\cdot 3-4\right],\\
f(u) &= \begin{cases}
\frac{1}{2}(a+1), & \text{if } u=x_{1,a} \text{ and } a \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m-1}+3\right)\gamma+a+2^{n-3}\cdot 3-2\right], & \text{if } u=x_{1,a} \text{ and } a \text{ is even},\\
\frac{1}{2}(2\gamma+a+1), & \text{if } u=x_{2,a} \text{ and } a \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m-1}+5\right)\gamma+a+2^{n-3}\cdot 3-2\right], & \text{if } u=x_{2,a} \text{ and } a \text{ is even},\\
\frac{1}{2}\left[\left(2^{i-2}+2\right)\gamma-a+1\right], & \text{if } u=x_{i,a}, 3\leq i\leq m, \text{ and } a \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m-1}+2^{i-2}+5\right)\gamma-a+2^{n-3}\cdot 3-2\right], & \text{if } u=x_{i,a}, 3\leq i\leq m \text{ and } a \text{ is even},\\
\frac{1}{2}(2\gamma-l+1), & \text{if } u=y_{l} \text{ and } l \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m-1}+5\right)\gamma-l+2^{n-3}\cdot 3-2\right], & \text{if } u=y_{l} \text{ and } l \text{ is even},\\
\frac{1}{2}\left[\left(2^{m-2}+2\right)\gamma+b+1\right], & \text{if } u=z_{1,b} \text{ and } b \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m-2}\cdot 3+5\right)\gamma+b+2^{n-3}\cdot 3-4\right], & \text{if } u=z_{1,b} \text{ and } b \text{ is even},\\
\frac{1}{2}\left[\left(2^{m-1}+3\right)\gamma-b+1\right], & \text{if } u=z_{2,b} \text{ and } b \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m}+6\right)\gamma-b+2^{n-3}\cdot 3-4\right], & \text{if } u=z_{2,b} \text{ and } b \text{ is even},\\
\frac{1}{2}\left[\left(2^{m-1}+3\right)\gamma+b+1\right], & \text{if } u=z_{3,b} \text{ and } b \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m}+6\right)\gamma+b+2^{n-3}\cdot 3-4\right], & \text{if } u=z_{3,b} \text{ and } b \text{ is even},
\end{cases}
\end{aligned}
\tag{21}
$$

and for $4 \leq j \leq n$,

$$
f\left(z_{j,b}\right)=\begin{cases}
\frac{1}{2}\left[\left(2^{m-2}\left(2^{j-4}+2\right)+2^{j-4}+3\right)\gamma-b+2^{j-3}\cdot 3-1\right], & \text{if } b \text{ is odd},\\
\frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m-2}\left(2^{j-4}+4\right)+2^{j-4}+6\right)\gamma-b+2^{n-3}\cdot 3+2^{j-3}\cdot 3-4\right], & \text{if } b \text{ is even}.
\end{cases}
\tag{22}
$$

We can examine that, for $n \geq 2$, the set of all edge-sums is $S = \{c, c+1, \ldots, c+p-2\}$, where

$$
c=\begin{cases}
\frac{1}{2}\left[\left(2^{m-1}+3\right)\gamma+4\right], & \text{for } n=2,\\
\frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m-1}+3\right)\gamma+2^{n-3}\cdot 3+2\right], & \text{for } n\geq 3.
\end{cases}
\tag{23}
$$

Hence, by Lemma 1, f extends to a SEM labeling of $G_{(m;s;n)}$ with magic constants $k = 2p + c - 1$. Therefore, for any $m \geq 3$, $n \geq 2$, and $s \equiv 1\,(\mathrm{mod}4)$, $\mu_s(G_{(m;s;n)}) = 0$.

Furthermore, since for any $n \geq 2$, $f(A) = \{1, 2, \ldots, d\}$, and $f(B) = \{d+1, d+2, \ldots, p\}$, where

$$d = \begin{cases} \frac{1}{2}\left[\left(2^{m-1} + 3\right)\gamma\right], & \text{if } n = 2, \\ \frac{1}{4}\left[\left(2^{m-2}\cdot 5 + 7\right)\gamma + 2\right], & \text{if } n = 3, \\ \frac{1}{2}\left[\left(2^{m-2}\left(2^{n-4} + 2\right) + 2^{n-4} + 3\right)\gamma + 2^{n-3}\cdot 3 - 2\right], & \text{if } n \geq 4, \end{cases} \tag{24}$$

then $G_{(m;s;n)}$ is a consecutively SEM graph. Hence, $\mu_c(G_{(m;s;n)}) = 0$ for any $m \geq 3$, $n \geq 2$, and $s \equiv 1\,(\mathrm{mod}4)$.

To clarify the proof of Theorem 6, see Figure 3. □

Theorem 7. *Let $\gamma \equiv 0\,(\mathrm{mod}4)$ be a positive integer. If $r_1 = \gamma$, $r_2 = s = \gamma - 1$, $r_i = 2^{i-3}\gamma$, $3 \leq i \leq m$, $t_1 = t_3 = (1/2)(2^{m-2} + 1)\gamma$, $t_2 = t_1 - 2$, and $t_j = 2^{j-4}(t_3 + 2)$, $4 \leq j \leq n$, then $\mu_s(G_{(m;s;n)}) = \mu_c(G_{(m;s;n)}) = 0$ for every $m \geq 3$, $n \geq 2$, and $s \equiv 3\,(\mathrm{mod}4)$.*

Proof

(i) Case 1: $n = 2$.

Define a labeling $f\colon V(G_{(m;s;2)}) \longrightarrow \{1, 2, \ldots, p\}$, where $p = (2^{m-1} + 3)\gamma - 2$, as follows:

$$\begin{aligned} f(x) &= \frac{1}{2}\left[\left(2^{m-1} + 5\right)\gamma - 2\right], \\ f(z) &= \left(2^{m-1} + 3\right)\gamma - 2, \\ f(u) &= \begin{cases} \frac{1}{2}(a+1), & \text{if } u = x_{1,a} \text{ and } a \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{m-1} + 3\right)\gamma + a - 2\right], & \text{if } u = x_{1,a} \text{ and } a \text{ is even}, \\ \frac{1}{2}(2\gamma + a + 1), & \text{if } u = x_{2,a} \text{ and } a \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{m-1} + 5\right)\gamma + a - 2\right], & \text{if } u = x_{2,a} \text{ and } a \text{ is even}, \\ \frac{1}{2}\left[\left(2^{i-2} + 2\right)\gamma - a + 1\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{m-1} + 2^{i-2} + 5\right)\gamma - a - 2\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is even}, \\ \frac{1}{2}(2\gamma - l + 1), & \text{if } u = y_l \text{ and } l \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{m-1} + 5\right)\gamma - l - 2\right], & \text{if } u = y_l \text{ and } l \text{ is even}, \\ \frac{1}{2}\left[\left(2^{m-2} + 2\right)\gamma + b + 1\right], & \text{if } u = z_{1,b} \text{ and } b \text{ is odd}, \\ \frac{1}{2}\left[\left(2^{m-2}\cdot 3 + 5\right)\gamma + b - 4\right], & \text{if } u = z_{1,b} \text{ and } b \text{ is even}, \\ \frac{1}{2}\left[\left(2^{m-1} + 3\right)\gamma - b - 1\right], & \text{if } u = z_{2,b} \text{ and } b \text{ is odd}, \\ \frac{1}{2}\left[(2^{m} + 6)\gamma - b - 4\right], & \text{if } u = z_{2,b} \text{ and } b \text{ is even}. \end{cases} \end{aligned} \tag{25}$$

(ii) Case 2: $n \geq 3$.

Define a labeling $f\colon V(G_{(m;s;n)}) \longrightarrow \{1, 2, \ldots, p\}$, where $p = (2^{n-4}(2^{m-2}+1) + 2^{m-1}+3)\gamma + 2^{n-2} - 4$, as follows:

$$f(x) = \frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right) + 2^{m-1} + 5\right)\gamma + 2^{n-2} - 4\right],$$

$$f(z) = \left(2^{n-5}\left(2^{m-2}+1\right) + 2^{m-1} + 3\right)\gamma + 2^{n-3} - 3,$$

$$f(u) = \begin{cases} \frac{1}{2}(a+1), & \text{if } u = x_{1,a} \text{ and } a \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right) + 2^{m-1}+3\right)\gamma + a + 2^{n-2} - 4\right], & \text{if } u = x_{1,a} \text{ and } a \text{ is even},\\ \frac{1}{2}(2\gamma + a + 1), & \text{if } u = x_{2,a} \text{ and } a \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right) + 2^{m-1}+5\right)\gamma + a + 2^{n-2} - 4\right], & \text{if } u = x_{2,a} \text{ and } a \text{ is even},\\ \frac{1}{2}\left[\left(2^{i-2}+2\right)\gamma - a + 1\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right) + 2^{m-1} + 2^{i-2}+5\right)\gamma - a + 2^{n-2} - 4\right], & \text{if } u = x_{i,a},\ 3 \leq i \leq m \text{ and } a \text{ is even},\\ \frac{1}{2}(2\gamma - l + 1), & \text{if } u = y_l \text{ and } l \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right) + 2^{m-1}+5\right)\gamma - l + 2^{n-2} - 4\right], & \text{if } u = y_l \text{ and } l \text{ is even},\\ \frac{1}{2}\left[\left(2^{m-2}+2\right)\gamma + b + 1\right], & \text{if } u = z_{1,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right) + 2^{m-2}\cdot 3+5\right)\gamma + b + 2^{n-2} - 6\right], & \text{if } u = z_{1,b} \text{ and } b \text{ is even},\\ \frac{1}{2}\left[\left(2^{m-1}+3\right)\gamma - b - 1\right], & \text{if } u = z_{2,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right) + 2^{m}+6\right)\gamma - b + 2^{n-2} - 6\right], & \text{if } u = z_{2,b} \text{ and } b \text{ is even},\\ \frac{1}{2}\left[\left(2^{m-1}+3\right)\gamma + b - 1\right], & \text{if } u = z_{3,b} \text{ and } b \text{ is odd},\\ \frac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right) + 2^{m}+6\right)\gamma + b + 2^{n-2} - 6\right], & \text{if } u = z_{3,b} \text{ and } b \text{ is even}, \end{cases} \tag{26}$$

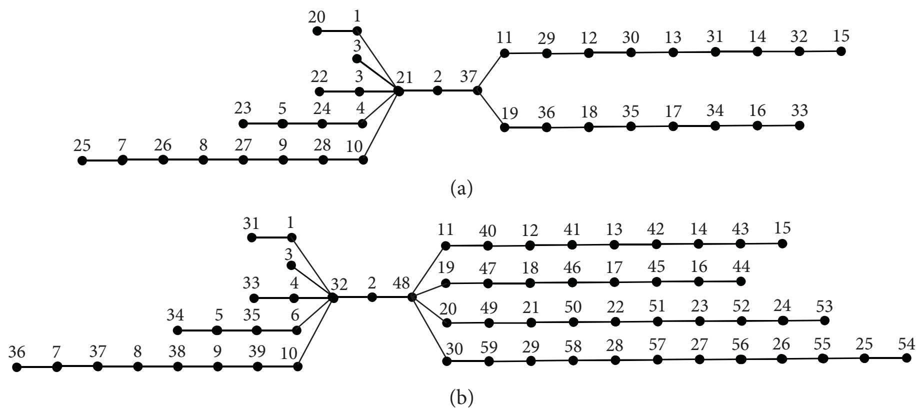

FIGURE 3: The (consecutively) SEM labelings of DT(2, 1, 2, 4, 8; 1; 9, 8) and DT(2, 1, 2, 4, 8; 1; 9, 8, 10, 12).

and for $4 \leq j \leq n$,

$$f\left(z_{j,b}\right) = \begin{cases} \dfrac{1}{2}\left[\left(2^{m-2}\left(2^{j-4}+2\right)+2^{j-4}+3\right)\gamma - b + 2^{j-2} - 3\right], & \text{if } b \text{ is odd}, \\ \\ \dfrac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m-2}\left(2^{j-4}+4\right)+2^{j-4}+6\right)\gamma - b + 2^{n-2} + 2^{j-2} - 6\right], & \text{if } b \text{ is even}. \end{cases} \quad (27)$$

It can be verified that, for $n \geq 2$, the set of all edge-sums is $S = \{c, c+1, \ldots, c+p-2\}$, where

$$c = \begin{cases} \dfrac{1}{2}\left[\left(2^{m-1}+3\right)\gamma + 2\right], & \text{for } n = 2, \\ \\ \dfrac{1}{2}\left[\left(2^{n-4}\left(2^{m-2}+1\right)+2^{m-1}+3\right)\gamma + 2^{n-2}\right], & \text{for } n \geq 3. \end{cases} \quad (28)$$

Hence, by Lemma 1, f extends to a SEM labeling of $G_{(m;s;n)}$ with magic constants $k = 2p + c - 1$. Therefore, for every $m \geq 3$, $n \geq 2$, and $s \equiv 3\,(\mathrm{mod}\,4)$, $\mu_s\left(G_{(m;s;2)}\right) = 0$.

Moreover, since for any $n \geq 2$, $f(A) = \{1, 2, \ldots, d\}$ and $f(B) = \{d+1, d+2, \ldots, p\}$, where

$$d = \begin{cases} \dfrac{1}{2}\left[\left(2^{m-1}+3\right)\gamma - 2\right], & \text{if } n = 2, \\ \\ \dfrac{1}{4}\left[\left(2^{m-2}\cdot 5+7\right)\gamma - 4\right], & \text{if } n = 3, \\ \\ \dfrac{1}{2}\left[\left(2^{m-2}\left(2^{n-4}+2\right)+2^{n-4}+3\right)\gamma + 2^{n-2} - 4\right], & \text{if } n \geq 4, \end{cases} \quad (29)$$

then $G_{(m;s;n)}$ is a consecutively SEM graph. Hence, $\mu_c\left(G_{(m;s;n)}\right) = 0$ for every $m \geq 3$, $n \geq 2$, and $s \equiv 3\,(\mathrm{mod}\,4)$. □

According to Theorems 5–7, we present the following open problems.

Open Problems 3. Find (consecutively) SEMD of the graphs $G_{(m;s;n)}$ for the remaining cases.

Data Availability

The topic of our research is a graph labelling theory based on related articles. The datasets used to support the findings of the study are cited in the text as references and are available at https://doi.org/10.1016/S0012-365X(00)00314-9 [9], https://doi.org/10.1016/S1571-0653(04)00074-5 [14], https://doi.org/10.5614/ejgta.2020.8.1.6 [15], and DOI: 10.1109/ACCESS.2019.2927244 reference number [11].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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