Research Article

Some New Connection Relations Related to Classical Orthogonal Polynomials

Wathek Chammam ^{1,2,3} and Wasim Ul-Haq ^{1,4}

¹Department of Mathematics, College of Science Al-Zulfi, Majmaah University, P.O. Box 66, Al-Majmaah 11952, Saudi Arabia

²Mathematics and Applications Laboratory, Faculty of Science, Gabès University, 6072 Gabès, Tunisia

³Department of Electro Mechanics, Higher Institute of Industrial Systems of Gabès, 6072 Gabès, Tunisia

⁴Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22010, Pakistan

Correspondence should be addressed to Wathek Chammam; w.chammam@mu.edu.sa

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In this paper, we deal with a problem of positivity of linear functionals in the linear space $\mathbb{P}$ of polynomials in one variable with complex coefficients. Some new results of connection relations between the corresponding sequences of monic orthogonal polynomials of classical character are established.

1. Introduction

The criterion of positivity is important in special functions theory, such as approximation theory, moment problem, and orthogonal polynomials and applications, among others, because the solutions of many problems depend on the determination of when a specific function is positive or nonnegative. For example, several authors have focused on the study of linear positive operators and their applications in problems and inequalities. Here, we can mention Aloui [1, 2], Bennett [3], and others. Also, we quote the work of Chen et al. [4] in the total positivity of Riordan arrays.

The linear functional $\mathcal{U}$ is positive definite if and only if $\Delta_n(\mathcal{U}) > 0$, for every integer $n \geq 0$, where $\Delta_n(\mathcal{U})$ is the Hankel determinants of order n of $\mathcal{U}$ (see [5]). In order to provide a construction process of positive-definite linear functional from positive-definite linear data, Sfaxi has proved a new construction process and he also gave an example a second-order positive-definite linear functional from a Laguerre positive-definite linear functional [6].

The outline of this paper is as follows. In Section 2, we introduce the basic background and notations to be used

throughout the paper. In Section 3, we present our main results. Indeed, using Oppenheim's inequality we prove that, for any pair $(\mu, c) \in]0, +\infty[\times \mathbb{R}$ and any positive-definite linear functional $\mathcal{U}$, the linear functional $\mathcal{U}(\mu, c)$ given by $(\mu - 1)\mathcal{U}(\mu, c) - (x - c)(\mathcal{U}(\mu, c))' = \mu\mathcal{U}$ is also positive definite. Finally, some new results of connection relations between the corresponding sequences of monic classical orthogonal polynomials are presented.

2. Orthogonality and Positive-Definite Linear Functional Character

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients and $\mathbb{P}'$ its algebraic dual space. We denote by $\langle \mathcal{U}, p \rangle$ the action of $\mathcal{U} \in \mathbb{P}'$ on $p \in \mathbb{P}$ and by $(\mathcal{U})_n := \langle \mathcal{U}, x^n \rangle$, $n \geq 0$, the sequence of moments of $\mathcal{U}$ with respect to the polynomial sequence $\{x^n\}_{n \geq 0}$. Let us define the following operations in $\mathbb{P}'$. For any linear functionals $\mathcal{U}$ and $\mathcal{V}$, any polynomial q , and any $(a, b, c) \in \mathbb{C}^* \times \mathbb{C}^2$, let $D\mathcal{U} = \mathcal{U}', q\mathcal{U}$, $(x - c)^{-1}\mathcal{U}$, $\tau_{-b}\mathcal{U}$, $h_a\mathcal{U}$, and $\mathcal{U}\mathcal{V}$ be the linear functionals defined by duality (see [7, 8]):

$$\begin{aligned}
\langle \mathcal{U}', p \rangle &:= -\langle \mathcal{U}, p' \rangle, \\
\langle q\mathcal{U}, p \rangle &:= \langle \mathcal{U}, qp \rangle, \\
\langle (x-c)^{-1}\mathcal{U}, p \rangle &:= \langle \mathcal{U}, \theta_c p \rangle = \langle \mathcal{U}, \frac{p(x)-p(c)}{x-c} \rangle, \\
\langle \tau_{-b}\mathcal{U}, p \rangle &:= \langle \mathcal{U}, \tau_b p \rangle = \langle \mathcal{U}, p(x-b) \rangle, \\
\langle h_a\mathcal{U}, p \rangle &:= \langle \mathcal{U}, h_a p \rangle = \langle \mathcal{U}, p(ax) \rangle, \\
\langle \mathcal{U}\mathcal{V}, p \rangle &:= \langle \mathcal{U}, \mathcal{V}p \rangle, \quad p \in \mathbb{P},
\end{aligned} \tag{1}$$

where the right-multiplication of $\mathcal{V}$ by p is a polynomial given by

$$(\mathcal{V}p)(x) := \langle \mathcal{V}, \frac{xp(x) - yp(y)}{x-y} \rangle, \quad p \in \mathbb{P}. \tag{2}$$

For any $c \in \mathbb{C}$, $(x-c)^{-1}(x-c)\mathcal{U} = \mathcal{U} - (\mathcal{U})_0\delta_c$, where δ_c is the Dirac functional defined by $\langle \delta_c, p \rangle = p(c)$, $p \in \mathbb{P}$.

Notice that if $\mathcal{U} \in \mathbb{P}'$ is such that $\mathcal{U}' = 0$, then $\mathcal{U} = 0$.

A linear functional $\mathcal{U}$ is said to be quasi-definite (regular) if we can associate with it a monic polynomial sequence (MPS) $\{B_n\}_{n \geq 0}$ with $\deg B_n = n$, $n \geq 0$, such that

$$\langle \mathcal{U}, B_n B_m \rangle = r_n \delta_{n,m}, \quad m, n \geq 0, \text{ and where } r_n \neq 0, n \geq 0, \tag{3}$$

where $\delta_{n,m}$ is the Kronecker delta.

In such a situation, $\{B_n\}_{n \geq 0}$ is said to be the monic orthogonal polynomial sequence (MOPS) with respect to $\mathcal{U}$. The MOPS $\{B_n\}_{n \geq 0}$ satisfies a three-term recurrence relation (TTRR):

$$\begin{cases} B_{-1}(x) = 0, \\ B_0(x) = 1, \\ B_{n+1}(x) = (x - \beta_n)B_n(x) - \gamma_n B_{n-1}(x), \quad n \geq 0, \end{cases} \tag{4}$$

where $\beta_n \in \mathbb{C}$ and $\gamma_n \in \mathbb{C}^*$ for every integer $n \geq 0$ and with $\gamma_0 = (\mathcal{U})_0$. Furthermore, we need the following formula:

$$\begin{aligned}
\beta_n &= \frac{\langle \mathcal{U}, xB_n^2 \rangle}{\langle \mathcal{U}, B_n^2 \rangle}, \quad n \geq 0, \\
\gamma_{n+1} &= \frac{\langle \mathcal{U}, B_{n+1}^2 \rangle}{\langle \mathcal{U}, B_n^2 \rangle}, \quad n \geq 0, \\
\langle \mathcal{U}, B_n^2 \rangle &= \prod_{\nu=0}^n \gamma_\nu, \quad n \geq 0, \quad (\gamma_0 = (\mathcal{U})_0).
\end{aligned} \tag{5}$$

When $\mathcal{U}$ is quasi-definite and $A \in \mathbb{P}$ such that $A\mathcal{U} = 0$, then $A = 0$.

A linear functional $\mathcal{U}$ is said to be positive-definite, if it satisfies (3) with $r_n > 0$, for every integer $n \geq 0$.

The quasi-definiteness of a linear functional $\mathcal{U}$ is equivalent to (see [9, 10])

$$\Delta_n(\mathcal{U}) := \begin{vmatrix} (\mathcal{U})_0 & (\mathcal{U})_1 & (\mathcal{U})_2 & \dots & (\mathcal{U})_n \\ (\mathcal{U})_1 & (\mathcal{U})_2 & (\mathcal{U})_3 & \dots & (\mathcal{U})_{n+1} \\ (\mathcal{U})_2 & (\mathcal{U})_3 & (\mathcal{U})_4 & \dots & (\mathcal{U})_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ (\mathcal{U})_n & (\mathcal{U})_{n+1} & (\mathcal{U})_{n+2} & \dots & (\mathcal{U})_{2n} \end{vmatrix} \neq 0, \quad \forall n \in \mathbb{N}_0, \tag{6}$$

where $\Delta_n(\mathcal{U})$ is the Hankel determinant of order $n+1$ of $\mathcal{U}$.

In this way, we have the following formula [5]:

$$\langle \mathcal{U}, B_n^2 \rangle = \frac{\Delta_n(\mathcal{U})}{\Delta_{n-1}(\mathcal{U})}, \quad n \geq 0, \quad (\Delta_{-1}(\mathcal{U}) = 1). \tag{7}$$

As a consequence, $\mathcal{U}$ is positive-definite if and only if $\Delta_n(\mathcal{U}) > 0$, for every integer $n \geq 0$.

Lemma 1 (see [8]). Let $\{B_n\}_{n \geq 0}$ be a MOPS with respect to the linear functional $\mathcal{U}$ and let A be a monic polynomial with $\deg A = t$. The following statements are equivalent:

- (i) The linear functional $A\mathcal{U}$ is quasi-definite.
- (ii) There exists a MPS $\{Q_n\}_{n \geq 0}$ and parameters $\lambda_{n,\nu}$ satisfying

$$\begin{aligned}
AQ_n(x) &= \sum_{\nu=n}^{n+t} \lambda_{n,\nu} B_\nu(x), \quad n \geq 0, \\
\lambda_{n,n} &\neq 0, \quad n \geq 0.
\end{aligned} \tag{8}$$

In this case, $\{Q_n\}_{n \geq 0}$ is the MOPS with respect to $A\mathcal{U}$.

The Hadamard product of two $n \times n$ matrices $A = (a_{ij})$ and (b_{ij}) is defined by $A \circ B \equiv (a_{ij}b_{ij})$. The Hadamard product arises in a wide variety of ways. It was perhaps the first significant result published about the Hadamard product that the class of positive semidefinite matrices of a given size is closed under the Hadamard product. For more information, the authors may refer to [11]. Oppenheim in [12] proved the following:

Lemma 2 (see [12]). Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ positive semidefinite matrices. Then,

$$\det(A \circ B) \geq \det A \prod_{i=1}^n b_{ii}. \tag{9}$$

3. Main Results

For any $(\mu, c) \in \mathbb{C}^2$ with $\mu \neq -n$, $n \geq 0$ and $\mathcal{U} \in \mathbb{P}'$, let us consider the linear functional $\mathcal{U}(\mu, c)$ where its moments with respect to the MPS $\{(x-c)^n\}_{n \geq 0}$ that we denote by $(\mathcal{U}(\mu, c))_{n,c}$, $n \geq 0$, are given by

$$(\mathcal{U}(\mu, c))_{n,c} = \frac{\mu}{n+\mu} (\mathcal{U})_{n,c}, \quad n \geq 0. \tag{10}$$

Equivalently, $\mathcal{U}(\mu, c)$ satisfies the distributional equation:

$$(\mu-1)\mathcal{U}(\mu, c) - (x-c)(\mathcal{U}(\mu, c))' = \mu\mathcal{U}. \tag{11}$$

3.1. Preservation of the Positive-Definite Property. Assuming that $\mathcal{U}$ is positive definite, our purpose is to find the necessary and sufficient condition on the real parameter μ , in order that the linear functional $\mathcal{U}(\mu, c)$ to be also positive definite.

Theorem 1. Let μ be a real number with $\mu \neq -n$, $n \geq 0$, let $c \in \mathbb{R}$, and let $\mathcal{U}$ be a positive-definite linear functional. The following statements are equivalent:

(i) $\mu \in]0, +\infty[$.

(ii) the linear functional $\mathcal{U}(\mu, c)$ satisfying

$$(\mu - 1)\mathcal{U}(\mu, c) - (x - c)(\mathcal{U}(\mu, c))' = \mu\mathcal{U} \text{ is positive definite.} \quad (12)$$

Proof. Taking (10) into account, we can always decompose the determinant of Hankel associate to linear functional $\mathcal{U}(\mu, c)$ thanks to Hadamard's product as follows:

$$\Delta_n(\mathcal{U}(\mu, c)) := \begin{vmatrix} (\mathcal{U}(\mu, c))_{0,c} & (\mathcal{U}(\mu, c))_{1,c} & (\mathcal{U}(\mu, c))_{2,c} & \dots & (\mathcal{U}(\mu, c))_{n,c} \\ (\mathcal{U}(\mu, c))_{1,c} & (\mathcal{U}(\mu, c))_{2,c} & (\mathcal{U}(\mu, c))_{3,c} & \dots & (\mathcal{U}(\mu, c))_{n+1,c} \\ (\mathcal{U}(\mu, c))_{2,c} & (\mathcal{U}(\mu, c))_{3,c} & (\mathcal{U}(\mu, c))_{4,c} & \dots & (\mathcal{U}(\mu, c))_{n+2,c} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\mathcal{U}(\mu, c))_{n,c} & (\mathcal{U}(\mu, c))_{n+1,c} & (\mathcal{U}(\mu, c))_{n+2,c} & \dots & (\mathcal{U}(\mu, c))_{2n,c} \end{vmatrix}$$

$$= \begin{vmatrix} (\mathcal{U})_{0,c} & (\mathcal{U})_{1,c} & (\mathcal{U})_{2,c} & \dots & (\mathcal{U})_{n,c} \\ (\mathcal{U})_{1,c} & (\mathcal{U})_{2,c} & (\mathcal{U})_{3,c} & \dots & (\mathcal{U})_{n+1,c} \\ (\mathcal{U})_{2,c} & (\mathcal{U})_{3,c} & (\mathcal{U})_{4,c} & \dots & (\mathcal{U})_{n+2,c} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\mathcal{U})_{n,c} & (\mathcal{U})_{n+1,c} & (\mathcal{U})_{n+2,c} & \dots & (\mathcal{U})_{2n,c} \end{vmatrix} \circ \begin{vmatrix} 1 & \frac{\mu}{1+\mu} & \frac{\mu}{2+\mu} & \dots & \frac{\mu}{n+\mu} \\ \frac{\mu}{1+\mu} & \frac{\mu}{2+\mu} & \frac{\mu}{3+\mu} & \dots & \frac{\mu}{n+\mu+1} \\ \frac{\mu}{2+\mu} & \frac{\mu}{3+\mu} & \frac{\mu}{4+\mu} & \dots & \frac{\mu}{n+\mu+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\mu}{n+\mu} & \frac{\mu}{n+\mu+1} & \frac{\mu}{n+\mu+2} & \dots & \frac{\mu}{2n+\mu} \end{vmatrix}. \quad (13)$$

Using Lemma 2 (Oppenheim's inequality) and $\Delta_n(\mathcal{U}) > 0$, for every integer $n \geq 0$ ($\mathcal{U}$ be a positive-definite linear functional), we find $\prod_{k=0}^n (\mu/2k + \mu) > 0$, if and only if $\mu > 0$. Hence, the required result is as follows. $\square$

3.2. Classical Case. A quasi-definite linear functional $\mathcal{W}$ is said to be *classical* if it satisfies a functional equation (Pearson equation):

$$(\phi\mathcal{W})' + \psi\mathcal{W} = 0, \quad (14)$$

where ϕ and ψ are polynomials such that ϕ is monic, $\deg\phi = t \leq 2$, and $\deg\psi = 1$.

The corresponding $\{B_n\}_{n \geq 0}$ is said to be *classical* (for more details, see [8, 13] and the literature therein). The classical character of a linear functional is invariant by shifting. Indeed, if $\mathcal{W}$ is a classical linear functional satisfying (10), then for any pair $(a, b) \in \mathbb{C}^2$ with $a \neq 0$, the shifted linear functional $\tilde{\mathcal{W}} = (h_{a^{-1}} \circ \tau_{-b})\mathcal{W}$ is also classical and satisfies $(\tilde{\phi}\tilde{\mathcal{W}})' + \tilde{\psi}\tilde{\mathcal{W}} = 0$ with $\tilde{\phi}(x) = a^{-t}\phi(ax + b)$ and

$\tilde{\psi}(x) = a^{1-t}\psi(ax + b)$, where $t = \deg\phi$. We find in [8] a description of the classical linear functional (*Hermite, Laguerre, Bessel, and Jacobi*).

Proposition 1. If $\mathcal{W}$ is a classical linear functional with $\deg\phi = 1$, then

$$(\mu^{-1}(\psi + \mu)\mathcal{W})(\mu, c) = \mathcal{W}. \quad (15)$$

Proof. $\deg\phi = 1$. So, we can write $\phi(x) = x - c$. Let $\mu \in \mathbb{C}$ with $\mu \neq -n$, $n \in \mathbb{N}$. From (14), it follows that

$$-(x - c)\mathcal{W}' + (\mu - 1)\mathcal{W} = (\psi + \mu)\mathcal{W}. \quad (16)$$

Clearly, we have $((\psi + \mu)\mathcal{W})_{0,c} = \mu \neq 0$. Accordingly, one has

$$(\mu^{-1}(\psi + \mu)\mathcal{W})(\mu, c) = \mathcal{W}. \quad (17)$$

Now, let us study the particular situation:

Namely, $c = 0$, $\mu = \alpha + 1$, $\psi(x) = x - \alpha - 1$, and $\mathcal{W} = \mathcal{L}^\alpha$. The Laguerre linear functional with parameter $\alpha \neq -n$, $n \geq 1$, which is the unique linear functional satisfying (see [8])

$$(x\mathcal{L}^\alpha)' + (x - \alpha - 1)\mathcal{L}^\alpha = 0. \quad (18)$$

Since $(\alpha + 1)^{-1}x\mathcal{L}^\alpha = \mathcal{L}^{\alpha+1}$ and $\alpha\mathcal{L}^\alpha - x(\mathcal{L}^\alpha)' = (\alpha + 1)\mathcal{L}^{\alpha+1}$, accordingly we obtain

$$\mathcal{L}^{\alpha+1}(\alpha + 1, 0) = \mathcal{L}^\alpha. \quad (19)$$

The MOPS $\{\hat{L}_n^\alpha(x)\}_{n \geq 0}$ of Laguerre satisfies the first and second structure relations [6, 8]:

$$x(\hat{L}_n^\alpha)'(x) = n\hat{L}_n^{\alpha+1}(x) + n(n + \alpha)\hat{L}_{n-1}^\alpha(x), \quad n \geq 0, \quad (20)$$

$$\hat{L}_n^\alpha(x) = \hat{L}_n^{\alpha+1}(x) + n\hat{L}_{n-1}^{\alpha+1}(x), \quad n \geq 0. \quad (21)$$

Corollary 1. *The polynomials $\hat{L}_n^\alpha(x)$ and $\hat{L}_n^{\alpha+1}(x)$ satisfy the following connection relation:*

$$x(\hat{L}_n^{\alpha+1})'(x) + (\alpha + 1)\hat{L}_n^{\alpha+1}(x) = (n + \alpha + 1)\hat{L}_n^\alpha(x), \quad n \geq 0. \quad (22)$$

Proof. Using (20) and (21) and by taking (19) into account,

$$\begin{aligned} & x(\hat{L}_n^{\alpha+1})'(x) + (\alpha + 1)\hat{L}_n^{\alpha+1}(x) \\ &= (n + \alpha + 1)\left(\hat{L}_n^{\alpha+1}(x) + n\hat{L}_{n-1}^{\alpha+1}(x)\right), \\ &= (n + \alpha + 1)\hat{L}_n^\alpha(x), \quad n \geq 0. \end{aligned} \quad (23)$$

□

Proposition 2. *If $\mathcal{W}$ is a classical linear functional with $\deg \phi = 2$, then*

$$(\mu^{-1}\rho\mathcal{W})(\mu, c) = \theta_c(\phi)\mathcal{W}. \quad (24)$$

Proof. $\deg \phi = 2$ and $s = \deg \phi - 2 = \deg \psi - 1$. Let c be a zero of ϕ . From the Euclidean division of ψ by $\theta_c(\phi)$, we always have

$$\psi(x) = -\mu\theta_c(\phi)(x) + \rho(x), \quad (25)$$

where $\rho \in \mathbb{P}$ with $\deg \rho \leq \deg(\psi) - 1$. Clearly, μ is given by

$$\mu = -\frac{\psi^{(s+1)}(0)}{(s+1)!} \neq 0. \quad (26)$$

Since $\deg \psi = \deg \phi - 1$, then μ is the leading coefficient of $-\psi$.

From (14) and (25), we can write

$$(\mu - 1)(\theta_c(\phi)\mathcal{W}) - (x - c)(\theta_c(\phi)\mathcal{W})' = \rho\mathcal{W}. \quad (27)$$

Notice that $\deg \rho \geq 0$, otherwise the linear functional $\theta_c(\phi)\mathcal{W}$ will satisfy $(\mu - 1)(\theta_c(\phi)\mathcal{W}) - (x - c)(\theta_c(\phi)\mathcal{W})' = 0$ and then its moments with respect to the MPS $\{(x - c)^n\}_{n \geq 0}$ are such that $(n + \mu)(\theta_c(\phi)\mathcal{W})_{n,c} = 0$, $n \geq 0$. So, there exists an integer $n_0 \geq 0$ such that

$(\theta_c(\phi)\mathcal{W})_{n,c} = 0$, $n \geq n_0$, i.e., $(x - c)^{n_0}\theta_c(\phi)\mathcal{W} = 0$ which contradicts the quasi-definiteness of $\mathcal{W}$.

Directly, we have

$$(\mu^{-1}\rho\mathcal{W})(\mu, c) = \theta_c(\phi)\mathcal{W}. \quad (28)$$

□

Now, let us study some particular situations.

I_1 (Bessel case): namely, $c = 0$, $\mu = 2\alpha$, $\psi(x) = -2(\alpha x + 1)$, and $\mathcal{W} = \mathcal{B}^\alpha$. The Bessel quasi-definite linear functional with parameter $\alpha \neq -(n/2)$, $n \geq 0$, which is the unique monic linear functional satisfying

$$(x^2\mathcal{B}^\alpha)' - 2(\alpha x + 1)\mathcal{B}^\alpha = 0. \quad (29)$$

We can easily show that

$$-\alpha x\mathcal{B}^\alpha = \mathcal{B}^{\alpha+(1/2)}. \quad (30)$$

It follows that

$$(2\alpha - 1)\mathcal{B}^{\alpha+(1/2)} - x(\mathcal{B}^{\alpha+(1/2)})' = 2\alpha\mathcal{B}^\alpha, \quad (31)$$

so that

$$\mathcal{B}^\alpha(2\alpha, 0) = \mathcal{B}^{\alpha+(1/2)}. \quad (32)$$

Denote by $\{\hat{B}_n^{(\alpha)}(x)\}_{n \geq 0}$ the MOPS with respect to the Bessel linear functional $\mathcal{B}^\alpha$. The MOPS $\{\hat{B}_n^\alpha(x)\}_{n \geq 0}$ satisfies the first structure relation [8]:

$$x^2(\hat{B}_{n+1}^{(\alpha)})'(x) = (n + 1)\left(x - \frac{1}{n + \alpha}\right)\hat{B}_{n+1}^{(\alpha)}(x) - \delta_n\hat{B}_n^{(\alpha)}, \quad n \geq 0, \quad (33)$$

where

$$\delta_n = \frac{(n + 1)(n + 2\alpha - 1)}{(2n + 2\alpha - 1)(n + \alpha)^2}, \quad n \geq 0. \quad (34)$$

Corollary 2. *The polynomials $\hat{B}_{n+1}^{(\alpha)}(x)$ and $\hat{B}_n^{\alpha+(1/2)}(x)$ satisfy the following relation:*

$$x(\hat{B}_{n+1}^{(\alpha)})'(x) - (n + 1)\hat{B}_{n+1}^{(\alpha)}(x) = -\left(\frac{n + 1}{n + \alpha}\right)\hat{B}_n^{\alpha+(1/2)}(x), \quad n \geq 0. \quad (35)$$

Proof. Using (33), we obtain for any integer $n \geq 0$

$$xQ_n(x) = \hat{B}_{n+1}^{(\alpha)}(x) - \frac{(n + 2\alpha - 1)}{(2n + 2\alpha - 1)(n + \alpha)}\hat{B}_n^{(\alpha)}(x), \quad n \geq 0, \quad (36)$$

where $Q_n(x) = -(n + 1/n + \alpha)(x(\hat{B}_{n+1}^{(\alpha)})'(x) - (n + 1)\hat{B}_{n+1}^{(\alpha)}(x))$, $n \geq 0$.

By Lemma 1, with $A(x) = x$, $B_n = \hat{B}_n^{(\alpha)}$, and $\lambda_{n,n} = -((n + 2\alpha - 1)/(2n + 2\alpha - 1)(n + \alpha))$, $n \geq 0$, ($\lambda_{0,0} = -(1/\alpha)$) and by taking (22) and (24) into account, the MPS $\{Q_n(x)\}_{n \geq 0}$ is orthogonal with respect to the linear functional $\mathcal{B}^{\alpha+(1/2)}$. Thus, $Q_n(x) = \hat{B}_n^{\alpha+(1/2)}(x)$, $n \geq 0$. This leads to (35).

I_2 (Jacobi case): namely, $c = 1$, $\mu = \alpha + \beta + 2$, $\psi(x) = -(\alpha + \beta + 2)x + \alpha - \beta$, and $\mathcal{W} = \mathcal{J}^{\alpha,\beta}$. The Jacobi

quasi-definite linear functional with parameters α and β where $\alpha, \beta \neq -n$ and $\alpha + \beta \neq -n - 1$, $n \geq 1$, which is the unique monic linear functional satisfying

$$\left((x^2 - 1)\mathcal{J}^{(\alpha, \beta)}\right)' + (-(\alpha + \beta + 2)x + \alpha - \beta)\mathcal{J}^{(\alpha, \beta)} = 0. \quad (37)$$

Since

$$\frac{\alpha + \beta + 2}{2(\alpha + 1)}(x + 1)\mathcal{J}^{(\alpha, \beta)} = \mathcal{J}^{(\alpha+1, \beta)}, \quad (38)$$

it follows that

$$(\alpha + \beta + 1)\mathcal{J}^{(\alpha+1, \beta)} - (x - 1)(\mathcal{J}^{(\alpha+1, \beta)})' = (\alpha + \beta + 2)\mathcal{J}^{(\alpha, \beta)}, \quad (39)$$

So, we can deduce that

$$\mathcal{J}^{(\alpha, \beta)}(\alpha + \beta + 2, 0) = \mathcal{J}^{(\alpha+1, \beta)}. \quad (40)$$

Denote by $\{\hat{J}_n^{(\alpha, \beta)}(x)\}_{n \geq 0}$ the MOPS with respect to the Jacobi linear functional $\mathcal{J}^{(\alpha, \beta)}$. The MOPS $\{\hat{J}_n^{(\alpha, \beta)}(x)\}_{n \geq 0}$ satisfies the first structure relation for $n \geq 0$ [8]:

$$\begin{aligned} (x^2 - 1)\left(\hat{J}_{n+1}^{(\alpha, \beta)}\right)'(x) &= (n + 1)\left(x + \frac{\alpha - \beta}{2n + \alpha + \beta + 2}\right)\hat{J}_{n+1}^{(\alpha, \beta)}(x) \\ &\quad - (2n + \alpha + \beta + 3)\gamma_{n+1}\hat{J}_n^{(\alpha, \beta)}, \end{aligned} \quad (41)$$

where

$$\gamma_{n+1} = \frac{4(n + 1)(n + \alpha + \beta + 1)(n + \alpha + 1)(n + \beta + 1)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 3)}, \quad n \geq 0. \quad (42)$$

Corollary 3. The polynomials $\hat{J}_{n+1}^{(\alpha, \beta)}(x)$ and $\hat{J}_n^{(\alpha+1, \beta)}(x)$ satisfy the following relation:

$$(x - 1)\left(\hat{J}_{n+1}^{(\alpha, \beta)}\right)'(x) - (n + 1)\hat{J}_{n+1}^{(\alpha, \beta)}(x) = \varepsilon_n \hat{J}_n^{(\alpha+1, \beta)}(x), \quad (43)$$

where for any integer $n \geq 0$,

$$\varepsilon_n = -2 \frac{(n + 1)(n + \beta + 1)}{2n + \alpha + \beta + 2}, \quad n \geq 0. \quad (44)$$

Proof. Using (41), we obtain for any integer $n \geq 0$,

$$\begin{aligned} (x + 1)Q_n(x) &= \hat{J}_{n+1}^{(\alpha, \beta)}(x) \\ &\quad + \frac{2(n + \alpha + 1)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)}\hat{J}_n^{(\alpha, \beta)}(x), \\ &\quad n \geq 0, \end{aligned} \quad (45)$$

where

$$\begin{aligned} Q_n(x) &= \frac{2n + \alpha + \beta + 2}{2(n + 1)(n + \beta + 1)} \\ &\quad \cdot \left((x - 1)\left(\hat{J}_{n+1}^{(\alpha, \beta)}\right)'(x) - (n + 1)\hat{J}_{n+1}^{(\alpha, \beta)}(x)\right), \quad n \geq 0. \end{aligned} \quad (46)$$

By Lemma 1, with $A(x) = x + 1$, $B_n = \hat{J}_n^{(\alpha, \beta)}$, $\lambda_{n,n} = (2(n + \alpha + 1)(n + \alpha + \beta + 1)/(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)) \neq 0$, $n \geq 0$, and by taking (38) and (40) into account, the MPS $\{Q_n(x)\}_{n \geq 0}$ is orthogonal with respect the linear functional $\mathcal{J}^{(\alpha+1, \beta)}$. Thus, $Q_n(x) = \hat{J}_n^{(\alpha+1, \beta)}(x)$, $n \geq 0$. This leads to (43). $\square$

Data Availability

All data required for this paper are included within this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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