Hindawi
Journal of Mathematics
Volume 2020, Article ID 4639260, 4 pages
https://doi.org/10.1155/2020/4639260

Research Article

A Note on Edge-Group Choosability of Planar Graphs without 5-Cycles

Amir Khamseh

Department of Mathematics, Kharazmi University, 15719-14911 Tehran, Iran

Correspondence should be addressed to Amir Khamseh; amirkhamseh114@gmail.com

Received 20 June 2020; Revised 11 August 2020; Accepted 11 October 2020; Published 22 October 2020

Academic Editor: Andrei V. Kelarev

This paper is devoted to a study of the concept of edge-group choosability of graphs. We say that G is edge-k-group choosable if its line graph is k-group choosable. In this paper, we study an edge-group choosability version of Vizing conjecture for planar graphs without 5-cycles and for planar graphs without noninduced 5-cycles (2010 Mathematics Subject Classification: 05C15, 05C20).

1. Introduction

We consider only simple graphs in this paper unless otherwise stated. For a graph G, we denote its vertex set, edge set, minimum degree, and maximum degree by $V(G)$, $E(G)$, $\delta(G)$, and $\Delta(G)$, respectively. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane. We denote the set of faces of a plane graph G by $F(G)$. For a plane graph G and $f \in F(G)$, we write $f = [u_1\ u_2\ \cdots\ u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk of f enumerated clockwise. The *degree of a face* is the number of edges on the boundary walk. Let $d_G(x)$, or simply $d(x)$, denote the degree of a vertex (or face) x in G. A vertex (or face) of degree k is called a *k-vertex* (or *k-face*). For $v \in V(G)$, $N_G(v)$ is the set of all vertices of G that are adjacent to v in G. We denote the line graph of a graph G by $\ell(G)$.

A *k-coloring* of a graph G is a mapping ϕ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every edge xy. A graph G is *k-colorable* if it has a k-coloring. The *chromatic number* $\chi(G)$ is the smallest integer k such that G is k-colorable. A mapping L is said to be a *list assignment* for G if it supplies a list $L(v)$ of possible colors to each vertex v. A *k-list assignment* of G is a list assignment L with $|L(v)| = k$ for each vertex $v \in V(G)$. If G has some k-coloring ϕ such that $\phi(v) \in L(v)$ for each vertex v, then G is *L-colorable* or ϕ is an *L-coloring* of G. We say that G is *k-choosable* if it is L-colorable for every *k-list assignment* L. The *choice number* or *list chromatic number* $\chi_l(G)$ is the smallest k such that G is *k-choosable*. For edge-colorings of G, we can define analogous notions such as *edge-k-colorability*, *edge-k-choosability*, the *chromatic index* $\chi'(G)$, and the *choice index* $\chi'_l(G)$. Clearly, we have $\chi'(G) = \chi(\ell(G))$ and $\chi'_l(G) = \chi_l(\ell(G))$. The notion of list coloring of graphs has been introduced by Erdős et al. [1] and Vizing [2]. The following conjecture, which first appeared in [3], is well-known as the List Edge Coloring Conjecture.

Conjecture 1. *If G is a multigraph, then $\chi'_l(G) = \chi'(G)$.*

Although Conjecture 1 has been proved for a few special cases such as bipartite multigraphs, complete graphs of odd order, multicircuits, graphs with $\Delta(G) \geq 12$ that can be embedded in a surface of nonnegative characteristic, and outerplanar graphs, it is regarded as very difficult. Vizing proposed the following weaker conjecture (see [4]).

Conjecture 2. *Every graph G is edge-$(\Delta(G)+1)$-choosable.*

Assume A is an Abelian group, and $F(G, A)$ denotes the set of all functions $f\colon E(G) \longrightarrow A$. Consider an arbitrary orientation of G. The graph G is A-colorable if, for every $f \in F(G, A)$, there is a vertex coloring $c\colon V(G) \longrightarrow A$ such that $c(x) - c(y) \neq f(xy)$ for each directed edge from x to y. The *group chromatic number* of G, $\chi_g(G)$, is the minimum k such that G is A-colorable for any Abelian group A of order at least k. The notion of group coloring of graphs was first introduced by Jaeger et al. [5].

The concept of *group choosability* was introduced by Král and Nejedlý in [6], and some first results in this area were obtained in [7, 8]. Let A be an Abelian group of order at least k and $L\colon V(G) \longrightarrow 2^A$ be a list assignment of G. For $f \in F(G, A)$, an (A, L, f)-*coloring* under an orientation D of G is an L-coloring $c\colon V(G) \longrightarrow A$ such that $c(x) - c(y) \neq f(xy)$ for every edge $e = xy$, where e is directed from x to y. If for each $f \in F(G, A)$, there exists an (A, L, f)-coloring for G, and then we say that G is (A, L)-*colorable*. The graph G is k-group choosable if G is (A, L)-colorable for each Abelian group A of order at least k and any *k-list assignment* $L\colon V(G) \longrightarrow \binom{A}{k}$. The minimum k for which G is k-group choosable is called the *group choice number* of G and is denoted by $\chi_{gl}(G)$. It is clear that the concept of group choosability is independent of the orientation on G. Graph G is called *edge-k-group choosable* if its line graph is k-group choosable. The *group-choice index* of G, $\chi'_{gl}(G)$, is the smallest k such that G is edge-*k-group choosable*, i.e., $\chi'_{gl}(G) = \chi_{gl}(\ell(G))$. It is easily seen that an even cycle is not edge-*2-group choosable*. This example shows that $\chi'_{gl}(G)$ is not generally equal to $\chi'(G)$. But we can extend the Vizing conjecture as follows.

Conjecture 3. *If G is a multigraph, then $\chi'_{gl}(G) \leq \Delta(G) + 1$.*

Since $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, as a sufficient condition, we have the following weaker conjecture.

Conjecture 4. *If G is a multigraph, then $\chi'_{gl}(G) \leq \chi'(G) + 1$.*

Some early results concerning edge-group choosability of graphs were presented by the authors in a series of lectures in Annual Iranian Mathematical Conferences (see [9–11]). Conjecture 3 has been proved for graphs with maximum degree $\Delta \leq 3$ [9], planar graphs with maximum degree $\Delta \geq 11$ [9], planar graphs without 4-cycles with maximum degree $\Delta \geq 5$ [11], outerplanar graphs [12], simple series-parallel graphs [12], $(K_2^c \wedge (K_1 \cup K_2))$-minor-free graphs [12], and planar graphs with maximum degree $\Delta(G) = 4$ that has no cycles of length from 4 to 14 [10]. For further reference, we add here some related details.

Theorem 1 (see [9]). *Let l be a natural number, v be a vertex of degree at most 2 of G, and e be an edge incident to v. If $\chi'_{gl}(G - e) \leq \Delta(G) + l$, then $\chi'_{gl}(G) \leq \Delta(G) + l$.*

Theorem 2 (see [9]). *Let G be a graph with $\chi'_{gl}(G - e) < \chi'_{gl}(G)$, for each $e \in E(G)$. Then, $\delta(\ell(G)) \geq \chi'_{gl}(G) - 1$.*

Theorem 3 (see [9]). *Let G be a graph with maximum degree $\Delta(G)$. If $\Delta(G) \leq 3$, then $\chi'_{gl}(G) \leq \Delta(G) + 1$, and if $\Delta(G) = 4$, then $\chi'_{gl}(G) \leq 6$.*

Theorem 4 (see [7])

(a) Let P_n and C_n denote a path and a cycle of length n, respectively. Then, $\chi_{gl}(P_n) = 2$ and $\chi_{gl}(C_n) = 3$.

(b) For any connected simple graph G, we have $\chi_{gl}(G) \leq \Delta(G) + 1$, with equality holds if and only if G is either a cycle or a complete graph.

Immediately from Theorem 4, we see that $\chi'_{gl}(P_n) = \Delta(P_n) = 2$ and $\chi'_{gl}(C_n) = \Delta(C_n) + 1 = 3$. In this paper, we show that any planar graph G without 5-cycles with maximum degree Δ is edge-$(\Delta + 2)$-group choosable. If in addition $\Delta(G) \geq 6$, we can show that G is edge-$(\Delta + 1)$-group choosable. This proves in advance that Conjecture 3 and, consequently, Conjecture 4 holds for this class of planar graphs. Moreover, we show that if G is a planar graph without noninduced 5-cycles, then $\chi'_{gl}(G) \leq \max\{7, \Delta + 2\}$.

2. Main Results

First we need a lemma, which we will discuss below. It is a structural lemma for plane graphs without 5-cycles.

Lemma 1 (see [13]). *If a plane graph G with $\delta(G) \geq 3$ has no five cycles, then there exists an edge xy of G such that $d(x) = 3$ and $d(y) \leq 5$.*

Note that G is a minimal counterexample to a theorem if G is a counterexample, that is, G satisfies the hypotheses but not the conclusion of the theorem, and there is no counterexample G' satisfying either $|V(G')| < |V(G)|$ or $|V(G')| = |V(G)|$ and $|E(G')| < |E(G)|$.

Theorem 5. *If G is a planar graph without 5-cycles with maximum degree Δ, then G is edge-$(\Delta + 2)$-group choosable.*

Proof. We saw in Theorem 4 that if P_n and C_n denote a path and a cycle of length n, respectively, then $\chi_{gl}(P_n) = 2$ and $\chi_{gl}(C_n) = 3$. Moreover, for any connected simple graph G, we have $\chi_{gl}(G) \leq \Delta(G) + 1$, with equality holds if and only if G is either a cycle or a complete graph. Immediately, we see that $\chi'_{gl}(P_n) = \Delta(P_n) = 2$ and $\chi'_{gl}(C_n) = \Delta(C_n) + 1 = 3$. Hence, if $\Delta(G) \leq 3$, then $\chi'_{gl}(G) \leq \Delta(G) + 1$, and if $\Delta(G) = 4$, then $\chi'_{gl}(G) \leq 6$. Here, we used the observation that, for a connected graph G, if $\Delta(G) = 1$, then $G = P_2$; if $\Delta(G) = 2$, then $G = P_n$ or $G = C_n$; if $\Delta(G) \leq 3$, then $\Delta(\ell(G)) \leq 4$; and if $\Delta(G) \leq 4$, then $\Delta(\ell(G)) \leq 6$. Now, let G be a minimal counterexample to Theorem 5 for some Abelian group A with $|A| \geq \Delta(G) + 2$, a $(\Delta(G) + 2)$-list assignment $L\colon V(\ell(G)) \longrightarrow \binom{A}{\Delta(G) + 2}$ and $f \in F(\ell(G), A)$. Then, G is connected, $\delta(G) \geq 3$, and $\Delta(G) \geq 5$. By Lemma 1, there exists a vertex $e \in V(\ell(G))$ with $d_{\ell(G)}(e) \leq 6$. Suppose that $G' = G - e$. Then, $\ell(G') = \ell(G) - e$, and since $\chi'_{gl}(G') \leq \Delta + 2$, there exists an (A, L, f)-coloring $c\colon V(\ell(G')) \longrightarrow A$. For each $e' \in N_{\ell(G)}(e)$ we can consider, without loss of generality, ee' to be directed from e to e'. Then, since $|L(e)| = \Delta + 2 \geq 7$ and $d_{\ell(G)}(e) \leq 6$, $|L(e) - \{c(e') + f(ee')\colon e' \in N_{\ell(G)}(e)\}| \geq 1$. In other words, there is at least one color available to color e. Thus, we can color all edges of G. This contradiction completes the proof of theorem.

The above proof shows that the only critical case is $\Delta = 5$. If remove it, we can prove a stronger result.

Theorem 6. *If G is a planar graph with maximum degree $\Delta \geq 6$ and without 5-cycles, then $\chi'_{gl}(G) \leq \Delta + 1$.*

Proof. Let G be a minimal counterexample to this theorem for some Abelian group A with $|A| \geq \Delta(G) + 1$, a $(\Delta(G) + 1)$-list assignment $L: V(\ell(G)) \longrightarrow \binom{A}{\Delta(G)+1}$ and $f \in F(\ell(G), A)$. Then, $\delta(G) \geq 3$. By Lemma 1, there exists a vertex $e \in V(\ell(G))$ with $d_{\ell(G)}(e) \leq 6$. Suppose that $G' = G - e$. Then, $\ell(G') = \ell(G) - e$, and since $\chi'_{gl}(G') \leq \Delta + 1$, there exists an (A, L, f)-coloring $c: V(\ell(G')) \longrightarrow A$. For each $e' \in N_{\ell(G)}(e)$, we can consider, without loss of generality, ee' to be directed from e to e'. Then, since $|L(e)| = \Delta + 1 \geq 7$ and $d_{\ell(G)}(e) \leq 6$, $|L(e) - \{c(e') + f(ee'): e' \in N_{\ell(G)}(e)\}| \geq 1$. In other words, there is at least one color available to color e. Thus, we can color all edges of G. This contradiction completes the proof of theorem.

The structure of planar graphs without noninduced 5-cycles is given in the following lemma.

Lemma 2 (see [14]). *Let G be a planar graph without noninduced 5-cycles. Then, G contains one of the following configurations:*

(1) An edge uv with $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$

(2) An even cycle C: $v_1, v_2, \ldots, v_{2n}$ with $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$ and $d(v_2) = d(v_4) = \cdots = d(v_{2n}) = \Delta(G)$

Theorem 7. *If G is a planar graph without noninduced 5-cycles, then*

$$\chi'_{gl}(G) \leq \max\{7, \Delta + 2\}. \tag{1}$$

Proof. Using Lemma 2, the proof is straightforward and is similar to the proof of Theorem 5. We leave the details to the reader.

If a planar graph G without noninduced 5-cycles in addition contains no even cycles, we can replace $\Delta + 2$ by $\Delta + 1$ in Theorem 7.

Theorem 8. *If G is a planar graph without noninduced 5-cycles and without even cycle C: $v_1, v_2, \ldots, v_{2n}$ with $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$ and $d(v_2) = d(v_4) = \cdots = d(v_{2n}) = \Delta(G)$, then $\chi'_{gl}(G) \leq \max\{7, \Delta + 1\}$.*

Proof. Let $k = \max\{7, \Delta + 1\}$ and G be a minimal counterexample to this theorem for some Abelian group A with $|A| \geq k$, a *k-list assignment* $L: V(\ell(G)) \longrightarrow \binom{A}{k}$ and $f \in F(\ell(G), A)$. By Theorem 1 and Lemma 2, there exists a vertex $e \in V(\ell(G))$ with $d_{\ell(G)}(e) \leq \Delta$. Suppose that $G' = G - e$. Then, $\ell(G') = \ell(G) - e$, and since $\chi'_{gl}(G') \leq \Delta + 1$, there exists an (A, L, f)-coloring $c: V(\ell(G')) \longrightarrow A$. For each $e' \in N_{\ell(G)}(e)$, we can consider, without loss of generality, ee' to be directed from e to e'. Then, since $|L(e)| \geq \Delta + 1$ and $d_{\ell(G)}(e) \leq \Delta$, $|L(e) - \{c(e') + f(ee'): e' \in N_{\ell(G)}(e)\}| \geq 1$. In other words, there is at least one color available to color e. Thus, we can color all edges of G. This contradiction completes the proof of theorem.

Data Availability

No data were used to support this study.

Disclosure

This research was performed as part of the employment of the author at Kharazmi University.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

References

[1] P. Erdős, A. L. Robin, and H. Taylor, "Choosability in graphs," in *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing, Congress Number 26*, pp. 125–157, Arcata, CA, USA, September 1979.

[2] V. G. Vizing, "Coloring the vertices of a graph in prescribed colors (in Russian)," *Methody Diskret. Analiz*, vol. 29, pp. 3–10, 1976.

[3] B. Bollobás and A. J. Harris, "List-colourings of graphs," *Graphs and Combinatorics*, vol. 1, no. 1, pp. 115–127, 1985.

[4] A. V. Kostochka, "List edge chromatic number of graphs with large girth," *Discrete Mathematics*, vol. 101, no. 1-3, pp. 189–201, 1992.

[5] F. Jaeger, N. Linial, C. Payan, and M. Tarsi, "Group connectivity of graphs—a nonhomogeneous analogue of nowhere-zero flow properties," *Journal of Combinatorial Theory, Series B*, vol. 56, no. 2, pp. 165–182, 1992.

[6] D. Král and P. Nejedlý, "Group coloring and list group coloring are Π_2^P-complete," *Lecture Notes in Computer Science*, vol. 3153, pp. 274–287, 2004.

[7] H. Chuang, H.-J. Lai, G. R. Omidi, and N. Zakeri, "On group choosability of graphs I," *Ars Combinatoria*, vol. 126, pp. 195–209, 2016.

[8] G. R. Omidi, "A note on group choosability of graphs with girth at least 4," *Graphs and Combinatorics*, vol. 27, no. 2, pp. 269–273, 2011.

[9] A. Khamseh, "Edge group choosability of planar graphs with maximum degree 11," in *Proceedings of the 46th Annual Iranian Mathematical Conferences*, pp. 25–28, Yazd University, Yazd, Iran, August 2015.

[10] A. Khamseh, "Planar graphs with maximum degree $\Delta = 4$ without cycles of length from 4 to 14 are edge-group

choosable," in *Proceedings of the 50th Annual Iranian Mathematical Conferences*, pp. 26–29, Shiraz University, Shiraz, Iran, August 2019.

[11] A. Khamseh and G. R. Omidi, "Edge group choosability of planar graphs without 4-cycles," in *Proceedings of the 49th Annual Iranian Mathematical Conferences*, pp. 23–26, Iran University of Science and Technology, Tehran, Iran, August 2018.

[12] A. Khamseh, "Edge-group choosability of outerplanar and near-outerplanar graphs," *Transactions on Combinatorics*, vol. 9, pp. 211–216, 2020.

[13] W. Wang and K.-W. Lih, "Choosability and edge choosability of planar graphs without five cycles," *Applied Mathematics Letters*, vol. 15, no. 5, pp. 561–565, 2002.

[14] J. Cai, J. Hou, X. Zhang, and G. Liu, "Edge-choosability of planar graphs without non-induced 5-cycles," *Information Processing Letters*, vol. 109, no. 7, pp. 343–346, 2009.