Research Article

Some Existence Results for a System of Nonlinear Fractional Differential Equations

Eskandar Ameer,¹ Hassen Aydi ,^{2,3} Hüseyin Işık ,⁴ Muhammad Nazam ,⁵
Vahid Parvaneh ,⁶ and Muhammad Arshad ,⁷

¹Department of Mathematics, Taiz University, Taiz, Yemen

²Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia

³China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

⁴Department of Mathematics, Muş Alparslan University, Muş 49250, Turkey

⁵Department of Mathematics, Allama Iqbal Open University, H-8, Islamabad, Pakistan

⁶Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran

⁷Department of Mathematics and Statistics, International Islamic University, Islamabad, Pakistan

Correspondence should be addressed to Hassen Aydi; hassen.aydi@isima.rnu.tn and Vahid Parvaneh; zam.dalahoo@gmail.com

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In this paper, we show that a sequence satisfying a Suzuki-type JS-rational contraction or a generalized Suzuki-type Ćirić JS-contraction, under some conditions, is a Cauchy sequence. This paper presents some common fixed point theorems and an application to resolve a system of nonlinear fractional differential equations. Some examples and consequences are also given.

1. Introduction

The application of Banach contraction principle (BCP) [1] is wide spread. Recently, fixed point theory is being applied to show the existence of solutions of different mathematical models expressed in the forms of differential, integral, functional, fractional differential, and matrix equations (both linear and nonlinear). There are several common fixed point theorems, in the literature, which generalize BCP and have been applied to show the existence of solutions of different mathematical models involving two or more functions (see [2–19], for details).

Suzuki [20] presented a new generalization of BCP, so called a Suzuki-contraction, and established an existence theorem which characterized the metric completeness. Piri and Kumam [21, 22] investigated results in [20] under the F -contraction structure both in metric and b -metric spaces, respectively. Further generalizations of Suzuki contractions have been studied by Aydi et al. [23]

who introduced a Suzuki-type multivalued contraction to obtain fixed point theorems in the setting of weak partial metric spaces. Abbas et al. [2] discussed generalized Suzuki-type multivalued contractions under the effect of a binary relation and obtained some fixed point results.

On the contrary, Jleli et al. [24, 25] introduced another generalization of BCP, known as a JS-contraction (also known as a θ -contraction) and Li and Jiang [26] modified the JS-contraction to JS-quasi contractions in order to obtain more general fixed point results, as compared to [24]. Altun et al. [27] obtained some fixed point theorems for JS-contraction type mappings via a binary operation. Several honorable researchers have published their valuable investigations on JS-contractions in well ranked journals (see [28–31]).

In this paper, we structure two common fixed point theorems comprising four self-mappings involving generalized Suzuki-type θ -rational contractions and generalized

Suzuki-type Ćirić θ -contractions. The existence of the solution of a mathematical model in terms of fractional differential equations is shown through a common fixed point theorem.

2. Advances on θ -Contractions

Let $\Theta := \{\theta: (0, \infty) \longrightarrow (1, \infty) \text{ such that } \theta \text{ satisfies } \Theta_1, \Theta_2, \Theta_3\}$, where

- (Θ_1) θ is nondecreasing.
- (Θ_2) For each sequence $\{\tilde{t}_n\} \subset (0, \infty)$,

$$\lim_{n \rightarrow \infty} \theta(\tilde{t}_n) = 1, \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} \tilde{t}_n = 0. \quad (1)$$

- (Θ_3) There exist $q \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{\tilde{t} \rightarrow 0^+} \theta(\tilde{t}) - 1/\tilde{t}^q = \ell$.

Jleli and Samet [24] introduced the following.

Definition 1. Let (χ, ρ) be a metric space. The mapping $T: \chi \longrightarrow \chi$ is called a θ -contraction (or a JS-contraction) if there exist a constant $k \in [0, 1)$ and $\theta \in \Theta$ such that

$$\rho(r, j) \leq k \cdot \theta(\rho(\tilde{T}r, \tilde{T}j)) \quad (2)$$

The following examples show that the set Θ is not empty.

Example 1. Let $\Phi, \psi: (0, \infty) \longrightarrow (1, \infty)$ be defined by, for all $t \in (0, \infty)$,

$$\begin{aligned} \Phi(t) &= e^{\sqrt{t}}, \\ \psi(t) &= e^{\sqrt{te^t}}. \end{aligned} \quad (3)$$

Then, Φ and ψ are in Θ .

Jleli and Samet [24] established the following fixed point theorem.

Theorem 1 (see [24]). *Let (χ, ρ) be a complete metric space and $T: \chi \longrightarrow \chi$ be a θ -contraction. Then, T has a unique fixed point.*

Consistent with [28], let $\mu := \{\theta: (0, \infty) \longrightarrow (1, \infty) \mid \theta \text{ satisfy } (\Theta_1), (\Theta_2), (\Theta'_3)\}$, where

- (Θ'_3) θ is continuous on $(0, \infty)$.

Note that (Θ_3) and (Θ'_3) are independent of each other. For this, see examples in [28]. The set of functions μ is not empty, as shown in the following.

Example 2 (see [28]). Let $\phi, \varphi, \psi, F, G, H: (0, \infty) \longrightarrow (1, \infty)$ be defined by, for all $t \in (0, \infty)$,

$$\begin{aligned} \phi(t) &= e^t, \\ \varphi(t) &= \cos ht, \\ \psi(t) &= e^{\sqrt{te^t}}, \\ F(t) &= 1 + \ln(t + 1), \\ G(t) &= e^{\sqrt{t}}, \\ H(t) &= e^{te^t}. \end{aligned} \quad (4)$$

Here, $\phi, \varphi, \psi, F, G, H \in \mu$.

Hussain et al. [29] modified the class Θ of mappings as follows:

$$\text{Let } \Psi = \{\theta: [0, \infty) \longrightarrow [1, \infty): \theta \text{ satisfies } (\Psi_1) - (\Psi_5)\}, \quad (5)$$

where

- (Ψ_1) θ is non decreasing.
- (Ψ_2) $\theta(\tilde{t}) = 1$ if and only if $\tilde{t} = 0$.
- (Ψ_3) For each sequence $\{\tilde{t}_n\} \subset (0, \infty)$:

$$\lim_{n \rightarrow \infty} \theta(\tilde{t}_n) = 1, \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} \tilde{t}_n = 0. \quad (6)$$

- (Ψ_4) There exist $q \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{\tilde{t} \rightarrow 0^+} \theta(\tilde{t}) - 1/\tilde{t}^q = \ell$.

$$(\Psi_5) \theta(\tilde{t}_1 + \tilde{t}_2) \leq \theta(\tilde{t}_1)\theta(\tilde{t}_2).$$

The functions $f(t) = e^{\sqrt{t}}$ and $g(t) = 5^{\sqrt{t}}$ (for all $t \in [0, \infty)$) are in Ψ . (E^*): for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ with $t_1 \cdot t_2 \cdot t_3 \cdot t_4 = 0$, there exists $k \in [0, 1)$ such that $E(t_1, t_2, t_3, t_4) = k$.

Let Δ_E denote the set of all functions $E: \mathbb{R}^{+4} \longrightarrow \mathbb{R}^+$ satisfying the condition (E^*).

Example 3 (see [31]). If $E(t_1, t_2, t_3, t_4) = \lambda e^{v \min\{t_1, t_2, t_3, t_4\}}$, where $v \in \mathbb{R}^+$ and $\lambda \in [0, 1)$, then $E \in \Delta_E$.

Example 4 (see [31]). If $E(t_1, t_2, t_3, t_4) = m \cdot \min\{t_1, t_2, t_3, t_4\} + \lambda$, where $m \in \mathbb{R}^+$ and $\lambda \in [0, 1)$, then $E \in \Delta_E$.

The following class of mappings was considered in [32] to develop some fixed point theorems. We also use this class of mappings in our investigations:

$$\Phi = \left\{ \Upsilon: \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R} \text{ and } \Upsilon(s, t) = \frac{1}{2}s - t \right\}. \quad (7)$$

Definition 2 (see [5]). Let (χ, ρ) be a metric space. The pair $\{f, g\}$ is said to be compatible if and only if

$$\lim_{n \rightarrow \infty} \rho(f(g(r_n)), g(f(r_n))) = 0, \quad (8)$$

whenever $\{r_n\}$ is a sequence in χ so that

$$\lim_{n \rightarrow \infty} f(r_n) = \lim_{n \rightarrow \infty} g(r_n) = t, \quad \text{for some } t \in \chi. \quad (9)$$

The following lemma is important in the sequel.

Lemma 1 (see [5]). *Let (χ, ρ) be a metric space. If there exist two sequences $\{r_n\}$ and $\{u_n\}$ such that*

$$\begin{aligned} \lim_{n \rightarrow \infty} \rho(r_n, u_n) &= 0, \\ \lim_{n \rightarrow \infty} r_n &= t, \quad \text{for some } t \in \chi, \end{aligned} \quad (10)$$

then

$$\lim_{n \rightarrow \infty} u_n = t. \quad (11)$$

3. Generalized Suzuki-Type Contractions and Fixed Point Results

Jleli and Samet [24, 25] have employed θ -contractions to obtain some fixed point theorems. Suzuki [20] extended Edelstein contraction to develop a new generalization of Banach contraction called Suzuki contraction. The contractions developed in [20, 24] were then followed by Liu

et al. [33] to introduce a Suzuki-type θ -contraction. In this section, we introduce more general forms of Suzuki-type θ -contractions involving four self-mappings. We construct some conditions under which a sequence, whose terms satisfy generalized Suzuki-type θ -rational contractions or generalized Suzuki-type Ćirić θ -contractions, is a Cauchy sequence. We obtain two common fixed point theorems, which then are applied to show the existence of the solution of a system of fractional differential equations. We start with the following definition.

Definition 3. Let (χ, ρ) be a metric space, and $f, g, \widehat{S}, \widetilde{T}: \chi \rightarrow \chi$ be four single-valued maps. These maps form a new generalized Suzuki-type (θ, E) -rational contraction if, for all $r, j \in \chi$, with $\rho(fr, gj) > 0$, for some $\theta \in \mu$, $E \in \Delta_E$, and $\Upsilon \in \Phi$,

$$\Upsilon(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \widetilde{T}j)) < 0 \Rightarrow \theta(\rho(fr, gj)) \leq [\theta(A(r, j))]^{E(G(r, j))}, \quad (12)$$

where

$$\begin{aligned} A(r, j) &= \max \left\{ \begin{aligned} &\rho(\widehat{S}r, \widetilde{T}j), \rho(fr, \widehat{S}r), \rho(gj, \widetilde{T}j), \frac{\rho(fr, \widetilde{T}j) + \rho(gj, \widehat{S}r)}{2}, \\ &\frac{(1 + \rho(fr, \widehat{S}r))\rho(gj, \widetilde{T}j)}{1 + \rho(\widehat{S}r, \widetilde{T}j)} \end{aligned} \right\}, \\ G(r, j) &= \{\rho(fr, \widehat{S}r), \rho(gj, \widetilde{T}j), \rho(fr, \widetilde{T}j), \rho(gj, \widehat{S}r)\}. \end{aligned} \quad (13)$$

Our first result is as follows.

Theorem 2. *Let (χ, ρ) be a complete metric space, and $f, g, \widehat{S}, \widetilde{T}: \chi \rightarrow \chi$ be four single-valued maps. Suppose that the mappings $f, g, \widehat{S}, \widetilde{T}$ form a new generalized Suzuki-type (θ, E) -rational contraction with $f(\chi) \subseteq \widetilde{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$. If $(f, \widehat{S})$ and $(g, \widetilde{T})$ are compatible pair and $\widetilde{T}$ and $\widehat{S}$ are continuous on (χ, ρ) , then $f, g, \widehat{S}$, and $\widetilde{T}$ have a unique common fixed point in χ .*

Proof. Let $r_0 \in \chi$ be an arbitrary point. As $f(\chi) \subseteq \widetilde{T}(\chi)$, there exists $r_1 \in \chi$ such that

$$f(r_0) = \widetilde{T}(r_1). \quad (14)$$

Since $g(r_1) \in \widehat{S}(\chi)$, we can choose $r_2 \in \chi$ such that $g(r_1) = \widehat{S}(r_2)$. In general r_{2n+1} and r_{2n+2} are chosen in χ such that

$$\begin{aligned} f(r_{2n}) &= \widetilde{T}(r_{2n+1}), \\ g(r_{2n+1}) &= \widehat{S}(r_{2n+2}). \end{aligned} \quad (15)$$

We construct a sequence $\{\mathfrak{R}_n\}$ in χ such that

$$\mathfrak{R}_{2n} = f(r_{2n}) = \widetilde{T}(r_{2n+1}), \quad (16)$$

$$\mathfrak{R}_{2n+1} = g(r_{2n+1}) = \widehat{S}(r_{2n+2}),$$

for $n = 0, 1, 2, \dots$. Assume that $\rho(f(r_{2n}), g(r_{2n+1})) > 0$ and since

$$\begin{aligned} \frac{1}{2}\rho(f(r_{2n}), \widehat{S}(r_{2n})) &= \frac{1}{2}\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}) < \rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}) \\ &= \rho(\widehat{S}(r_{2n}), \widetilde{T}(r_{2n+1})). \end{aligned} \quad (17)$$

We have

$$\Upsilon(\rho(f(r_{2n}), \widehat{S}(r_{2n})), \rho(\widehat{S}(r_{2n}), \widetilde{T}(r_{2n+1}))) < 0. \quad (18)$$

Hence, from the contractive condition (12),

$$\begin{aligned} 1 < \theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) &= \theta(\rho(f(r_{2n}), g(r_{2n+1}))) \\ &\leq [\theta(A(r_{2n}, r_{2n+1}))]^{E(G(r_{2n}, r_{2n+1}))}. \end{aligned} \quad (19)$$

For every $n \geq 1$, where

$$G(r_{2n}, r_{2n+1}) = \begin{cases} \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}), \\ \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1}) \end{cases} \quad (20)$$

$$= \begin{cases} \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}), \\ 0, \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1}) \end{cases}.$$

As $\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}) \cdot \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}) \cdot \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1}) \cdot 0 = 0$, so by (E^*) , there exists $z \in [0, 1]$ such that

$$A(r_{2n}, r_{2n+1}) = \max \left\{ \begin{array}{l} \rho(\hat{S}r_{2n}, \tilde{T}r_{2n+1}), \rho(fr_{2n}, \hat{S}r_{2n}), \rho(gr_{2n+1}, \tilde{T}r_{2n+1}), \\ \frac{\rho(fr_{2n}, \tilde{T}r_{2n+1}) + \rho(gr_{2n+1}, \hat{S}r_{2n})}{2}, \frac{(1 + \rho(fr_{2n}, \hat{S}r_{2n}))\rho(gr_{2n+1}, \tilde{T}r_{2n+1})}{1 + \rho(\hat{S}r_{2n}, \tilde{T}r_{2n+1})} \end{array} \right\} \quad (22)$$

$$= \max \left\{ \begin{array}{l} \rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}), \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}), \\ \frac{\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n}) + \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1})}{2}, \frac{(1 + \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}))\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n})}{1 + \rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n})} \end{array} \right\}$$

$$= \max \left\{ \rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}), \frac{\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1})}{2} \right\}.$$

Since

$$\frac{\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1})}{2} \leq \frac{\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1})}{2} + \frac{\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n})}{2} \leq \max\{\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n})\}. \quad (23)$$

$$\begin{aligned} 1 &< \theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) \\ &\leq [\theta(\rho(\mathfrak{R}_{n-1}, \mathfrak{R}_n))]^{z^2} \\ &\vdots \\ &\leq [\theta(\rho(\mathfrak{R}_0, \mathfrak{R}_1))]^{z^n}, \end{aligned} \quad (28)$$

one writes

$$A(r_{2n}, r_{2n+1}) = \max\{\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n})\}. \quad (24)$$

If, for some n , $A(r_{2n}, r_{2n+1}) = \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})$, then from (19), (21), and (24), we have

$$\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) \leq [\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1}))]^z < \theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})), \quad (25)$$

a contradiction. Therefore,

$$\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) \leq [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}))]^z, \quad \text{for } n \geq 1. \quad (26)$$

Thus, for all $n \in \mathbb{N}$, we have

$$\theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) \leq [\theta(\rho(\mathfrak{R}_{n-1}, \mathfrak{R}_n))]^z, \quad \text{for } n \geq 1. \quad (27)$$

It implies that

$$E(G(r_{2n}, r_{2n+1})) = E(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}), 0, \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1})) = z, \quad (21)$$

for all $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$ in (28) and using the fact that $\theta \in \mu$, we have

$$\lim_{n \rightarrow \infty} \theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) = 1. \quad (29)$$

By $(\Theta 2)$, we obtain

$$\lim_{n \rightarrow \infty} \rho(\mathfrak{R}_n, \mathfrak{R}_{n+1}) = 0. \quad (30)$$

Now, we will show that $\{\mathfrak{R}_n\}$ is a Cauchy sequence. We suppose on the contrary that $\{\mathfrak{R}_n\}$ is not a Cauchy sequence; then, there exist $\varepsilon > 0$ and subsequences $\{n(k)\}_{k=1}^{\infty}$ and $\{m(k)\}_{k=1}^{\infty}$ of natural numbers such that, for $m(k) > n(k) > k$, we have $\rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)}) \geq \varepsilon$. Then, $\rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)-1}) < \varepsilon$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} \varepsilon &\leq \rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)}) \leq \rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)-1}) + \rho(\mathfrak{R}_{m(k)-1}, \mathfrak{R}_{m(k)}) \\ &< \varepsilon + \rho(\mathfrak{R}_{m(k)-1}, \mathfrak{R}_{m(k)}). \end{aligned} \quad (31)$$

By taking the upper limit as $k \rightarrow \infty$ in (31) and using (30), we obtain

$$\lim_{k \rightarrow \infty} \rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)}) = \varepsilon. \quad (32)$$

Using the triangular inequality, we have

$$\rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)}) \leq \rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{n(k)+1}) + \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)}), \quad (33)$$

$$\rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)}) \leq \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{n(k)}) + \rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)}). \quad (34)$$

By taking upper limit as $k \rightarrow \infty$ in (33) and (34) and applying (30) and (32),

$$\varepsilon \leq \lim_{k \rightarrow \infty} \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)}) \leq \varepsilon. \quad (35)$$

Thus,

$$\lim_{k \rightarrow \infty} \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)}) = \varepsilon. \quad (36)$$

Similarly, we can obtain

$$\lim_{k \rightarrow \infty} \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)+1}) = \lim_{k \rightarrow \infty} \rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)+1}) = \varepsilon. \quad (37)$$

Assume that $\rho(f(r_{n(k)+1}), g(r_{m(k)+1})) > 0$, and since

$$\frac{1}{2} \rho(f(r_{n(k)}), \widehat{S}(r_{m(k)})) = \frac{1}{2} \rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)-1})$$

$$< \rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)-1}) \quad (38)$$

$$= \rho(\widehat{S}(r_{m(k)}), \widetilde{T}(r_{n(k)+1})),$$

we obtain

$$\Upsilon(\rho(f(r_{n(k)}), \widehat{S}(r_{m(k)})), \rho(\widehat{S}(r_{n(k)}), \widetilde{T}(r_{m(k)+1}))) < 0. \quad (39)$$

Hence, from (12),

$$\begin{aligned} \theta(\rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)+1})) &= \theta(\rho(f(r_{n(k)+1}), g(r_{m(k)+1}))) \\ &\leq [\theta(A(r_{n(k)}, r_{m(k)}))]^{E(G(r_{n(k)}, r_{m(k)}))}, \end{aligned} \quad (40)$$

where

$$G(r_{n(k)}, r_{m(k)}) = \left\{ \begin{array}{l} \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{n(k)}), \rho(\mathfrak{R}_{m(k)+1}, \mathfrak{R}_{m(k)}), \\ \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)}), \rho(\mathfrak{R}_{m(k)+1}, \mathfrak{R}_{n(k)}) \end{array} \right\}. \quad (41)$$

Taking limit at $k \rightarrow \infty$ in (41), we obtain

$$\lim_{k \rightarrow \infty} G(r_{n(k)}, r_{m(k)}) = \left\{ \begin{array}{l} \lim_{k \rightarrow \infty} \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{n(k)}), \lim_{k \rightarrow \infty} \rho(\mathfrak{R}_{m(k)+1}, \mathfrak{R}_{m(k)}), \\ \lim_{k \rightarrow \infty} \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)}), \lim_{k \rightarrow \infty} \rho(\mathfrak{R}_{m(k)+1}, \mathfrak{R}_{n(k)}) \end{array} \right\} = \{0, 0, \varepsilon, \varepsilon\}. \quad (42)$$

By E^* , there exists $z \in [0, 1)$ such that $E(0, 0, \varepsilon, \varepsilon) = z$. Using the continuity of E and (41),

$$E\left(\lim_{k \rightarrow \infty} G(r_{n(k)}, r_{m(k)})\right) \leq z. \quad (43)$$

Also,

$$A(r_{n(k)}, r_{m(k)}) = \max \left\{ \begin{array}{l} \rho(\widehat{S}(r_{n(k)+1}), \widetilde{T}(r_{m(k)+1})), \rho(f(r_{n(k)+1}), \widehat{S}(r_{n(k)+1})), \\ \rho(g(r_{m(k)+1}), \widetilde{T}(r_{m(k)+1})), \\ \frac{\rho(f(r_{n(k)+1}), \widetilde{T}(r_{m(k)+1})) + \rho(g(r_{m(k)+1}), \widehat{S}(r_{n(k)+1}))}{2}, \\ \frac{(1 + \rho(f(r_{n(k)+1}), \widehat{S}(r_{n(k)+1}))) \rho(g(r_{m(k)+1}), \widetilde{T}(r_{m(k)+1}))}{1 + \rho(\widehat{S}(r_{n(k)+1}), \widetilde{T}(r_{m(k)+1}))} \end{array} \right\},$$

$$= \max \left\{ \begin{array}{c} \rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)}), \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{n(k)}), \\ \rho(\mathfrak{R}_{m(k)+1}, \mathfrak{R}_{m(k)}), \\ \frac{\rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)}) + \rho(\mathfrak{R}_{m(k)+1}, \mathfrak{R}_{n(k)})}{2}, \\ \frac{(1 + \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{n(k)}))\rho(\mathfrak{R}_{m(k)+1}, \mathfrak{R}_{m(k)})}{1 + \rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)})} \end{array} \right\}. \quad (44)$$

Taking $k \rightarrow \infty$ and using (30), (32), (36), and (37), we obtain

$$\lim_{k \rightarrow \infty} A(r_{n(k)}, r_{m(k)}) \leq \varepsilon. \quad (45)$$

Thus, from $(\Theta'3)$, (37), (40), (43), and (45), we have

$$\begin{aligned} \theta(\varepsilon) &= \theta\left(\lim_{k \rightarrow \infty} \rho(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)+1})\right) \\ &\leq \left[\theta\left(\lim_{k \rightarrow \infty} A(r_{n(k)}, r_{m(k)})\right)\right]^{E(\lim_{k \rightarrow \infty} G(r_{n(k)}, r_{m(k)}))} \\ &\leq [\theta(\varepsilon)]^z < \theta(\varepsilon), \end{aligned} \quad (46)$$

which is a contradiction. Thus, $\{\mathfrak{R}_n\}$ is a Cauchy sequence. Since χ is a complete metric space, there exists $r^* \in \chi$ such that $\lim_{n \rightarrow \infty} \rho(\mathfrak{R}_n, r^*) = 0$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} f(r_{2n}) &= \lim_{n \rightarrow \infty} \tilde{T}(r_{2n+1}) = \lim_{n \rightarrow \infty} g(r_{2n+1}) \\ &= \lim_{n \rightarrow \infty} \hat{S}(r_{2n+2}) = r^*. \end{aligned} \quad (47)$$

Now, we shall prove that $\mathfrak{R}^*$ is a common fixed point of $f, g, \hat{S}$, and $\tilde{T}$. As $\hat{S}$ is continuous, so

$$\lim_{n \rightarrow \infty} \hat{S}(f(r_{2n})) = \hat{S}(r^*) = \lim_{n \rightarrow \infty} \hat{S}(\hat{S}(r_{2n+2})). \quad (48)$$

Since $(f, \hat{S})$ is a compatible pair,

$$\lim_{n \rightarrow \infty} \rho\left(g(\tilde{T}(r_{2n+1})), \tilde{T}(g(r_{2n+1}))\right) = 0. \quad (49)$$

From Lemma 1, we have

$$\lim_{n \rightarrow \infty} f(\hat{S}(r_{2n})) = \hat{S}(r^*). \quad (50)$$

Put $r_1 = \hat{S}(r_{2n})$ and $r_2 = r_{2n+1}$ in (12) and if $\rho(\hat{S}(r^*), r^*) > 0$, we obtain

$$\frac{1}{2} \rho(f(\hat{S}(r_{2n})), \hat{S}(\hat{S}(r_{2n}))) < \rho(\hat{S}(\hat{S}(r_{2n})), \tilde{T}(r_{2n+1})). \quad (51)$$

Hence,

$$\Upsilon\left(\rho(f(\hat{S}(r_{2n})), \hat{S}(\hat{S}(r_{2n}))), \rho(\hat{S}(\hat{S}(r_{2n})), \tilde{T}(r_{2n+1}))\right) < 0, \quad (52)$$

and by (12), we obtain

$$\theta(\rho(f(\hat{S}(r_{2n})), g(r_{2n+1}))) \leq [\theta(A(r_{2n}, r_{2n+1}))]^{E(G(r_{2n}, r_{2n+1}))}, \quad (53)$$

where

$$A(r_{2n}, r_{2n+1}) = \max \left\{ \begin{array}{c} \rho(\hat{S}(\hat{S}(r_{2n})), \tilde{T}(r_{2n+1})), \rho(f(\hat{S}(r_{2n})), \hat{S}(\hat{S}(r_{2n}))), \\ \rho(g(r_{2n+1}), \tilde{T}(r_{2n+1})), \\ \frac{\rho(f(\hat{S}(r_{2n})), \tilde{T}(r_{2n+1})) + \rho(g(r_{2n+1}), \hat{S}(\hat{S}(r_{2n})))}{2}, \\ \frac{(1 + \rho(f(\hat{S}(r_{2n})), \hat{S}(\hat{S}(r_{2n}))))\rho(g(r_{2n+1}), \tilde{T}(r_{2n+1}))}{1 + \rho(\hat{S}(\hat{S}(r_{2n})), \tilde{T}(r_{2n+1}))} \end{array} \right\},$$

$$G(r_{2n}, r_{2n+1}) = \left\{ \begin{array}{l} \rho(f(\widehat{S}(r_{2n}), \widehat{S}(\widehat{S}(r_{2n}))), \rho(g(r_{2n+1}), \widetilde{T}(r_{2n+1})), \\ \rho(f(\widehat{S}(r_{2n}), \widetilde{T}(r_{2n+1})), \rho(g(r_{2n+1}), \widehat{S}(\widehat{S}(r_{2n}))) \end{array} \right\}. \quad (54)$$

Setting the upper limit in (53), we obtain

$$\theta(\rho(\widehat{S}(r^*), r^*)) \leq [\theta(\rho(\widehat{S}(r^*), r^*))]^z < \theta(\rho(\widehat{S}(r^*), r^*)), \quad (55)$$

a contradiction. Thus, $\rho(\widehat{S}(r^*), r^*) = 0$ and $\widehat{S}(r^*) = r^*$. Again since T is continuous,

$$\lim_{n \rightarrow \infty} \widetilde{T}(g(r_{2n+1})) = \widetilde{T}(r^*) = \lim_{n \rightarrow \infty} \widetilde{T}(\widetilde{T}(r_{2n+1})). \quad (56)$$

Since $(g, \widetilde{T})$ is a compatible pair,

$$\lim_{n \rightarrow \infty} \rho(g(\widetilde{T}(r_{2n+1}), \widetilde{T}(g(r_{2n+1}))) = 0. \quad (57)$$

From Lemma 1, we have

$$\lim_{n \rightarrow \infty} g(\widetilde{T}(r_{2n+1})) = \widetilde{T}(r^*). \quad (58)$$

Put $r_1 = r_{2n}$ and $r_2 = \widetilde{T}(r_{2n+1})$ in (12) and if $\rho(r^*, \widetilde{T}(r^*)) > 0$, we obtain

$$\frac{1}{2} \rho(f(r_{2n}), \widehat{S}(r_{2n})) < \rho(\widehat{S}(r_{2n}), \widetilde{T}(\widetilde{T}(r_{2n+1}))). \quad (59)$$

Therefore,

$$\Upsilon \left(\begin{array}{l} \rho(f(r_{2n}), \widehat{S}(r_{2n})), \\ \rho(\widehat{S}(r_{2n}), \widetilde{T}(\widetilde{T}(r_{2n+1}))) \end{array} \right) < 0. \quad (60)$$

By (12), one obtains

$$\theta(\rho(f(r_{2n}), g(\widetilde{T}(r_{2n+1})))) \leq [\theta(A(r_{2n}, r_{2n+1}))]^{E(G(r_{2n}, r_{2n+1}))}, \quad (61)$$

where

$$A(r_{2n}, r_{2n+1}) = \max \left\{ \begin{array}{l} \rho(\widehat{S}(r_{2n}), \widetilde{T}(\widetilde{T}(r_{2n+1}))), \rho(f(r_{2n}), \widehat{S}(r_{2n})), \\ \rho(g(\widetilde{T}(r_{2n+1}), \widetilde{T}(\widetilde{T}(r_{2n+1})))), \\ \frac{\rho(f(r_{2n}), \widetilde{T}(\widetilde{T}(r_{2n+1}))) + \rho(g(\widetilde{T}(r_{2n+1}), \widehat{S}(r_{2n})))}{2}, \\ \frac{(1 + \rho(f(r_{2n}), \widehat{S}(r_{2n}))) \rho(g(\widetilde{T}(r_{2n+1}), \widetilde{T}(\widetilde{T}(r_{2n+1}))))}{1 + \rho(\widehat{S}(r_{2n}), \widetilde{T}(\widetilde{T}(r_{2n+1})))} \end{array} \right\}, \quad (62)$$

$$G(r_{2n}, r_{2n+1}) = \left\{ \begin{array}{l} \rho(f(r_{2n}), \widehat{S}(r_{2n})), \rho(g(\widetilde{T}(r_{2n+1}), \widetilde{T}(\widetilde{T}(r_{2n+1})))), \\ \rho(f(r_{2n}), \widetilde{T}(\widetilde{T}(r_{2n+1}))), \rho(g(\widetilde{T}(r_{2n+1}), \widehat{S}(r_{2n}))) \end{array} \right\}.$$

Setting the upper limit, we obtain

$$\theta(\rho(r^*, \widetilde{T}(r^*))) \leq [\theta(\rho(r^*, \widetilde{T}(r^*)))]^z < \theta(\rho(r^*, \widetilde{T}(r^*))), \quad (63)$$

a contradiction. Thus, $\rho(r^*, \widetilde{T}(r^*)) = 0$ and so $r^* = \widetilde{T}(r^*)$. Suppose $\rho(f(r^*), r^*) > 0$, we obtain

$$\frac{1}{2} \rho(\widehat{S}(r^*), \widetilde{T}(r_{2n+1})) < \rho(f(r^*), \widehat{S}(r^*)). \quad (64)$$

Hence,

$$\Upsilon \left(\begin{array}{l} \rho(f(r^*), \widehat{S}(r^*)), \\ \rho(\widehat{S}(r^*), \widetilde{T}(r_{2n+1})) \end{array} \right) < 0, \quad (65)$$

and by (12), we obtain

$$\theta(\rho(f(r^*), g(r_{2n+1}))) \leq [\theta(A(r^*, r_{2n+1}))]^{E(G(r^*, r_{2n+1}))}, \quad (66)$$

where

$$A(r^*, r_{2n+1}) = \max \left\{ \begin{array}{l} \rho(\widehat{S}(r^*), \widetilde{T}(r_{2n+1})), \rho(f(r^*), \widehat{S}(r^*)), \\ \rho(g(r_{2n+1}), \widetilde{T}(r_{2n+1})), \frac{\rho(f(r^*), \widetilde{T}(r_{2n+1})) + \rho(g(r_{2n+1}), \widehat{S}(r^*))}{2}, \\ \frac{(1 + \rho(f(r^*), \widehat{S}(r^*)))\rho(g(r_{2n+1}), \widetilde{T}(r_{2n+1}))}{1 + \rho(\widehat{S}(r^*), \widetilde{T}(r_{2n+1}))} \end{array} \right\}, \quad (67)$$

$$G(r^*, r_{2n+1}) = \left\{ \begin{array}{l} \rho(f(r^*), \widehat{S}(r^*)), \rho(g(r_{2n+1}), \widetilde{T}(r_{2n+1})), \\ \rho(f(r^*), \widetilde{T}(r_{2n+1})), \rho(g(r_{2n+1}), \widehat{S}(r^*)) \end{array} \right\}.$$

Taking the upper limit in (66), and as $r^* = \widetilde{T}(r^*) = \widehat{S}(r^*)$, so we obtain

$$\theta(\rho(f(r^*), r^*)) \leq [\theta(\rho(f(r^*), r^*))]^z < \theta(\rho(f(r^*), r^*)), \quad (68)$$

a contradiction. Thus, $\rho(f(r^*), r^*) = 0$ and $r^* = f(r^*)$. Finally, suppose $\rho(r^*, g(r^*)) > 0$, and as $r^* = \widetilde{T}(r^*) = \widehat{S}(r^*) = f(r^*)$, so, we obtain

$$\Upsilon(0, \rho(r^*, g(r^*))) < 0. \quad (69)$$

Thus, from (12)

$$\begin{aligned} \theta(\rho(r^*, g(r^*))) &= \theta(\rho(f(r^*), g(r^*))) \leq [\theta(\rho(r^*, g(r^*)))]^z \\ &< \theta(\rho(r^*, g(r^*))), \end{aligned} \quad (70)$$

a contradiction. Thus, $\rho(r^*, g(r^*)) = 0$ and $r^* = g(r^*)$. Therefore, r^* is a common of the four mappings $\widehat{S}$, g , f and T . Next, assume that v^* is another common of the four mappings $\widehat{S}$, g , f , and T such that $r^* \neq v^*$, one obtains

$$\Upsilon(0, \rho(\widehat{S}(r^*), \widetilde{T}(v^*))) < 0. \quad (71)$$

Then, by (12), we obtain

$$\theta(\rho(r^*, v^*)) = \theta(\rho(f(r^*), g(v^*))) \leq [\theta(A(r^*, v^*))]^{E(G(r^*, v^*))}, \quad (72)$$

where

$$A(r^*, v^*) = \max \left\{ \begin{array}{l} \rho(\widehat{S}(r^*), \widetilde{T}(v^*)), \rho(f(r^*), \widehat{S}(r^*)), \rho(g(v^*), \widetilde{T}(v^*)), \\ \frac{\rho(f(r^*), \widetilde{T}(v^*)) + \rho(g(v^*), \widehat{S}(r^*))}{2}, \frac{(1 + \rho(f(r^*), \widehat{S}(r^*)))\rho(g(v^*), \widetilde{T}(v^*))}{1 + \rho(\widehat{S}(r^*), \widetilde{T}(v^*))} \end{array} \right\}, \quad (73)$$

$$G(r^*, v^*) = \left\{ \begin{array}{l} \rho(f(r^*), \widehat{S}(r^*)), \rho(g(v^*), \widetilde{T}(v^*)), \\ \rho(f(r^*), \widetilde{T}(v^*)), \rho(g(v^*), \widehat{S}(r^*)) \end{array} \right\}.$$

It implies that

$$\theta(\rho(r^*, v^*)) \leq [\theta(\rho(r^*, v^*))]^z < \theta(\rho(r^*, v^*)), \quad (74)$$

a contradiction. Thus, $r^* = v^*$. Therefore, r^* is the unique common fixed point of $\widehat{S}, g, f$, and T . $\square$

Corollary 1. Let (χ, ρ) be a complete metric space, and $f, g, \widehat{S}, T: \chi \longrightarrow \chi$ be four single-valued maps. Suppose that

if, for all $r, j \in \chi$ with $\rho(fr, gj) > 0$ for some $\theta \in \mu, z \in [0, 1]$ and $Y \in \Phi$,

$$Y(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \widetilde{T}j)) < 0 \implies \theta(\rho(fr, gj)) \leq [\theta(A(r, j))]^z, \quad (75)$$

where

$$A(r, j) = \max \left\{ \begin{array}{l} \rho(\widehat{S}r, \widetilde{T}j), \rho(fr, \widehat{S}r), \rho(gj, \widetilde{T}j), \frac{\rho(fr, \widetilde{T}j) + \rho(gj, \widehat{S}r)}{2}, \\ \frac{(1 + \rho(fr, \widehat{S}r))\rho(gj, \widetilde{T}j)}{1 + \rho(\widehat{S}r, \widetilde{T}j)} \end{array} \right\}, \quad (76)$$

with $f(\chi) \subseteq \widetilde{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$. If $(f, \widehat{S})$ and $(g, \widetilde{T})$ are compatible pairs and T and $\widehat{S}$ are continuous on (χ, ρ) , then $f, g, \widehat{S}$, and T have a unique common fixed point in χ .

Example 5. Let $\chi = [0, 1]$ and define the function $\rho: \chi \times \chi \longrightarrow [0, +\infty)$ by $\rho(r, j) = |r - j|$. Clearly, (χ, ρ) is a complete metric space. Let $\theta(t) = e^t, t > 0$, and $Y(s, t) = (s/2) - t$, then $\theta \in \mu$ and $Y \in \Phi$. Define the mappings $\widehat{S}, g, f, T: \chi \longrightarrow \chi$ by

$$\begin{aligned} \widehat{S}(r) &= \left(\frac{r}{3}\right)^8, \\ g(r) &= \left(\frac{r}{3}\right)^8, \\ f(r) &= \left(\frac{r}{3}\right)^{16}, \\ \widetilde{T}(r) &= \left(\frac{r}{3}\right)^4. \end{aligned} \quad (77)$$

Clearly, $f(\chi) \subseteq \widetilde{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$, if $\{r_n\}$ is a sequence in χ such that

$$\lim_{n \rightarrow \infty} f(r_n) = \lim_{n \rightarrow \infty} \widehat{S}(r_n) = t, \quad \text{for some } t \in \chi, \quad (78)$$

then

$$\lim_{n \rightarrow \infty} |f(r_n) - t| = \lim_{n \rightarrow \infty} |\widehat{S}(r_n) - t| = 0, \quad (79)$$

and equivalently,

$$\lim_{n \rightarrow \infty} \left| \left[\frac{r_n}{3} \right]^{16} - t \right| = \lim_{n \rightarrow \infty} \left| \left[\frac{r_n}{3} \right]^8 - t \right| = 0. \quad (80)$$

Thus,

$$\lim_{n \rightarrow \infty} |r_n|^{16} - 3^{16}t = \lim_{n \rightarrow \infty} |r_n|^8 - 3^8t = 0. \quad (81)$$

We conclude $t^{1/16} = t^{1/8}$ (by uniqueness of limit); hence, $t \in \{0, 1\}$. Using continuity of f and $\widehat{S}$, one obtains

$$\begin{aligned} \lim_{n \rightarrow \infty} \rho(f\widehat{S}(r_n), \widehat{S}f(r_n)) &= \lim_{n \rightarrow \infty} |f\widehat{S}(r_n) - \widehat{S}f(r_n)| \\ &= |f(t) - \widehat{S}(t)| = |0 - 0| = 0, \end{aligned} \quad (82)$$

for $t = 0 \in \chi$. Hence, the pair $(f, \widehat{S})$ is compatible. Similarly, the pair (g, T) is compatible. Define $E: \mathbb{R}^4 \longrightarrow \mathbb{R}^+$ as $E(s_1, s_2, s_3, s_4) = (9/10)$. Next, for all $r, j \in \chi$ with $\rho(fr, gj) > 0$,

$$Y(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \widetilde{T}j)) < 0, \quad (83)$$

and so

$$\theta(\rho(fr, gj)) \leq [\theta(A(r, j))]^{E(G(r, j))}. \quad (84)$$

Hence, all hypotheses of Theorem 2 are satisfied. Thus, $f, g, \widehat{S}$, and T have a unique common fixed point, which is $r^* = 0$.

We can easily obtain the following results by chasing the proof of Theorem 2.

Corollary 2. Let (χ, ρ) be a complete metric space, and $f, g, \widehat{S}, T: \chi \longrightarrow \chi$ be four single-valued maps. Suppose that if, for all $r, j \in \chi$, with $\rho(fr, gj) > 0$, for some $\theta \in \mu, Y \in \Phi$ and $E: \mathbb{R}^4 \longrightarrow \mathbb{R}^+$,

$$\begin{aligned} Y(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \widetilde{T}j)) < 0 &\implies \theta(\rho(fr, gj)) \\ &\leq [\theta(\max\{\rho(fr, \widehat{S}r), \rho(gj, \widetilde{T}j)\})]^{E(G(r, j))}, \end{aligned} \quad (85)$$

where

$$G(r, j) = \left\{ \rho(fr, \widehat{S}r), \rho(gj, \widetilde{T}j), \rho(fr, \widetilde{T}j), \rho(gj, \widehat{S}r) \right\}, \quad (86)$$

with $f(\chi) \subseteq \widetilde{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$. If $(f, \widehat{S})$ and (g, T) are compatible pairs, and T and $\widehat{S}$ are continuous on (χ, ρ) , then $f, g, \widehat{S}$, and T have a unique common fixed point in χ .

The following corollary generalizes the result introduced by Sehgal [34].

Corollary 3. Let (χ, ρ) be a complete metric space, and $f, g, \widehat{S}, \widetilde{T}: \chi \longrightarrow \chi$ be four single-valued maps. Suppose that if, for all $r, j \in \chi$, with $\rho(fr, gj) > 0$, for some $\theta \in \mu$, $\Upsilon \in \Phi$ and $E: \mathbb{R}^{+4} \longrightarrow \mathbb{R}^{+}$, $\Upsilon(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \widetilde{T}j)) < 0$ implies that

$$\theta(\rho(fr, gj)) \leq [\theta(\max\{\rho(\widehat{S}r, \widetilde{T}j), \rho(fr, \widehat{S}r), \rho(gj, \widetilde{T}j)\})]^{E(G(r, j))}, \quad (87)$$

where

$$G(r, j) = \left\{ \rho(fr, \widehat{S}r), \rho(gj, \widetilde{T}j), \rho(fr, \widetilde{T}j), \rho(gj, \widehat{S}r) \right\}, \quad (88)$$

with $f(\chi) \subseteq \widetilde{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$. If $(f, \widehat{S})$ and $(g, \widetilde{T})$ are compatible pairs, and $\widetilde{T}$ and $\widehat{S}$ are continuous on (χ, ρ) , then $f, g, \widehat{S}$, and $\widetilde{T}$ have a unique common fixed point in χ .

4. On Suzuki-Type Ćirić JS-Contractions

We start this section with the following definition.

Definition 4. Let (χ, ρ) be a metric space and $f, g, \widehat{S}, \widetilde{T}: \chi \longrightarrow \chi$ be four single-valued maps. These maps form a new generalized Suzuki-type Ćirić JS-contraction if, for all $r, j \in \chi$, for some $\theta \in \Psi$, $\Upsilon \in \Phi$, $\Upsilon(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \widetilde{T}j)) < 0$ implies that

$$\begin{aligned} \theta(\rho(fr, gj)) &\leq [\theta(\rho(\widehat{S}r, \widetilde{T}j))]^a \cdot [\theta(\rho(fr, \widehat{S}r))]^b \\ &\quad \cdot [\theta(\rho(gj, \widetilde{T}j))]^c \cdot [\theta(\rho(fr, \widetilde{T}j) + \rho(gj, \widehat{S}r))]^d, \end{aligned} \quad (89)$$

where $a, b, c, d \geq 0$ with $a + b + c + 2d < 1$.

Our second result is as follows.

Theorem 3. Let (χ, ρ) be a complete metric space and $f, g, \widehat{S}, \widetilde{T}: \chi \longrightarrow \chi$ be four single-valued maps. Suppose that the mappings $f, g, \widehat{S}, \widetilde{T}$ form a generalized Suzuki-type Ćirić JS-contraction with $f(\chi) \subseteq \widetilde{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$. If $(f, \widehat{S})$ and $(g, \widetilde{T})$ are compatible pairs and $\widetilde{T}$ and $\widehat{S}$ are continuous on (χ, ρ) , then $f, g, \widehat{S}$, and $\widetilde{T}$ have a unique common fixed point in χ .

Proof. Let $r_0 \in \chi$ be an arbitrary point. As $f(\chi) \subseteq \widetilde{T}(\chi)$, there exists $r_0 \in \chi$ such that

$$f(r_0) = \widetilde{T}(r_1). \quad (90)$$

Since $g(r_1) \in \widehat{S}(\chi)$, we can choose $r_2 \in \chi$ such that $g(r_1) = \widehat{S}(r_2)$. In general, r_{2n+1} and r_{2n+2} are chosen in χ such that

$$\begin{aligned} f(r_{2n}) &= \widetilde{T}(r_{2n+1}), \\ g(r_{2n+1}) &= \widehat{S}(r_{2n+2}). \end{aligned} \quad (91)$$

We construct a sequence $\{\mathfrak{R}_n\}$ in χ such that

$$\begin{aligned} \mathfrak{R}_{2n} &= f(r_{2n}) = \widetilde{T}(r_{2n+1}), \\ \mathfrak{R}_{2n+1} &= g(r_{2n+1}) = \widehat{S}(r_{2n+2}), \end{aligned} \quad (92)$$

for $n = 0, 1, 2, \dots$. Since

$$\begin{aligned} \frac{1}{2}\rho(f(r_{2n}), \widehat{S}(r_{2n})) &= \frac{1}{2}\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}) < \rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}) \\ &= \rho(\widehat{S}(r_{2n}), \widetilde{T}(r_{2n+1})), \end{aligned} \quad (93)$$

we have

$$\Upsilon\left(\rho(f(r_{2n}), \widehat{S}(r_{2n})), \rho(\widehat{S}(r_{2n}), \widetilde{T}(r_{2n+1}))\right) < 0. \quad (94)$$

Then, from (89),

$$\begin{aligned} \theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) &= \theta(\rho(f(r_{2n}), g(r_{2n+1}))) \leq [\theta(\rho(\widehat{S}(r_{2n}), \widetilde{T}(r_{2n+1})))]^a \cdot [\theta(\rho(f(r_{2n}), \widehat{S}(r_{2n})))]^b \\ &\quad \cdot [\theta(\rho(g(r_{2n+1}), \widetilde{T}(r_{2n+1})))]^c \cdot [\theta(\rho(f(r_{2n}), \widetilde{T}(r_{2n+1})) + \rho(g(r_{2n+1}), \widehat{S}(r_{2n})))]^d \\ &= [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}))]^a \cdot [\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}))]^b \cdot [\theta(\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}))]^c \cdot [\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n}) + \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1}))]^d \\ &= [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}))]^{a+b} \cdot [\theta(\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}))]^c \cdot [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}) + \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1}))]^d. \end{aligned} \quad (95)$$

By (Ψ_5) , we have, for every $n \in \mathbb{N}$,

$$\begin{aligned} \theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}) + \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) &\leq \theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n})) \\ &\quad \cdot \theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})). \end{aligned} \quad (96)$$

Hence, (95) becomes

$$\begin{aligned} 1 < \theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) &\leq [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}))]^{a+b+d} \\ &\quad \cdot [\theta(\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}))]^{c+d}. \end{aligned} \quad (97)$$

It implies that

$$[\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1}))]^{1-c-d} \leq [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}))]^{a+b+d}. \quad (98)$$

Therefore,

$$\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) \leq [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}))]^{(a+b+d)/(1-c-d)}, \quad (99)$$

and so

$$\theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) \leq [\theta(\rho(\mathfrak{R}_{n-1}, \mathfrak{R}_n))]^{(a+b+d)/(1-c-d)}. \quad (100)$$

It implies that

$$\begin{aligned} 1 &< \theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) \\ &\leq [\theta(\rho(\mathfrak{R}_{n-1}, \mathfrak{R}_n))]^{((a+b+d)/(1-c-d))^2} \\ &\vdots \\ &\leq [\theta(\rho(\mathfrak{R}_0, \mathfrak{R}_1))]^{((a+b+d)/(1-c-d))^n}, \end{aligned} \quad (101)$$

for all $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$ in (101) and knowing that $\theta \in \Psi$, we have

$$\lim_{n \rightarrow \infty} \theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) = 1. \quad (102)$$

By (Ψ_3) , we obtain

$$\lim_{n \rightarrow \infty} \rho(\mathfrak{R}_n, \mathfrak{R}_{n+1}) = 0. \quad (103)$$

From condition (Ψ_4) , there exist $q \in (0, 1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{n \rightarrow \infty} \frac{\theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) - 1}{[\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})]^q} = \ell. \quad (104)$$

Suppose that $\ell < \infty$. Let $V = (\ell/2) > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) - 1}{[\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})]^q} - \ell \right| \leq V, \quad \text{for all } n \geq n_0. \quad (105)$$

This implies

$$\frac{\theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) - 1}{[\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})]^q} \geq \ell - V = V, \quad \text{for all } n \geq n_0. \quad (106)$$

Then,

$$n[\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})]^q \leq Pn[\theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) - 1], \quad \text{for all } n \geq n_0, \quad (107)$$

where $P = (1/V)$. Suppose now that $\ell = \infty$. Let $V > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \geq 1$ such that

$$\frac{\theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) - 1}{[\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})]^q} \geq V, \quad \text{for all } n \geq n_0, \quad (108)$$

which implies

$$n[\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})]^q \leq Pn[\theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) - 1], \quad n \geq n_0, \quad (109)$$

where $P = (1/V)$. Thus, in all cases, there exist $P > 0$ and $n_0 \geq 1$ such that

$$n[\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})]^q \leq Pn[\theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) - 1], \quad n \geq n_0. \quad (110)$$

By using (101), we obtain

$$n[\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})]^q \leq Pn \left([\theta(\rho(\mathfrak{R}_0, \mathfrak{R}_1))]^{((a+b+d)/(1-c-d))^n} - 1 \right), \quad \text{for all } n \geq n_0. \quad (111)$$

Setting $n \rightarrow \infty$ in inequality (111), we obtain

$$\lim_{n \rightarrow \infty} n[\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})]^q = 0. \quad (112)$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1}) \leq \frac{1}{n^{(1/q)}}, \quad \text{for all } n \geq n_1. \quad (113)$$

To prove $\{\mathfrak{R}_n\}$ is a Cauchy sequence, we use (113) and for $m > n \geq n_1$,

$$\rho(\mathfrak{R}_n, \mathfrak{R}_m) \leq \sum_{i=n}^{m-1} \rho(\mathfrak{R}_i, \mathfrak{R}_{i+1}) \leq \sum_{i=n}^{\infty} \rho(\mathfrak{R}_i, \mathfrak{R}_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{(1/q)}}. \quad (114)$$

The convergence of the series $\sum_{i=n}^{\infty} 1/i^{(1/q)}$ entails $\lim_{n,m \rightarrow \infty} \rho(\mathfrak{R}_n, \mathfrak{R}_m) = 0$. Thus, $\{\mathfrak{R}_n\}$ is a Cauchy sequence. Since χ is a complete metric space, there exists $r^* \in \chi$ such that $\lim_{n \rightarrow \infty} d(\mathfrak{R}_n, r^*) = 0$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} f(r_{2n}) &= \lim_{n \rightarrow \infty} \tilde{T}(r_{2n+1}) = \lim_{n \rightarrow \infty} g(r_{2n+1}) \\ &= \lim_{n \rightarrow \infty} \hat{S}(r_{2n+2}) = r^*. \end{aligned} \quad (115)$$

Now, we shall prove that $\mathfrak{R}^*$ is a common fixed point of $f, g, \hat{S}$, and $\tilde{T}$. As $\hat{S}$ is continuous, so

$$\lim_{n \rightarrow \infty} \hat{S}(f(r_{2n})) = \hat{S}(r^*) = \lim_{n \rightarrow \infty} \hat{S}(\hat{S}(r_{2n+2})). \quad (116)$$

Since $(f, \hat{S})$ is a compatible pair,

$$\lim_{n \rightarrow \infty} \rho(f(\hat{S}(r_{2n})), \hat{S}(f(r_{2n}))) = 0. \quad (117)$$

From Lemma 1, we have

$$\lim_{n \rightarrow \infty} f(\hat{S}(r_{2n})) = \hat{S}(r^*). \quad (118)$$

Put $r = \hat{S}(r_{2n})$ and $j = r_{2n+1}$ in (89) and if $\rho(\hat{S}(r^*), r^*) > 0$, we obtain

$$\frac{1}{2} \rho(f(\hat{S}(r_{2n})), \hat{S}(\hat{S}(r_{2n}))) < \rho(\hat{S}(\hat{S}(r_{2n})), \tilde{T}(r_{2n+1})). \quad (119)$$

Hence,

$$\Upsilon \left(\begin{array}{c} \rho(f(\hat{S}(r_{2n})), \hat{S}(\hat{S}(r_{2n}))), \\ \rho(\hat{S}(\hat{S}(r_{2n})), \tilde{T}(r_{2n+1})) \end{array} \right) < 0. \quad (120)$$

Then, from (89), we obtain

$$\begin{aligned}
\theta(\rho(f(\widehat{S}(r_{2n})), g(r_{2n+1}))) &\leq \left[\theta(\rho(\widehat{S}(\widehat{S}(r_{2n})), \widetilde{T}(r_{2n+1}))) \right]^a \\
&\cdot \left[\theta(\rho(f(\widehat{S}(r_{2n})), \widehat{S}(\widehat{S}(r_{2n})))) \right]^b \\
&\cdot \left[\theta(\rho(g(r_{2n+1}), \widetilde{T}(r_{2n+1}))) \right]^c \\
&\cdot \left[\theta(\rho(f(\widehat{S}(r_{2n})), \widetilde{T}(r_{2n+1})) \right. \\
&\quad \left. + \rho(g(r_{2n+1}), \widehat{S}(\widehat{S}(r_{2n})))) \right]^d.
\end{aligned} \tag{121}$$

Setting the upper limit, we obtain

$$\begin{aligned}
\theta(\rho(\widehat{S}(r^*), r^*)) &\leq [\theta(\rho(\widehat{S}r^*, r^*))]^a \\
&\cdot [\theta(\rho(\widehat{S}r^*, r^*) + \rho(r^*, \widehat{S}r^*))]^d \\
&\leq [\theta(\rho(\widehat{S}r^*, r^*))]^{a+2d} < \theta(\rho(\widehat{S}(r^*), r^*)),
\end{aligned} \tag{122}$$

a contradiction. Thus, $\rho(\widehat{S}(r^*), r^*) = 0$, and so $\widehat{S}(r^*) = r^*$. Again, since $\widetilde{T}$ is continuous,

$$\lim_{n \rightarrow \infty} \widetilde{T}(g(r_{2n+1})) = \widetilde{T}(r^*) = \lim_{n \rightarrow \infty} \widetilde{T}(\widetilde{T}(r_{2n+1})). \tag{123}$$

Since $(g, \widetilde{T})$ is a compatible pair,

$$\lim_{n \rightarrow \infty} \rho(g(\widetilde{T}(r_{2n+1})), \widetilde{T}(g(r_{2n+1}))) = 0. \tag{124}$$

From Lemma 1, we have

$$\lim_{n \rightarrow \infty} g(\widetilde{T}(r_{2n+1})) = \widetilde{T}(r^*). \tag{125}$$

Put $r_1 = r_{2n}$ and $r_2 = \widetilde{T}(r_{2n+1})$ in (89) and if $\rho(r^*, \widetilde{T}(r^*)) > 0$, we obtain

$$\frac{1}{2} \rho(f(r_{2n}), \widehat{S}(r_{2n})) < \rho(\widehat{S}(r_{2n}), \widetilde{T}(\widetilde{T}(r_{2n+1}))). \tag{126}$$

Therefore,

$$Y \left(\begin{array}{c} \rho(f(r_{2n}), \widehat{S}(r_{2n})), \\ \rho(\widehat{S}(r_{2n}), \widetilde{T}(\widetilde{T}(r_{2n+1}))) \end{array} \right) < 0, \tag{127}$$

and from (89), one obtains

$$\begin{aligned}
\theta(\rho(f(r_{2n}), g(\widetilde{T}(r_{2n+1})))) &\leq \left[\theta(\rho(\widehat{S}(r_{2n}), \widetilde{T}(\widetilde{T}(r_{2n+1})))) \right]^a \cdot [\theta(\rho(f(r_{2n}), \widehat{S}(r_{2n})))]^b \\
&\cdot \left[\theta(\rho(g(\widetilde{T}(r_{2n+1})), \widetilde{T}(\widetilde{T}(r_{2n+1})))) \right]^c \\
&\cdot \left[\theta(\rho(f(r_{2n}), \widetilde{T}(\widetilde{T}(r_{2n+1}))) + \rho(g(\widetilde{T}(r_{2n+1})), \widehat{S}(r_{2n}))) \right]^d.
\end{aligned} \tag{128}$$

Passing to the upper limit in (128), we obtain

$$\begin{aligned}
\theta(\rho(r^*, \widetilde{T}(r^*))) &\leq [\theta(\rho(r^*, \widetilde{T}(r^*)))]^{a+2d} \\
&< \theta(\rho(r^*, \widetilde{T}(r^*))),
\end{aligned} \tag{129}$$

a contradiction. Thus, $\rho(r^*, \widetilde{T}(r^*)) = 0$ and hence $r^* = \widetilde{T}(r^*)$. Suppose that $\rho(f(r^*), r^*) > 0$, we obtain

$$\frac{1}{2} \rho(\widehat{S}(r^*), \widetilde{T}(r_{2n+1})) < \rho(f(r^*), \widehat{S}(r^*)), \quad \text{for all } n \in \mathbb{N}, \tag{130}$$

and so

$$Y \left(\begin{array}{c} \rho(f(r^*), \widehat{S}(r^*)), \\ \rho(\widehat{S}(r^*), \widetilde{T}(r_{2n+1})) \end{array} \right) < 0. \tag{131}$$

By (89), we obtain

$$\begin{aligned}
\theta(\rho(f(r^*), g(r_{2n+1}))) &\leq \left[\theta(\rho(\widehat{S}(r^*), \widetilde{T}(r_{2n+1}))) \right]^a \cdot [\theta(\rho(f(r^*), \widehat{S}(r^*)))]^b \\
&\cdot \left[\theta(\rho(g(r_{2n+1}), \widetilde{T}(r_{2n+1}))) \right]^c \cdot \left[\theta(\rho(f(r^*), \widetilde{T}(r_{2n+1})) + \rho(g(r_{2n+1}), \widehat{S}(r^*))) \right]^d.
\end{aligned} \tag{132}$$

Taking the upper limit in (132) and as $r^* = \widetilde{T}(r^*) = \widehat{S}(r^*)$, we obtain

$$\theta(\rho(f(r^*), r^*)) \leq [\theta(\rho(f(r^*), r^*))]^{b+d} < \theta(\rho(f(r^*), r^*)), \tag{133}$$

a contradiction. Thus, $\rho(f(r^*), r^*) = 0$ and hence $r^* = \underline{f}(r^*)$. Finally, suppose $\rho(r^*, g(r^*)) > 0$ and as $r^* = \underline{T}(r^*) = \widehat{S}(r^*) = f(r^*)$, we have

$$Y(0, \rho(r^*, g(r^*))) < 0. \quad (134)$$

From (89), we deduce

$$\theta(\rho(r^*, g(r^*))) = \theta(\rho(f(r^*), g(r^*))) < \theta(\rho(r^*, g(r^*))), \quad (135)$$

a contradiction. Thus, $\rho(r^*, g(r^*)) = 0$ and $r^* = g(r^*)$. Therefore, r^* is a common of the four mappings $\widehat{S}$, g , f , and $\underline{T}$. Next, assume that v^* is another common of the four mappings $\widehat{S}$, g , f , and $\underline{T}$ such that $r^* \neq v^*$, one obtains

$$Y\left(0, \rho\left(\widehat{S}(r^*), \underline{T}(v^*)\right)\right) < 0. \quad (136)$$

Then, by (89),

$$\begin{aligned} \theta(\rho(r^*, v^*)) &= \theta\left(\rho\left(\widehat{S}(r^*), \underline{T}(v^*)\right)\right) \leq \left[\theta\left(\rho\left(\widehat{S}(r^*), \underline{T}(v^*)\right)\right)\right]^a \cdot \left[\theta\left(\rho\left(f(r^*), \widehat{S}(r^*)\right)\right)\right]^b \\ &\quad \cdot \left[\theta\left(\rho\left(g(v^*), \underline{T}(v^*)\right)\right)\right]^c \cdot \left[\theta\left(\rho\left(f(r^*), \underline{T}(v^*)\right) + \rho\left(g(v^*), \widehat{S}(r^*)\right)\right)\right]^d. \end{aligned} \quad (137)$$

It implies that

$$\theta(\rho(r^*, v^*)) \leq [\theta(\rho(r^*, v^*))]^{a+2d} < \theta(\rho(r^*, v^*)), \quad (138)$$

a contradiction. Thus, $r^* = v^*$, that is, r^* is the unique common fixed point of $\widehat{S}$, g , f , and $\underline{T}$. $\square$

Example 6. Let $\chi = [0, \infty)$ and define the function $\rho: \chi \times \chi \rightarrow [0, +\infty)$ by $\rho(r, j) = |r - j|$. Clearly, (χ, ρ) is a complete metric space. Let $\theta(t) = e^{\sqrt{t}}$, and $Y(s, t) = (s/2) - t$, then $\theta \in \Psi$ and $Y \in \Phi$. Define the mappings $\widehat{S}, g, f, \underline{T}: \chi \rightarrow \chi$ by

$$\begin{aligned} \widehat{S}(r) &= e^{7r} - 1, \\ g(r) &= \ln\left(1 + \frac{r}{6}\right), \\ f(r) &= \ln\left(1 + \frac{r}{6}\right), \\ \underline{T}(r) &= e^{6r} - 1. \end{aligned} \quad (139)$$

Clearly, $f(\chi) = \underline{T}(\chi) = g(\chi) = \widehat{S}(\chi)$ if $\{r_n\}$ is a sequence in χ such that

$$\lim_{n \rightarrow \infty} f(r_n) = \lim_{n \rightarrow \infty} \widehat{S}(r_n) = t, \quad \text{for some } t \in \chi. \quad (140)$$

Then,

$$\lim_{n \rightarrow \infty} |f(r_n) - t| = \lim_{n \rightarrow \infty} |\widehat{S}(r_n) - t| = 0, \quad (141)$$

and equivalently,

$$\lim_{n \rightarrow \infty} \left| \ln\left(1 + \frac{r_n}{6}\right) - t \right| = \lim_{n \rightarrow \infty} |e^{7r_n} - 1 - t| = 0. \quad (142)$$

Thus,

$$\lim_{n \rightarrow \infty} \left| r_n - (6e^t - 6) \right| = \lim_{n \rightarrow \infty} \left| r_n - \frac{\ln(1+t)}{7} \right| = 0. \quad (143)$$

It gives that $6e^t - 6 = (\ln(1+t)/7)$ (by uniqueness of limit); hence, $t = 0$. Using continuity of f and $\widehat{S}$, one obtains

$$\begin{aligned} \lim_{n \rightarrow \infty} \rho(f\widehat{S}(r_n), \widehat{S}f(r_n)) &= \lim_{n \rightarrow \infty} |f\widehat{S}(r_n) - \widehat{S}f(r_n)| \\ &= |f(t) - \widehat{S}(t)| = |0 - 0| = 0, \end{aligned} \quad (144)$$

for $t = 0 \in \chi$. Hence, the pair $(f, \widehat{S})$ is compatible. Similarly, the pair $(g, \underline{T})$ is compatible. Next, for all $r, j \in \chi$ with

$$Y(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \underline{T}j)) < 0. \quad (145)$$

So, for $a = b = c = 0.1$ and $d = 0.3$, we have

$$\begin{aligned} \theta(\rho(fr, gj)) &\leq [\theta(\rho(\widehat{S}r, \underline{T}j))]^a \cdot [\theta(\rho(fr, \widehat{S}r))]^b \\ &\quad \cdot [\theta(\rho(gj, \underline{T}j))]^c \\ &\quad \cdot [\theta(\rho(fr, \underline{T}j) + \rho(gj, \widehat{S}r))]^d. \end{aligned} \quad (146)$$

Hence, all hypotheses of Theorem 3 are satisfied. Thus, $f, g, \widehat{S}$, and $\underline{T}$ have a unique common fixed point.

5. An Application

We apply the result given by Theorem 2 to study the existence of a solution for a system of nonlinear fractional differential equations. Let $\chi = C([0, 1], \mathbb{R})$ be the space of all continuous functions on $[0, 1]$. The metric on χ is given by

$$\rho(r, j) = \|r - j\|_{\infty} = \max_{t \in [0, 1]} |r(t) - j(t)|, \quad r, j \in \chi. \quad (147)$$

Then, $\chi = C([0, 1], \mathbb{R})$ is complete metric space.

Consider the following system of fractional differential equations:

$$\begin{cases} {}^C D^\alpha r(t) = K_1(t, \widehat{S}(r(t))), {}^C D^\alpha j(t) = K_2(t, \underline{T}(j(t))), \end{cases} \quad (148)$$

with boundary conditions

$$\begin{cases} r(0) = 0, Ir(1) = r'(0), \\ j(0) = 0, Qj(1) = j'(0). \end{cases} \quad (149)$$

Note that ${}^C D^\alpha$ denotes the Caputo fractional derivative of order α , defined by

$$\left\{ {}^C D^\alpha K_1(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t ((t-s)^{n-\alpha-1} K_1^n(s)) ds, {}^C D^\alpha K_2(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t ((t-s)^{n-\alpha-1} K_2^n(s)) ds, \right. \quad (150)$$

where we consider

$$\begin{aligned} n-1 < \alpha, \\ \alpha < 1, \\ n = [\alpha] + 1, \end{aligned} \quad (151) \quad \left\{ \begin{aligned} I^\alpha K_1(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_1(s) ds, \quad \text{with } \alpha > 0, \\ Q^\alpha K_2(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_2(s) ds, \quad \text{with } \alpha > 0. \end{aligned} \right. \quad (152)$$

and $I^\alpha K_1$ and $I^\alpha K_2$ denote the Riemann–Liouville fractional integral of order α of continuous functions K_1 and K_2 , given by

System (148) can be written in the following integral form:

$$\left\{ \begin{aligned} r(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_1(s, \widehat{S}(r(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} K_1(u, \widehat{S}(r(u))) du ds, \\ j(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_2(s, \widetilde{T}(j(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} K_2(u, \widetilde{T}(j(u))) du ds. \end{aligned} \right. \quad (153)$$

Define the mappings $f, g: \chi \longrightarrow \chi$ by

$$\begin{aligned} f(r(t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_1(s, \widehat{S}(r(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} K_1(u, \widehat{S}(r(u))) du ds, \\ g(j(t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_2(s, \widetilde{T}(j(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} K_2(u, \widetilde{T}(j(u))) du ds. \end{aligned} \quad (154)$$

Theorem 4. Assume that the following conditions hold:

- (i) $K_1, K_2[0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.
- (ii) $K_1(s, \cdot), K_2(s, \cdot): \mathbb{R} \longrightarrow \mathbb{R}$ are increasing functions.
- (iii) For all $r, j \in \chi$ with $f(r) \leq \widetilde{T}(j)$ and $|f(r(s)) - g(j(s))| > 0$, we have

$$\left| K_1(s, \widehat{S}(r(s))) - K_2(s, \widetilde{T}(j(s))) \right| \leq \frac{e^{-\tau} \Gamma(\alpha+1)}{4} A(r, j), \quad (155)$$

where

$$A(r, j) = \max \left[\begin{aligned} &|\widehat{S}(r(s)) - \widetilde{T}(j(s))|, |f(r(s)) - \widehat{S}(r(s))|, \\ &|g(j(s)) - \widetilde{T}(j(s))|, \frac{|f(r(s)) - \widetilde{T}(j(s))| + |g(j(s)) - \widehat{S}(r(s))|}{2}, \\ &\frac{(1 + |f(r(s)) - \widehat{S}(r(s))|) |g(j(s)) - \widetilde{T}(j(s))|}{1 + |\widehat{S}(r(s)) - \widetilde{T}(j(s))|} \end{aligned} \right]. \quad (156)$$

(iv) There exists $r_0, j_0 \in C([0, 1], \mathbb{R})$ such that, for all $t \in [0, 1]$, we have

$$\begin{cases} r_0(t) \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_1(s, \widehat{S}(r_0(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} K_1(u, \widehat{S}(r_0(u))) du ds, \\ j_0(t) \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_2(s, \widetilde{T}(j_0(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} K_2(u, \widetilde{T}(j_0(u))) du ds. \end{cases} \quad (157)$$

(v) If there exists a sequence $\{r_n\}$ in χ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \rho(f(\widehat{S}(r_n)), \widehat{S}(f(r_n))) &= 0, \\ \lim_{n \rightarrow \infty} \rho(g(\widetilde{T}(r_n)), \widetilde{T}(g(r_n))) &= 0, \end{aligned} \quad (158)$$

whenever

$$\begin{aligned} \lim_{n \rightarrow \infty} f(r_n) &= \lim_{n \rightarrow \infty} \widehat{S}(r_n) = t, \\ \lim_{n \rightarrow \infty} g(r_n) &= \lim_{n \rightarrow \infty} \widetilde{T}(r_n) = t, \end{aligned} \quad (159)$$

for some $t \in \chi$.

Then, system (148) has a solution.

Proof. Following assumptions (iii) and (iv), we have

$$\begin{aligned} & |f(r(t)) - g(j(t))| \\ &= \left| \begin{aligned} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_1(s, \widehat{S}(r(s))) ds \\ & - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_2(s, \widetilde{T}(j(s))) ds \\ & + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} K_1(u, \widehat{S}(r(u))) du ds \\ & - \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} K_2(u, \widetilde{T}(j(u))) du ds \end{aligned} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |K_1(s, \widehat{S}(r(s))) - K_2(s, \widetilde{T}(j(s)))| ds \\ &\quad + \frac{2}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} |K_1(s, \widehat{S}(r(s))) - K_2(u, \widetilde{T}(j(u)))| du ds \\ &\leq \frac{1}{\Gamma(\alpha)} \frac{e^{-\tau} \Gamma(\alpha+1)}{4} \cdot \int_0^t (t-s)^{\alpha-1} A(r, j) ds \\ &\quad + \frac{2}{\Gamma(\alpha)} \frac{e^{-\tau} \Gamma(\alpha+1)}{4} \cdot \int_0^1 \int_0^s (s-u)^{\alpha-1} A(r, j) du ds \\ &\leq \frac{1}{\Gamma(\alpha)} \frac{e^{-\tau} \Gamma(\alpha+1)}{4} \cdot A(r, j) \cdot \int_0^t (t-s)^{\alpha-1} ds \end{aligned}$$

$$\begin{aligned}
& + \frac{2}{\Gamma(\alpha)} \frac{e^{-\tau} \Gamma(\alpha+1)}{4} \cdot A(r, j) \cdot \int_0^1 \int_0^s (s-u)^{\alpha-1} du ds \\
& \leq \left(\frac{e^{-\tau} \Gamma(\alpha) \cdot \Gamma(\alpha+1)}{4\Gamma(\alpha) \cdot \Gamma(\alpha+1)} \right) \cdot A(r, j) + 2e^{-\tau} B(\alpha+1, 1) \frac{\Gamma(\alpha) \cdot \Gamma(\alpha+1)}{4\Gamma(\alpha) \cdot \Gamma(\alpha+1)} \cdot A(r, j) \\
& \leq \frac{e^{-\tau}}{4} A(r, j) + \frac{e^{-\tau}}{2} A(r, j),
\end{aligned} \tag{160}$$

where B is the beta function. From the above inequality, we obtain that

$$\rho(f(r), g(j)) \leq \frac{3}{4} e^{-\tau} A(r, j). \tag{161}$$

It implies that

$$\begin{aligned}
e^{\sqrt{\rho(f(r), g(j))}} & \leq e^{\sqrt{(3/4)e^{-\tau} A(r, j)}} = e^{\sqrt{(3/4)e^{-\tau}} \sqrt{A(r, j)}} \\
& = \left[e^{\sqrt{A(r, j)}} \right]^{\sqrt{(3/4)e^{-\tau}}} = \left[e^{\sqrt{A(r, j)}} \right]^{E(G(r, j))},
\end{aligned} \tag{162}$$

where $E(s_1, s_2, s_3, s_4) = \sqrt{3/4e^{-\tau}} < 1$, $G(r, j) = \{ \rho(fr, \widehat{Sr}), \rho(gj, \widetilde{T}j), \rho(fr, \widetilde{T}j), \rho(gj, \widehat{Sr}) \}$ and $\theta(t) = e^{\sqrt{t}}$.

Since this inequality holds for all $r, j \in \chi$ with $f(r) \leq \widetilde{T}(j)$, so it is true for any $Y \in \Phi$,

$$Y(\rho(fr, \widehat{Sr}), \rho(\widehat{Sr}, \widetilde{T}j)) < 0. \tag{163}$$

Hence, we have

$$\theta(\rho(f(r), g(j))) \leq [\theta(A(r, j))]^{E(G(r, j))}. \tag{164}$$

Thus, $f, g, \widehat{S}$, and $\widetilde{T}$ are generalized Suzuki-type (θ, E) -rational contraction mappings. Therefore, all hypotheses of Theorem 2 are satisfied. Hence, $f, g, \widehat{S}$, and $\widetilde{T}$ have a common fixed point, that is, system (148) has at least one solution. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors read and approved the final manuscript.

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Vahid Parvaneh dedicates this article to the first teacher, Mr. Allah Karam Parsaei, of the village where he was born, for his hard work and efforts in education during the years 1975-1977.

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