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Research Article

Hermite–Hadamard and Fractional Integral Inequalities for Interval-Valued Generalized p-Convex Function

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In the present paper, the new interval-valued generalized p convex functions are introduced. By using the notion of interval-valued generalized p convex functions and some auxiliary results of interval analysis, new Hermite–Hadamard and Fejér type inequalities are proved. The established results are more generalized than existing results in the literature. Moreover, fractional integral inequality for this generalization is also established.

1. Introduction

The theory of interval analysis introduced in numerical analysis by Moore in [1] had rapid development in last few decades. In computational problems, one can program a computer to find interval that contains the exact answer to the problems. Also, interval analysis provides rigorous enclosure of solution to the model equation. Moreover, the interval analysis is widely used in chemical and structured engineering, economics, control circuitry design, robotics, beam physics, behavioral ecology, constraint satisfaction, computer graphics, signal processing, asteroid orbits and global optimization [2], neural network output optimization [3], and many others. For interesting fundamental results, we refer [2, 4–8] to the readers.

Since the convexity play a vital role not only in convex analysis but also in almost all branches of mathematics. The famous inequalities in convex analysis are Jensen type, Hermite–Hadamard type, Fejér type, Ostrowski type, etc. For deeper insight about these inequalities, we refer [9–16] and references therein.

Furthermore, the definition of classical convexity enables us to tackle modern applied problems, because most of the problems are nonconvex in nature. Famous generalization of convexity are logarithmic convexity [16], p-convexity [17], η convexity [18], h-convexity [19], modified h-convexity [15], etc. For example, in [20], Nchama et al. used the Caputo-Fabrizio fractional integral and gave some new inequalities. For detailed applications of fractional calculus, we refer [21–28] to the readers and references therein.

In order to introduce the main definition of this paper, let us recall few generalizations of convexity present in the literature.

Definition 1 (see [17]). An interval I_1 is p-convex set, if for any $x_1, x_2 \in I_1$, $\alpha_1 \in [0,1]$, we have

$$\left[\alpha_1 x_1^p + (1-\alpha_1) x_2^p\right]^{(1/p)} \in I_1, \tag{1}$$

where $p = 2k_1 + 1$ or $p = (n_1/m_1)$, $n_1 = 2r_1 + 1$, $m_1 = 2t_1 + 1$, and $k_1, r_1, t_1 \in N$.

Definition 2 (see [17]). A mapping f defined from a p-convex set I_1 to $\mathbb{R}$ is said to be p-convex function, if

$$f\left(\left[\alpha_1^p + (1-\alpha_1) x_2^p\right]^{(1/p)}\right) \le \alpha_1 f(x_1) + (1-\alpha_1) f(x_2), \tag{2}$$

for each $x_1, x_2 \in I_1$ and $\alpha_1 \in [0,1]$ hold.

Definition 3 (see [29]). The mapping f defined from I_1 to $\mathbb{R}$ is said to be η-convex if

$$f(\alpha_1 x_1 + (1-\alpha_1)x_2) \le f(x_2) + \alpha_1 \eta(f(x_1), f(x_2)) \quad (3)$$

holds with respect to $\eta\colon B_1 \times B_1 \longrightarrow B_2$ for appropriate $B_1, B_2 \subseteq \mathbb{R}$, and for each $x_1, x_2 \in I_1$, $\alpha_1 \in [0,1]$.

Definition 4 (see [29]). A mapping is nonnegatively homogeneous if $\eta(\alpha x_1, \alpha x_2) = \alpha\eta(x_1, x_2)$ for each $x_1, x_2 \in \mathbb{R}$ and $\alpha \ge 0$.

Definition 5 (see [30]). A mapping f defined from a p-convex set I_1 to $\mathbb{R}$ is said to be generalized p convex function, if

$$f\left(\left[\alpha_1 x_1^p + (1-\alpha_1)x_2^p\right]^{(1/p)}\right) \le f(x_2) + \alpha_1 \eta(f(x_1), f(x_2)) \quad (4)$$

holds for $\eta\colon B_1 \times B_1 \longrightarrow B_2$ be a bifunction for appropriate $B_1, B_2 \subseteq \mathbb{R}$ and for each $x_1, x_2 \in I_1$ and $\alpha_1 \in [0,1]$.

Now, we present the concept of interval-valued generalized p convex function.

Definition 6. A mapping f defined from a p-convex set I_1 to $\mathbb{R}$ is said to be interval-valued generalized p-convex function, if

$$f\left(\left[\alpha_1 x_1^p + (1-\alpha_1)x_2^p\right]^{(1/p)}\right) \supseteq f(x_2) + \alpha_1 \eta(f(x_1), f(x_2)) \quad (5)$$

holds for $\eta\colon B_1 \times B_1 \longrightarrow B_2$ be a bifunction for appropriate $B_1, B_2 \subseteq \mathbb{R}$ and for each $x_1, x_2 \in I_1$ and $\alpha_1 \in [0,1]$.

Here, for $\overline{f} = \underline{f}$ and $p = 1$, (5) is an η-convexity, for $\overline{f} = \underline{f}$ and $\eta(x_1, x_2) = x_1 - x_2$ (5) is p-convexity, and for $p = 1$ and $\eta(x_1, x_2) = x_1 - x_2$, (5) is classical convexity.

This article is in the direction of the concepts and some results discussed in [30], but now we use interval-valued generalized p-convex function instead of generalized p convex function. After this introduction, in Section 2, we develop some basic properties of interval-valued generalized p convex functions. InSection 3, we make some new inequalities like Hermite–Hadamard's and Fejér type for interval-valued generalized p convex functions.

2. Basic Results

Here, we derive some operations which preserves interval-valued generalized p convex function.

Proposition 1. *Let f_1 and f_2 be two interval-valued generalized p convex functions:*

(1) If η is additive, then $f_1 + f_2$ is interval-valued generalized p convex

(2) If η is nonnegatively homogeneous, then λf_1 is interval-valued generalized p convex for any $\lambda \ge 0$.

Proof. The proof is straightforward. □

Theorem 1. *Let $f\colon [r,s] \longrightarrow R_I^+$ be an interval-valued function such that $f(\lambda) = [\underline{f}(\lambda), \overline{f}(\lambda)]$, then $\underline{f} \in SX((\eta,p),[r,s],R_I^+)$ iff $\underline{f} \in SX((\overline{\eta},p),[r,s],R_I^+)$ and $\overline{f} \in SV((\eta,p),[r,s],R_I^+)$.*

Proof. Let $f \in SX((\eta,p),[r,s],R_I^+)$, then for any $x, y \in [r,s]$ $\lambda \in (0,1)$, we have

$$f(y) + t\eta(f(x), f(y)) \subseteq f(\lambda x^p + (1-\lambda)y^p), \quad (6)$$

that is,

$$\begin{aligned}&\left[\underline{f}(y) + t\eta\left(\underline{f}(x), \underline{f}(y)\right), \overline{f}(y) + t\eta(\overline{f}(x), \overline{f}(y))\right]\\ &\subseteq \left[\underline{f}(\lambda x^p + (1-\lambda)y^p)^{(1/p)}, \overline{f}(\lambda x^p + (1-\lambda)y^p)^{(1/p)}\right].\end{aligned} \quad (7)$$

It follows that

$$\begin{aligned}&\underline{f}(y) + t\eta\left(\underline{f}(x), \underline{f}(y)\right) \ge \underline{f}(\lambda x^p + (1-\lambda)y^p)^{(1/p)},\\ &\overline{f}(y) + t\eta(\overline{f}(x), \overline{f}(y)) \le \overline{f}(\lambda x^p + (1-\lambda)y^p)^{(1/p)}.\end{aligned} \quad (8)$$

This shows that

$$\begin{aligned}&\underline{f} \in \mathrm{SX}((\eta,p),[r,s],R_I^+),\\ &\overline{f} \in \mathrm{SV}((\eta,p),[r,s],R_I^+).\end{aligned} \quad (9)$$

Conversely, suppose that

$$\begin{aligned}&\underline{f} \in \mathrm{SX}((\eta,p),[r,s],R_I^+),\\ &\overline{f} \in \mathrm{SV}((\eta,p),[r,s],R_I^+).\end{aligned} \quad (10)$$

Then, it follows that $f \in \mathrm{SX}((\eta,p),[r,s],R_I^+)$. This completes the proof. □

Theorem 2. *Let $f\colon [r,s] \longrightarrow R_I^+$ be an interval-valued function such that $f(\lambda) = [\underline{f}(\lambda), \overline{f}(\lambda)]$, then $\underline{f} \in \mathrm{SV}((\eta,p),[r,s],R_I^+)$ if $\underline{f} \in \mathrm{SV}((\eta,\overline{p}),[r,s],R_I^+)$ and $\overline{f} \in \mathrm{SX}((\eta,p),[r,s],R_I^+)$.*

Proof. The proof is similar to that of Theorem 1. □

3. Hermite–Hadamard-Type Inequality for Interval-Valued Generalized p Convex Function

In the following theorem, we present the Hermite–Hadamard type inequality for interval-valued generalized p convex function.

Theorem 3. *Let $f\colon I \longrightarrow \mathbb{R}$ be an interval-valued generalized p convex function for $\xi_1, \xi_2 \in I$ with condition $\xi_1 < \xi_2$, then we obtain the following inequality:*

$$f\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} - \frac{p}{2\left(\xi_2^p-\xi_1^p\right)}\int_{\xi_1}^{\xi_2} x^{p-1}\eta \cdot\left(f\left(\xi_1^p+\xi_2^p-x^p\right)^{(1/p)}, f(x)\mathrm{d}x\right)$$
$$\supseteq \frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1} f(x)\mathrm{d}x \supseteq \frac{f(\xi_1)+f(\xi_2)}{2} + \frac{1}{4}\left[\eta(f(\xi_1), f(\xi_2)) + \eta(f(\xi_2), f(\xi_1))\right]. \quad (11)$$

Proof. Take $u^p = t\xi_1^p + (1-t)\xi_2^p$ and $v^p = (1-t)\xi_1^p + t\xi_2^p$, it implies

$$\frac{\xi_1^p+\xi_2^p}{2} = \frac{u^p+v^p}{2}. \quad (12)$$

So,

$$f\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} = f\left(\frac{u^p+v^p}{2}\right)^{(1/p)}. \quad (13)$$

By definition of interval-valued generalized p convex functions, we have

$$f\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} = f\left(\frac{1}{2}\left(\left(t\xi_1^p+(1-t)\xi_2^p\right)^{(1/p)}\right)^p + \frac{1}{2}\left(\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)}\right)^p\right)^{(1/p)}$$
$$\supseteq f\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)} + \frac{1}{2}\eta\left(f\left(t\xi_1^p+(1-t)\xi_2^p\right)^{(1/p)}, f\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)}\right). \quad (14)$$

Now, by the definition of interval

$$f\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} = \left[\underline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)}, \overline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)}\right], \quad (15)$$

we have

$$\left[\underline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)}, \overline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)}\right],$$
$$\supseteq \left[\underline{f}\left((1-t)\xi_1^p+\xi_2^p\right)^{(1/p)} + \frac{1}{2}\eta\left(\underline{f}\left(t\xi_1^p+(1-t)\xi_2^p\right)^{(1/p)}, \underline{f}\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)}\right),\right.$$
$$\left.\overline{f}\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)} + \frac{1}{2}\eta\left(\overline{f}\left(t\xi_1^p+(1-t)\xi_2^p\right)^{(1/p)}, \overline{f}\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)}\right)\right]. \quad (16)$$

It follows that

$$\underline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} \le \underline{f}\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)}, + \frac{1}{2}\eta\left(\underline{f}\left(t\xi_1^p+(1-t)\xi_2^p\right)^{(1/p)}, \underline{f}\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)}\right), \quad (17)$$

$$\overline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} \ge \overline{f}\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)} + \frac{1}{2}\eta\left(\overline{f}\left(t\xi_1^p+(1-t)\xi_2^p\right)^{(1/p)}, \overline{f}\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)}\right). \quad (18)$$

Integrating (17) with respect to "x" on [0, 1], we get

$$\underline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} \le \int_0^1 \underline{f}\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)}\mathrm{d}t + \frac{1}{2}\int_0^1 \eta\left(\underline{f}\left(t\xi_1^p+(1-t)\xi_2^p\right)^{(1/p)}, \underline{f}\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)}\right)\mathrm{d}t, \quad (19)$$

which implies

$$\underline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} - \frac{p}{2\left(\xi_2^p-\xi_1^p\right)}\int_{\xi_1}^{\xi_2} x^{p-1}\eta\left(\underline{f}\left(\xi_1^p+\xi_2^p-x^p\right)^{(1/p)}, \underline{f}(x)\right)\mathrm{d}x,$$
$$\le \frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\underline{f}(x)\mathrm{d}x. \quad (20)$$

Now,

$$\int_{\xi_1}^{\xi_2} x^{p-1}\underline{f_-}(x)dx = \frac{\xi_2^p-\xi_1^p}{p}\int_0^1 \underline{f}\left(t\xi_1^p+(1-t)\xi_2^p\right)^{(1/p)}\mathrm{d}t,$$
$$\leq \frac{\xi_2^p-\xi_1^p}{p}\left(\underline{f}(\xi_2)+\int_0^1 t\eta\left(\underline{f}(\xi_1),\underline{f}((\xi_2))\right)\mathrm{d}t\right), \tag{21}$$
$$\frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\underline{f}(x)\mathrm{d}x \leq \underline{f}(\xi_2)+\int_0^1 t\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right)\mathrm{d}t.$$

Similarly,

$$\frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\underline{f}(x)\mathrm{d}x \leq \underline{f}(\xi_1)+\int_0^1 t\eta\left(\underline{f}(\xi_2),\underline{f}(\xi_1)\right)\mathrm{d}t. \tag{22}$$

Adding (21) and (22), we obtain

$$\frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\underline{f}(x)\mathrm{d}x \leq \frac{\underline{f}(\xi_1)+\underline{f}(\xi_2)}{2}+\frac{1}{4}\left[\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right)+\eta\left(\underline{f}(\xi_2),\underline{f}(\xi_1)\right)\right]. \tag{23}$$

Now, Integrating (18) with respect to "x" on [0, 1], we get

$$\overline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} \geq \int_0^1 \overline{f}\left((1-t)\xi_1^p+t\xi_2^p\right)^{\frac{1}{p}}\mathrm{d}t$$
$$+\frac{1}{2}\int_0^1 \eta\left(\overline{f}\left(t\xi_1^p+(1-t)\xi_2^p\right)^{(1/p)},\right.$$
$$\left.\overline{f}\left((1-t)\xi_1^p+t\xi_2^p\right)^{(1/p)}\right)\mathrm{d}t, \tag{24}$$

which implies

$$\overline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} - \frac{p}{2\left(\xi_2^p-\xi_1^p\right)}\int_{\xi_1}^{\xi_2} x^{p-1}\eta$$
$$\cdot\left(\overline{f}\left(\xi_1^p+\xi_2^p-x^p\right)^{(1/p)},\overline{f}(x)\right)\mathrm{d}x, \tag{25}$$
$$\geq \frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\overline{f}(x)\mathrm{d}x.$$

Now,

$$\int_{\xi_1}^{\xi_2} x^{p-1}\overline{f^-}(x)dx = \frac{\xi_2^p-\xi_1^p}{p}\int_0^1 \overline{f}\left(t\xi_1^p+(1-t)\xi_2^p\right)^{(1/p)}\mathrm{d}t,$$
$$\geq \frac{\xi_2^p-\xi_1^p}{p}$$
$$\cdot\left(\overline{f}(\xi_2)+\int_0^1 t\eta\left(\overline{f}(\xi_1),\overline{f}((\xi_2))\right)\mathrm{d}t\right),$$
$$\frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\overline{f}(x)\mathrm{d}x \leq \overline{f}(\xi_2)+\int_0^1 t\eta\left(\overline{f}(\xi_1),\overline{f}(\xi_2)\right)\mathrm{d}t. \tag{26}$$

Similarly,

$$\frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\overline{f}(x)\mathrm{d}x \leq \overline{f}(\xi_1)+\int_0^1 t\eta\left(\overline{f}(\xi_2),\overline{f}(\xi_1)\right)\mathrm{d}t. \tag{27}$$

Adding (26) and (27), we obtain

$$\frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\overline{f}(x)\mathrm{d}x \geq \frac{\overline{f}(\xi_1)+\overline{f}(\xi_2)}{2}, + \frac{1}{4}\left[\eta\left(\overline{f}(\xi_1),\overline{f}(\xi_2)\right) + \eta\left(\overline{f}(\xi_2),\overline{f}(\xi_1)\right)\right]. \tag{28}$$

Combining (20) and (21), we obtain

$$\begin{aligned}&\underline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} - \frac{p}{2\left(\xi_2^p-\xi_1^p\right)}\int_{\xi_1}^{\xi_2} x^{p-1}\eta \\ &\cdot\left(\underline{f}\left(\xi_1^p+\xi_2^p-x^p\right)^{(1/p)},\underline{f}(x)\right)\mathrm{d}x, \\ &\leq \frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\,\underline{f}(x)\mathrm{d}x, \\ &\leq \frac{\underline{f}(\xi_1)+\underline{f}(\xi_2)}{2} + \frac{1}{4}\left[\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right) + \eta\left(\underline{f}(\xi_2),\underline{f}(\xi_1)\right)\right].\end{aligned} \tag{29}$$

Combining (25) and (28), we obtain

$$\begin{aligned}&\overline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} - \frac{p}{2\left(\xi_2^p-\xi_1^p\right)}\int_{\xi_1}^{\xi_2} x^{p-1}\eta \\ &\cdot\left(\overline{f}\left(\xi_1^p+\xi_2^p-x^p\right)^{(1/p)},\overline{f}(x)\right)\mathrm{d}x, \\ &\geq \frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\overline{f}(x)\mathrm{d}x, \\ &\geq \frac{\overline{f}(\xi_1)+\overline{f}(\xi_2)}{2} + \frac{1}{4}\left[\eta\left(\overline{f}(\xi_1),\overline{f}(\xi_2)\right) \right. \\ &\left. + \eta\left(\overline{f}(\xi_2),\overline{f}(\xi_1)\right)\right].\end{aligned} \tag{30}$$

Equations (29) and (30) follows:

$$\begin{aligned}&\left[\underline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} - \frac{p}{2\left(\xi_2^p-\xi_1^p\right)}\int_{\xi_1}^{\xi_2} x^{p-1}\eta\left(\underline{f}\left(\xi_1^p+\xi_2^p-x^p\right)^{(1/p)},\underline{f}(x)\right)\mathrm{d}x,\right. \\ &\left.\underline{f}\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} - \frac{p}{2\left(\xi_2^p-\xi_1^p\right)}\int_{\xi_1}^{\xi_2} x^{p-1}\eta\left(\overline{f}\left(\xi_1^p+\xi_2^p-x^p\right)^{(1/p)},\overline{f}(x)\right)\mathrm{d}x\right], \\ &\supseteq \left[\frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\,\underline{f}(x)\mathrm{d}x, \frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\overline{f}(x)\mathrm{d}x\right], \\ &\supseteq \frac{\underline{f}(\xi_1)+\underline{f}(\xi_2)}{2} + \frac{1}{4}\left[\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right) + \eta\left(\underline{f}(\xi_2),\underline{f}(\xi_1)\right)\right], \frac{\overline{f}(\xi_1)+\overline{f}(\xi_2)}{2}, \\ &+ \frac{1}{4}\left[\eta\left(\overline{f}(\xi_1),\overline{f}(\xi_2)\right) + \eta\left(\overline{f}(\xi_2),\overline{f}(\xi_1)\right)\right].\end{aligned} \tag{31}$$

which completely follows (11) □

Remark 1. By putting $\overline{f}=\underline{f}$ and $p=1$, (11) becomes Hermite–Hadamard type inequality for η-convexity [18].

Remark 2. By putting $\overline{f}=\underline{f}$ and $\eta(\xi_1,\xi_2)=\xi_1-\xi_2$ in (11), we obtain Hermite–Hadamard type inequality for p-convexity [17].

Remark 3. By putting $\overline{f}=\underline{f}$, $p=1$ and $\eta(\xi_1,\xi_2)=\xi_1-\xi_2$ in (11), we get classical Hermite–Hadamard type inequality for convex functions.

Example 1. Consider $\eta(x,y)=x-y$, $[\xi_1,\xi_2]=[-1,1]$ and $f\colon [r,s]\longrightarrow R^+$ be defined by $f(\lambda)=[\lambda^p,4-e^{\lambda^p}]$ with p as an odd number, then we have

$$\begin{aligned}&f\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{(1/p)} - \frac{p}{2\left(\xi_2^p-\xi_1^p\right)}\int_{\xi_1}^{\xi_2} x^{p-1}\eta \\ &\cdot\left(f\left(\xi_1^p+\xi_2^p-x^p\right)^{(1/p)},f(x)\right)\mathrm{d}x = [0,3],\end{aligned} \tag{32}$$

$$\begin{aligned}\frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}f(x)\mathrm{d}x &= \frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\left[x^p,4-e^{x^p}\right]\mathrm{d}x, \\ &= \frac{p}{2}\int_{-1}^{1} x^{p-1}x^p\mathrm{d}x, \\ &\frac{p}{2}\int_{-1}^{1} x^{p-1}\left(4-e^{x^p}\right)\mathrm{d}x.\end{aligned} \tag{33}$$

Put $z=x^p$ and simplify, we get

$$\frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1} f(x)\mathrm{d}x = \frac{1}{2}\left[0,8-\left(e^1-e^{-1}\right)\right] = \left[0,4-\frac{\left(e^1-e^{-1}\right)}{2}\right], \tag{34}$$

$$\begin{aligned}&\frac{f(\xi_1)+f(\xi_2)}{2}+\frac{1}{4}[\eta(f(\xi_1),f(\xi_2))+\eta(f(\xi_2),f(\xi_1))],\\ &=\frac{\left[\xi_1^p,4-e^{\xi_1^p}\right]+\left[\xi_2^p,4-e^{\xi_2^p}\right]}{2}+\frac{1}{4}[\{f(\xi_1)-f(\xi_2)\}\\ &\quad+\{f(\xi_2)-f(a)\}],\\ &=\frac{\left[-1,4-e^{-1}\right]+\left[1,4-e^{1}\right]}{2}=\left[0,4-\frac{\left(e^1+e^{-1}\right)}{2}\right].\end{aligned} \tag{35}$$

Combining (32), (34), and (35), we get

$$[0,3]\supseteq\left[0,4-\frac{\left(e^1-e^{-1}\right)}{2}\right]\supseteq\left[0,4-\frac{\left(e^1+e^{-1}\right)}{2}\right]. \tag{36}$$

4. Fejér-Type Inequality for Interval-Valued Generalized p Convex Function

Now, we develop Fejér type inequality for interval-valued generalized p convex functions.

Theorem 4. *Let f and g be nonnegative interval-valued generalized p convex functions $\xi_1,\xi_2\in I$ $\xi_1<\xi_2$ such that $fg\in L_1[\xi_1,\xi_2]$, then*

$$\frac{p}{\xi_2^p-\xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1} f(x)g(x)\mathrm{d}x\supseteq M(\xi_1,\xi_2)+\frac{1}{2}N(\xi_1,\xi_2), \tag{37}$$

where

$$\begin{aligned}M(\xi_1,\xi_2)&=f(\xi_2)g(\xi_2)+\frac{1}{3}\eta(f(\xi_1),f(\xi_2))\eta(g(\xi_1),g(\xi_2)),\\ N(\xi_1,\xi_2)&=f(\xi_2)\eta(g(\xi_1),g(\xi_2))+g(\xi_2)\eta(f(\xi_1),f(\xi_2)).\end{aligned} \tag{38}$$

Proof. Since f and g are interval-valued generalized p convex functions, we have

$$\begin{aligned}f\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right)&\supseteq f(\xi_2)+t\eta(f(\xi_1),f(\xi_2)),\\ g\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right)&\supseteq g(\xi_2)+t\eta(g(\xi_1),g(\xi_2)),\end{aligned} \tag{39}$$

for all $t\in[\xi_1,\xi_2]$. Since f and g are nonnegative,

$$\begin{aligned}&f\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right)g\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right),\\ &\supseteq f(\xi_2)g(\xi_2)+tf(\xi_2)\eta(g(\xi_1),g(\xi_2))\\ &\quad+tg(\xi_2)\eta(f(\xi_1),f(\xi_2)),\\ &\quad+t^2\eta(f(\xi_1),f(\xi_2))\eta(g(\xi_1),g(\xi_2)).\end{aligned} \tag{40}$$

By the definition of interval, we have

$$\begin{aligned}&\left[\underline{f}\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right)\underline{g}\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right),\right.\\ &\left.\overline{f}\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right)\overline{g}\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right)\right],\\ &\supseteq\left[\underline{f}(\xi_2)\underline{g}(\xi_2)+t\underline{f}(\xi_2)\eta\left(\underline{g}(\xi_1),\underline{g}(\xi_2)\right)\right.\\ &\quad+t\underline{g}(\xi_2)\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right)\\ &\quad\left.+t^2\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right)\eta\left(\underline{g}(\xi_1),\underline{g}(\xi_2)\right)\right],\\ &\left[\overline{f}(\xi_2)\overline{g}(\xi_2)+t\overline{f}(\xi_2)\eta\left(\underline{g}(\xi_1),\overline{g}(\xi_2)\right)\right.\\ &\quad+t\overline{g}(\xi_2)\eta\left(\overline{f}(\xi_1),\overline{f}(\xi_2)\right)\\ &\quad\left.+t^2\eta\left(\overline{f}(\xi_1),\overline{f}(\xi_2)\right)\eta\left(\overline{g}(\xi_1),\overline{g}(\xi_2)\right)\right].\end{aligned} \tag{41}$$

It follows

$$\begin{aligned}&\underline{f}\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right)\underline{g}\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right),\\ &\le\underline{f}(\xi_2)\underline{g}(\xi_2)+t\underline{f}(\xi_2)\eta\left(\underline{g}(\xi_1),\underline{g}(\xi_2)\right)\\ &\quad+t\underline{g}(\xi_2)\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right),\\ &\quad+\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right)\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right)\mathrm{d}t.\end{aligned} \tag{42}$$

$$\begin{aligned}&\overline{f}\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right)\overline{g}\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right),\\ &\le\overline{f}(\xi_2)\overline{g}(\xi_2)+t\overline{f}(\xi_2)\eta\left(\overline{g}(\xi_1),\overline{g}(\xi_2)\right)\\ &\quad+t\overline{g}(\xi_2)\eta\left(\overline{f}(\xi_1),\overline{f}(\xi_2)\right),\\ &\quad+t^2\eta\left(\overline{f}(\xi_1),\overline{f}(\xi_2)\right)\eta\left(\overline{f}(\xi_1),\overline{f}(\xi_2)\right)\mathrm{d}t.\end{aligned} \tag{43}$$

Integrating (42) over $(0,1)$, we obtain the following inequality:

$$\begin{aligned}&\int_0^1\underline{f}\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right)\underline{g}\left(\left[t\xi_1^p+(1-t)\xi_2^p\right]^{(1/p)}\right)\mathrm{d}t,\\ &\le\int_0^1\underline{f}(\xi_2)\underline{g}(\xi_2)\mathrm{d}t+\int_0^1 t\underline{f}(\xi_2)\eta\left(\underline{g}(\xi_1),\underline{g}(\xi_2)\right)\mathrm{d}t\\ &\quad+\int_0^1 t\underline{g}(\xi_2)\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right)\mathrm{d}t,\\ &\quad+\int_0^1 t^2\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right)\eta\left(\underline{f}(\xi_1),\underline{f}(\xi_2)\right)\mathrm{d}t.\end{aligned} \tag{44}$$

Setting $x=[t\xi_1^p+(1-t)\xi_2^p]^{(1/p)}$, we get

$$\frac{p}{\xi_2^p - \xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\,\underline{f}(x)\,\underline{g}(x)\mathrm{d}x,$$
$$\le \underline{f}(\xi_2)\,\underline{g}(\xi_2) + \frac{1}{2}\,\underline{f}(\xi_2)\eta\big(\underline{g}(\xi_1), \underline{g}(\xi_2)\big)$$
$$+\frac{1}{2}\,\underline{g}(\xi_2)\eta\big(\underline{f}(\xi_1), \underline{f}(\xi_2)\big),$$
$$+\frac{1}{3}\eta\big(\underline{f}(\xi_1), \underline{f}(\xi_2)\big)\eta\big(\underline{g}(\xi_1), \underline{g}(\xi_2)\big). \tag{45}$$

Integrating (43) over (0, 1), we get

$$\int_0^1 \overline{f}\Big(\big[t\xi_1^p + (1-t)\xi_2^p\big]^{(1/p)}\Big)\overline{g}\Big(\big[t\xi_1^p + (1-t)\xi_2^p\big]^{(1/p)}\Big)\mathrm{d}t,$$
$$\ge \int_0^1 \overline{f}(\xi_2)\overline{g}(\xi_2)\mathrm{d}t + \int_0^1 t\overline{f}(\xi_2)\eta(\overline{g}(\xi_1), \overline{g}(\xi_2))\mathrm{d}t$$
$$+\int_0^1 t\overline{g}(\xi_2)\eta\big(\overline{f}(\xi_1), \overline{f}(\xi_2)\big)\mathrm{d}t,$$
$$+\int_0^1 t^2\eta\big(\overline{f}(\xi_1), \overline{f}(\xi_2)\big)\eta\big(\overline{f}(\xi_1), \overline{f}(\xi_2)\big)\mathrm{d}t. \tag{46}$$

Setting $x = [t\xi_1^p + (1-t)\xi_2^p]^{(1/p)}$, we get

$$\frac{p}{\xi_2^p - \xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\overline{f}(x)\overline{g}(x)\mathrm{d}x,$$
$$\le \overline{f}(\xi_2)\overline{g}(\xi_2) + \frac{1}{2}\overline{f}(\xi_2)\eta(\overline{g}(\xi_1), \overline{g}(\xi_2))$$
$$+\frac{1}{2}\overline{g}(\xi_2)\eta\big(\overline{f}(\xi_1), \overline{f}(\xi_2)\big),$$
$$+\frac{1}{3}\eta\big(\overline{f}(\xi_1), \overline{f}(\xi_2)\big)\eta(\overline{g}(\xi_1), \overline{g}(\xi_2)). \tag{47}$$

Using inequality, (47) we get

$$\left[\frac{p}{\xi_2^p - \xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\,\underline{f}(x)\,\underline{g}(x)\mathrm{d}x, \frac{p}{\xi_2^p - \xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\overline{f}(x)\overline{g}(x)\mathrm{d}x\right],$$
$$\supseteq \Big[\underline{f}(\xi_2)\,\underline{g}(\xi_2) + \frac{1}{2}\,\underline{f}(\xi_2)\eta\big(\underline{g}(\xi_1), \underline{g}(\xi_2)\big) + \frac{1}{2}\,\underline{g}(\xi_2)\eta\big(\underline{f}(\xi_1), \underline{f}(\xi_2)\big),$$
$$+\frac{1}{3}\eta\big(\underline{f}(\xi_1), \underline{f}(\xi_2)\big)\eta\big(\underline{g}(\xi_1), \underline{g}(\xi_2)\big), \overline{f}(\xi_2)\overline{g}(\xi_2) + \frac{1}{2}\overline{f}(\xi_2)\eta(\overline{g}(\xi_1), \overline{g}(\xi_2)),$$
$$+\frac{1}{2}\overline{g}(\xi_2)\eta\big(\overline{f}(\xi_1), \overline{f}(\xi_2)\big) + \frac{1}{3}\eta\big(\overline{f}(\xi_1), \overline{f}(\xi_2)\big)\eta(\overline{g}(\xi_1), \overline{g}(\xi_2))\Big], \tag{48}$$
$$\frac{p}{\xi_2^p - \xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1} f(x)g(x)\mathrm{d}x,$$
$$\supseteq f(\xi_2)g(\xi_2) + \frac{1}{2}f(\xi_2)\eta(g(\xi_1), g(\xi_2)) + \frac{1}{2}g(\xi_2)\eta(f(\xi_1), f(\xi_2)),$$
$$+\frac{1}{3}\eta(f(\xi_1), f(\xi_2))\eta(g(\xi_1), g(\xi_2)).$$

Then, we obtain the inequality (37). □

Remark 4. If we put $\overline{f} = \underline{f}$, $p = 1$ and $\eta(x, y) = x - y$ in (37), then it reduces to classical convex functions.

5. Fractional Hermite–Hadamard-Type Inequalities for Interval-Valued Generalized p Convex Functions

The fractional inequalities has applications in every field of science and engineering. The new fractional integral inequalities in analysis are always appreciable. Because of the wide applications of Hermite–Hadamard-type inequalities and fractional integrals, many researchers extended their studies to Hermite–Hadamard-type inequality involving fractional integral inequalities. For fractional integral inequalities for interval-valued function, we suggest the reader to refer [31, 32].

Definition 7 (see [33–35]). Let $\phi \in L[a, b]$. The right-hand side and left-hand side RiemannLiouville fractional integral of order $\alpha > 0$ with $b > a > 0$ are defined by

$$J_{a+}^{\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-k)^{\alpha-1}\phi(k)\mathrm{d}k, \quad x > a,$$
$$J_{b-}^{\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (k-x)^{\alpha-1}\phi(k)\mathrm{d}k, \quad x < b, \tag{49}$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined as $\Gamma\Gamma(\alpha) = \int_0^{\infty} e^{-k}k^{\alpha-1}\mathrm{d}k$.

It is to be noted that $J_{\alpha^+}^{0}\phi(x) = J_{b^-}^{0}\phi(x) = \phi(x)$

Reimann integral is reduced as classical integral for $\alpha = 1$.

Definition 8. let $p \in \mathbb{R}/0$. A function w: $[a,b] \subset (0,\infty) \longrightarrow \mathbb{R}$ is said to be p-symmetric with respect to $[(a^p + b^p/2)]^{(1/p)}$ if $w(x) = w([a^p + b^p - x^p]^{(1/p)})$ and holds for all $x, y \in I = [a,b]$.

Following lemma will help us in obtaining our fractional integrals inequalities which can be found in [36].

Lemma 1. *Let* $p \in (\mathbb{R}/0)$, $\alpha > 0$ *and* w: $[a,b] \subset (0,\infty) \longrightarrow \mathbb{R}$ *is integrable, p-symmetric with respect to* $[(a^p + b^p/2)]^{(1/p)}$:

(1) If $p > 0$,

$$\begin{aligned} J_{a^p+}^{\alpha}(\mathrm{wog})(b^p) &= J_{b^p-}^{\alpha}(\mathrm{wog})(a^p) \\ &= \frac{1}{2}\left[J_{a^p+}^{\alpha}(\mathrm{wog})(b^p) + J_{b^p-}^{\alpha}(\mathrm{wog})(a^p)\right]. \end{aligned} \tag{50}$$

with $g(x) = x^{(1/p)}$, $x \in [a^p, b^p]$.

(2) If $p < 0$,

$$\begin{aligned} J_{b^p+}^{\alpha}(\mathrm{wog})(a^p) &= J_{a^p-}^{\alpha}(\mathrm{wog})(a^p) \\ &= \frac{1}{2}\left[J_{b^p+}^{\alpha}(\mathrm{wog})(b^p) + J_{a^p-}^{\alpha}(\mathrm{wog})(b^p)\right], \end{aligned} \tag{51}$$

with $g(x) = x^{(1/p)}$, $x \in [b^p, a^p]$.

Now, we are ready to develop the Fractional Hermite–Hadamard-type inequalities for interval-valued generalized p convex functions.

Theorem 5. *Let* ϕ: $I \longrightarrow \mathbb{R}$ *be generalized p convex function and provided* η *is bounded above on* $\phi(I)\times\phi(I)$ *and* $\phi \in L[a,b]$. *Then, following fractional integral inequality holds, if* $p \in (\mathbb{R}/(0))$ *and* $p > 0$:

$$\begin{aligned} &\phi\left(\frac{a^p + b^p}{2}\right)^{(1/p)} - 2M_\eta, \\ &\supseteq \frac{\Gamma(\alpha+1)}{(b^p - a^p)^{\alpha}2^{1-\alpha}}\left[J_{(a^p+b^p/2)+}^{\alpha}\phi og(b^p) + J_{(a^p+b^p/2)-}^{\alpha}\phi og(a^p)\right], \\ &\supseteq \frac{\phi(a) + \phi(b)}{2} + 2N_\eta. \end{aligned} \tag{52}$$

Proof. Let ϕ be a generalized p convex function with $p \geq 0$ and η is bounded above by M_η.

Take $x = (ka^p + (1-k)b^p)^{(1/p)}$ and $y = (kb^p + (1-k)a^p)^{(1/p)}$.

Since

$$\phi\left(\frac{x^p + y^p}{2}\right)^{(1/p)} - M_\eta \supseteq \frac{\phi(x) + \phi(y)}{2} + M_\eta, \tag{53}$$

(53) becomes

$$\begin{aligned} \phi\left(\frac{a^p + b^p}{2}\right)^{(1/p)} - M_\eta &\supseteq \frac{\phi(ka^p + (1-k)b^p)^{(1/p)}}{2} \\ &+ \frac{\phi(kb^p + (1-k)a^p)^{(1/p)}}{2} + M_\eta. \end{aligned} \tag{54}$$

Multiplying both sides of (54) by $k^{\alpha-1}$ and then integrating the resulting inequality with respect to k over $[0, (1/2)]$, we obtain

$$\begin{aligned} &\int_0^{(1/2)}\phi\left(\frac{a^p + b^p}{2}\right)^{(1/p)}k^{\alpha-1}\mathrm{d}k - \int_0^{(1/2)}M_\eta k^{\alpha-1}\mathrm{d}k, \\ &\supseteq \frac{1}{2}\int_0^{(1/2)}\phi(ka^p + (1-k)b^p)^{(1/p)}k^{\alpha-1}\mathrm{d}k \\ &+ \frac{1}{2}\int_0^{(1/2)}\phi(kb^p + (1-k)a^p)^{(1/p)}k^{\alpha-1}\mathrm{d}k, \\ &+ \int_0^{(1/2)}M_\eta k^{\alpha-1}\mathrm{d}k. \end{aligned} \tag{55}$$

By definition of RiemannLiouville integrable function with $g(x) = x^{(1/p)}$, we obtain

$$\begin{aligned} \phi\left(\frac{a^p + b^p}{2}\right)^{(1/p)} - 2M_\eta &\supseteq \frac{\Gamma(\alpha+1)}{(b^p - a^p)^{\alpha}2^{1-\alpha}} \\ &\cdot\left[J_{(a^p+b^p/2)+}^{\alpha}\phi og(b^p) + J_{(a^p+b^p/2)-}^{\alpha}\phi og(a^p)\right]. \end{aligned} \tag{56}$$

which is the left-hand side of theorem (56).

To prove the right-hand side, we take $x = (ka^p + (1-k)b^p)^{(1/p)}$ and $y = (kb^p + (1-k)a^p)^{(1/p)}$:

$$\phi\left(ka^p + (1-k)b^p\right)^{(1/p)} \supseteq \phi(b) + k\eta(\phi(a), \phi(b)), \tag{57}$$

$$\phi\left(kb^p + (1-k)a^p\right)^{(1/p)} \supseteq \phi(a) + k\eta(\phi(b), \phi(a)). \tag{58}$$

Adding the (57) and (58) and multiplying the resulting inequality with $2k^{\alpha-1}$ and integrating with respect to k over $[0, (1/2)]$ we obtain

$$\begin{aligned} &2\int_0^{(1/2)} \phi\left(kb^p + (1-k)a^p\right)^{(1/p)} k^{\alpha-1} \mathrm{d}k \\ &\quad + 2\int_0^{(1/2)} \phi\left(kb^p + (1-k)a^p\right)^{(1/p)} k^{\alpha-1} \mathrm{d}k, \\ &\supseteq 2\int_0^{(1/2)} \phi(a) k^{\alpha-1} \mathrm{d}k + 2\int_0^{(1/2)} \phi(b) k^{\alpha-1} \mathrm{d}k, \\ &\quad + \frac{(\eta(\phi(b), \phi(a)) + \eta(\phi(a), \phi(a)))}{2^\alpha(\alpha+1)}. \end{aligned} \tag{59}$$

By definition of RiemannLiouville integrable function, we get

$$\begin{aligned} &\frac{2\Gamma(\alpha)}{\left(b^p - a^p\right)^\alpha}\left[J^\alpha_{(a^p+b^p/2)+}\phi og\left(b^p\right) + J^\alpha_{(a^p+b^p/2)-}\phi og\left(a^p\right)\right] \\ &\supseteq \frac{\phi(a)}{\alpha 2^{\alpha-1}} + \frac{\phi(b)}{\alpha 2^{\alpha-1}} + \frac{\alpha N_\eta}{(\alpha+1)2^{\alpha-2}}. \end{aligned} \tag{60}$$

Rearranging the above inequality, we get the right-hand side:

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{\left(b^p - a^p\right)^\alpha 2^{1-\alpha}}\left[J^\alpha_{(a^p+b^p/2)+}\phi og\left(b^p\right) + J^\alpha_{(a^p+b^p/2)-}\phi og\left(a^p\right)\right] \\ &\supseteq \frac{\phi(a) + \phi(b)}{2} + 2N_\eta. \end{aligned} \tag{61}$$

This completes the proof. □

Remark 5. If we put $p = 1, \eta(x, y) = x - y$, and $\phi = \overline{\phi}$, then we will get Hermite–Hadamard-type inequality for fractional function for classical convex function [37].

6. Conclusions

The convex functions and fractional calculus play an important role in applied sciences [38–43]. Here, the new interval-valued generalized convex functions are introduced. By using the notion of interval-valued generalized p convex functions and some auxiliary results of interval analysis, some new Hermite–Hadamard- and Fejér-type inequalities are presented. Our results can be considered as generalization of many existing results. Moreover, fractional integral inequality for this generalization is also established.

Data Availability

The data used to support the article are available within the article.

Conflicts of Interest

The authors declare that do not have any conflicts of interest.

Authors' Contributions

All the authors contributed equally to this paper.

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