Research Article

Composition Formulae for the k -Fractional Calculus Operators Associated with k -Wright Function

D. L. Suthar

Department of Mathematics, Wollo University, P.O. Box 1145, Dessie, Ethiopia

Correspondence should be addressed to D. L. Suthar; dlsuthar@gmail.com

Received 18 April 2020; Revised 29 May 2020; Accepted 1 June 2020; Published 21 July 2020

Academic Editor: Martin Bohner

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In this article, the k -fractional-order integral and derivative operators including the k -hypergeometric function in the kernel are used for the k -Wright function; the results are presented for the k -Wright function. Also, some of special cases related to fractional calculus operators and k -Wright function are considered.

1. Introduction and Preliminaries

Fractional calculus was introduced in 1695, but in the last two decades researchers have been able to use it properly on the account of availability of computational resources. In many areas of application of fractional calculus, the researchers found significant applications in science and engineering. In the literature, many applications of fractional calculus are available in astrophysics, biosignal processing, fluid dynamics, nonlinear control theory, stochastic dynamical system, and so on. Also, a number of researchers [1–10] have studied in-depth level of properties,

applications, and various directions of extensions of Gauss hypergeometric function of fractional integration.

Recently, in a series of research publications on generalized classical fractional calculus operators, research by Mubeen and Habibullah [11] has been published on the integral part of the Riemann-Liouville version and its applications; an alternative definition for the k -Riemann-Liouville fractional derivative was introduced by Dorrego [12]. The left- and right-hand operators of Saigo k -fractional integration and differentiation associated with the k -Gauss hypergeometric function defined by Gupta and Parihar [13] (see also [14]) are as follows:

$$(I_{0+,k}^{\tau,\delta,\gamma} f)(x) = \frac{x^{(-\tau-\delta)/k}}{k\Gamma_k(\tau)} \int_0^x (x-t)^{(\tau/k)-1} {}_2F_{1,k}\left((\tau+\delta, k), (-\gamma, k); (\tau, k); \left(1-\frac{t}{x}\right)\right) f(t) dt; \quad \Re(\tau) > 0, k > 0, \quad (1)$$

$$(I_{-,k}^{\tau,\delta,\gamma} f)(x) = \frac{1}{k\Gamma_k(\tau)} \int_x^\infty (t-x)^{(\tau/k)-1} t^{(-\tau-\delta)/k} {}_2F_{1,k}\left((\tau+\delta, k), (-\gamma, k); (\tau, k); \left(1-\frac{x}{t}\right)\right) f(t) dt; \quad \Re(\tau) > 0, k > 0, \quad (2)$$

where ${}_2F_{1,k}((\tau, k), (\delta, k); (\gamma, k); x)$ is the k -Gauss hypergeometric function defined by [11] for $x \in \mathbb{C}, |x| < 1$, $\Re(\gamma) > \Re(\delta) > 0$:

$${}_2F_{1,k}((\tau, k), (\delta, k); (\gamma, k); x) = \sum_{n=0}^{\infty} \frac{(\tau)_{n,k} (\delta)_{n,k} x^n}{(\gamma)_{n,k} n!}. \quad (3)$$

The corresponding fractional differential operators have their respective forms as

$$\begin{aligned}
 (D_{0+,k}^{\tau,\delta,\gamma} f)(x) &= \left(\frac{d}{dx} \right)^n \left(I_{0+,k}^{-\tau+n, -\delta-n, \tau+\gamma-n} f \right)(x); \quad \Re(\tau) > 0, k > 0; \quad n = [\Re(\tau) + 1] \\
 (D_{0+,k}^{\tau,\delta,\gamma} f)(x) &= \left(\frac{d}{dx} \right)^n \frac{x^{((\tau+\delta)/k)}}{k\Gamma_k(-\tau+n)} \int_0^x (x-t)^{-(\tau/k)+n-1} {}_2F_{1,k} \left((-\tau-\delta, k), (-\gamma-\tau+n, k); (-\tau+n, k); \left(1-\frac{t}{x}\right) \right) f(t) dt, \\
 (D_{-,k}^{\tau,\delta,\gamma} f)(x) &= \left(-\frac{d}{dx} \right)^n \left(I_{-,k}^{-\tau+n, -\delta-n, \tau+\gamma} f \right)(x); \quad \Re(\tau) > 0, k > 0; \quad n = [\Re(\tau) + 1] \\
 (D_{-,k}^{\tau,\delta,\gamma} f)(x) &= \left(-\frac{d}{dx} \right)^n \frac{1}{k\Gamma_k(-\tau+n)} \int_x^\infty (t-x)^{-((\tau+n)/k)-1} t^{(\tau+\delta)/k} {}_2F_{1,k} \left((-\tau-\delta, k), (-\gamma-\tau, k); (-\tau+n, k); \left(1-\frac{x}{t}\right) \right) f(t) dt,
 \end{aligned} \tag{4}$$

where $x > 0, \tau \in \mathbb{C}, \Re(\tau) > 0, k > 0$, and $[\Re(\tau)]$ is the integer part of $\Re(\tau)$.

Remark 1. If we set $k = 1$ in equations (1), (2), (4), and (5), operators reduce to Saigo's fractional integral and derivative operators stated in [5], respectively.

Now, we consider the following basic results for our study.

Lemma 1 (see [13], pp. 497, Eq. 4.2). Let $\tau, \delta, \gamma, \zeta \in \mathbb{C}, \Re(\zeta) > \max[0, \Re(\delta - \gamma)]$; then,

$$(I_{0+,k}^{\tau,\delta,\gamma} t^{(\zeta/k)-1})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\zeta)\Gamma_k(\zeta - \delta + \gamma)}{\Gamma_k(\zeta - \delta)\Gamma_k(\zeta + \tau + \gamma)} x^{((\zeta - \delta)/k) - 1}. \tag{6}$$

Lemma 2 (see [13], pp. 497, Eq. 4.3). Let $\tau, \delta, \gamma, \zeta \in \mathbb{C}, \Re(\tau) > 0, k \in \Re^+(0, \infty)$, and $\Re(\zeta) > \max[\Re(-\delta), \Re(-\gamma)]$; then,

$$(I_{-,k}^{\tau,\delta,\gamma} t^{-(\zeta/k)})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\zeta + \delta)\Gamma_k(\zeta + \gamma)}{\Gamma_k(\zeta)\Gamma_k(\zeta + \tau + \delta + \gamma)} x^{(-\zeta - \delta)/k}. \tag{7}$$

Lemma 3 (see [13], pp. 500, Eq. 6.2). Let $\tau, \delta, \gamma, \zeta \in \mathbb{C}, n = [\Re(\tau)] + 1, k \in \Re^+(0, \infty)$ such that $\Re(\zeta) > \max[0, \Re(-\tau - \delta - \gamma)]$; then,

$$(D_{0+,k}^{\tau,\delta,\gamma} t^{(\zeta/k)-1})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\zeta)\Gamma_k(\zeta + \delta + \gamma + \tau)}{\Gamma_k(\zeta + \gamma)\Gamma_k(\zeta + \delta + n - nk)} x^{((\zeta + \delta + n)/k) - n - 1}. \tag{8}$$

Lemma 4 (see [13], pp. 500, Eq. 6.3). Let $\tau, \delta, \gamma, \zeta \in \mathbb{C}$, and $n = [\Re(\tau)] + 1, k \in \Re^+$, and $\Re(\zeta) > \max[\Re(-\tau - \gamma), \Re(\delta - nk + n)]$; then,

$$(D_{-,k}^{\tau,\delta,\gamma} t^{-(\zeta/k)})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\zeta - \delta - n + nk)\Gamma_k(\zeta + \tau + \gamma)}{\Gamma_k(\zeta)\Gamma_k(\zeta - \delta + \gamma)} x^{(-\zeta + \delta + n)/k - n}. \tag{9}$$

Recently, Gehlot and Prajapati [15] studied the concept of the generalized k -Wright function, which is presented in the following definition, and its connection with other special functions. It is the generalization of Mittag-Leffler function and many other special functions (see also, [16–21]). These special functions have found many important applications in solving problems of physics, biology, engineering, and applied sciences.

The k -Wright function is defined for $k \in \mathbb{R}^+; x, a_i, b_j \in \mathbb{C}, \varepsilon_i, \varsigma_j \in \mathbb{R} (\varepsilon_i, \varsigma_j \neq 0; i = 1, 2, \dots, p; j = 1, 2, \dots, q)$, and $(a_i + \varepsilon_i n), (b_j + \varsigma_j n) \in \mathbb{C} \setminus kz^-$ as

$$\begin{aligned}
 {}_p\Psi_q^k(x) &= {}_p\Psi_q^k \left[\begin{matrix} (a_1, \varepsilon_1), \dots, (a_p, \varepsilon_p); \\ (b_1, \varsigma_1), \dots, (b_q, \varsigma_q); \end{matrix} x \right] \\
 &= {}_p\Psi_q^k \left[\begin{matrix} (a_i, \varepsilon_i)_{1,p}; \\ (b_j, \varsigma_j)_{1,q}; \end{matrix} x \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma_k(a_i + \varepsilon_i n)}{\prod_{j=1}^q \Gamma_k(b_j + \varsigma_j n)} \frac{(x)^n/n!}{(x)^n/n!},
 \end{aligned} \tag{10}$$

with the convergence conditions described as

$$\sum_{j=1}^q \frac{\varsigma_j}{k} - \sum_{i=1}^p \frac{\varepsilon_i}{k} > -1. \tag{11}$$

Remark 2. When we put $k = 1$ in (10), the k -Wright function reduces to Wright function which is stated in [22].

The following relation of the k -Wright function in terms of the generalized k -Mittag-Leffler function, k -Bessel function, k -hypergeometric function, and Mittag-Leffler

family function is defined as follows by giving the appropriate values of the parameters:

- (1) For $p = 1, q = 2$, the generalized k -Mittag-Leffler function from Gehlot [17] is

$${}_1\Psi_2^k(x) = {}_1\Psi_2^k \left[\begin{matrix} (\vartheta, \omega k); \\ (\varsigma, \varepsilon), (\vartheta, 0); \end{matrix} x \right] \\ = \sum_{n=0}^{\infty} \frac{(\vartheta)_{n\omega, k}}{\Gamma_k(\varepsilon n + \varsigma)} \frac{x^n}{n!} = E_{k, \varepsilon, \varsigma}^{\vartheta, \omega}(x), \quad (12)$$

Here, Díaz and Pariguan [23] introduced the k -Pochhammer symbol and k -gamma function as follows:

$$(\mathfrak{S})_{n, k} = \begin{cases} \frac{\Gamma_k(\mathfrak{S} + nk)}{\Gamma_k(\mathfrak{S})}, & k \in \mathfrak{R}, \mathfrak{S} \in \mathbb{C}/\{0\}; \\ \mathfrak{S}(\mathfrak{S} + k) \dots (\mathfrak{S} + (n-1)k), & n \in \mathbb{N}, \mathfrak{S} \in \mathbb{C}, \end{cases} \quad (13)$$

and the relation with classical Euler's gamma function is as follows:

$$\Gamma_k(\mathfrak{S}) = k^{(\mathfrak{S}/k)-1} \Gamma\left(\frac{\mathfrak{S}}{k}\right), \quad (14)$$

where $\mathfrak{S} \in \mathbb{C}$, $k \in \mathfrak{R}$, and $n \in \mathbb{N}$. For more information on the k -Pochhammer symbol, k -special functions, and fractional Fourier transforms, refer to Romero and Cerutti's [24] articles.

- (2) For $p = 1, q = 2$, the generalized k -Mittag-Leffler function from Dorrego and Cerutti [16] is

$${}_1\Psi_2^k(x) = {}_1\Psi_2^k \left[\begin{matrix} (\vartheta, k); \\ (\varsigma, \varepsilon), (\vartheta, 0); \end{matrix} x \right] \\ = \sum_{n=0}^{\infty} \frac{(\vartheta)_{nk}}{\Gamma_k(\varepsilon n + \varsigma)} \frac{x^n}{n!} = E_{k, \varepsilon, \varsigma}^{\vartheta}(x). \quad (15)$$

- (3) For $p = 1, q = 3$, the k -Bessel function of the first kind from Cerutti [25] is

$${}_1\Psi_3^k(x) = {}_1\Psi_3^k \left[\begin{matrix} (\vartheta, k); \\ (\varsigma + 1, \varepsilon), (\vartheta, 0)(k, k); \end{matrix} \frac{-xk}{2} \right] \\ = \sum_{n=0}^{\infty} \frac{(-1)^n (\vartheta)_{nk}}{\Gamma_k(\varepsilon n + \varsigma + 1)} \frac{(x/2)^n}{(n!)^2} = J_{k, \varsigma}^{\vartheta, \varepsilon}(x). \quad (16)$$

- (4) For $p = 3, q = 3$, the k -hypergeometric function with three parameters from Mubeen et al. [26] is

$${}_3\Psi_3^k(x) = {}_3\Psi_3^k \left[\begin{matrix} (\vartheta, k), (\varphi, k), (\varsigma, 0); \\ (\varsigma, k), (\vartheta, 0)(\varphi, 0); \end{matrix} x \right] \\ = \sum_{n=0}^{\infty} \frac{(\vartheta)_{nk} (\varphi)_{nk} x^n}{(\varsigma)_{nk} n!} = {}_2F_{1, k}((\vartheta, k), (\varphi, k); (\varsigma, k); x). \quad (17)$$

- (5) For $k = 1, p = 1, q = 2$, the generalized Mittag-Leffler function from Shukla and Prajapati [20] is

$${}_1\Psi_2^1(x) = {}_1\Psi_2^1 \left[\begin{matrix} (\vartheta, \omega); \\ (\varsigma, \varepsilon), (\vartheta, 0); \end{matrix} x \right] \\ = \sum_{n=0}^{\infty} \frac{(\vartheta)_{n\omega}}{\Gamma(\varepsilon n + \varsigma)} \frac{x^n}{n!} = E_{\varepsilon, \varsigma}^{\vartheta, \omega}(x). \quad (18)$$

- (6) For $k = 1, p = 1, q = 2$, the generalized Mittag-Leffler function from Prabhakar [19] is

$${}_1\Psi_2^1(x) = {}_1\Psi_2^1 \left[\begin{matrix} (\vartheta, 1); \\ (\varsigma, \varepsilon), (\vartheta, 0); \end{matrix} x \right] \\ = \sum_{n=0}^{\infty} \frac{(\vartheta)_n}{\Gamma(\varepsilon n + \varsigma)} \frac{x^n}{n!} = E_{\varepsilon, \varsigma}^{\vartheta}(x). \quad (19)$$

- (7) For $k = 1, p = 1$, and $q = 2$, the Mittag-Leffler function from Wiman [21] is

$${}_1\Psi_1^1(x) = {}_1\Psi_1^1 \left[\begin{matrix} (1, 1); \\ (\varsigma, \varepsilon); \end{matrix} x \right] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\varepsilon n + \varsigma)} \frac{x^n}{n!} = E_{\varepsilon, \varsigma}(x). \quad (20)$$

- (8) For $k = 1, p = 1$, and $q = 2$, the Mittag-Leffler function from Mittag-Leffler [18] is

$${}_1\Psi_1^1(x) = {}_1\Psi_1^1 \left[\begin{matrix} (1, 1); \\ (1, \varepsilon); \end{matrix} x \right] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\varepsilon n + 1)} \frac{x^n}{n!} = E_{\varepsilon}(x). \quad (21)$$

2. Saigo k -Fractional Integration in terms of k -Wright Function

In this section, we present the composition formulas of k -fractional integrals (1) and (2), involving the k -Wright function.

Theorem 1. Let $\tau, \delta, \gamma \in \mathbb{C}; k \in \mathbb{R}^+$, $c, \varepsilon_i, \varsigma_j \in \mathbb{R} (\varepsilon_i, \varsigma_j \neq 0; i = 1, 2, \dots, p; j = 1, 2, \dots, q)$, and $\nu > 0$ such that $\Re(\tau) > 0$, $\Re(\zeta) > \max[0, \Re(\delta - \gamma)]$, and $\Re(\zeta + \gamma - \delta) > 0$. If condition (11) is satisfied and $I_{0+, k}^{\tau, \delta, \gamma}$ be the left-sided integral operator of the generalized k -fractional integration associated

with k -Wright function, then the following equation holds true:

$$\begin{aligned} & \left(I_{0+,k}^{\tau,\delta,\gamma} \left\{ t^{(\zeta/k)-1} {}_p\Psi_q^k \left[\begin{matrix} (a_i, \varepsilon_i)_{1,p}; \\ (b_j, \varsigma_j)_{1,q}; \end{matrix} ; ct^{v/k} \right] \right\} \right) (x) \\ &= x^{(\zeta-\delta/k)-1} {}_{p+2}\Psi_{q+2}^k \left[\begin{matrix} (a_1, \varepsilon_1), \dots, (a_p, \varepsilon_p), (\zeta, v), (\zeta + \gamma - \delta, v); \\ (b_1, \varsigma_1), \dots, (b_q, \varsigma_q), (\zeta - \delta, v), (\zeta + \tau + \gamma, v); \end{matrix} ; kcx^{v/k} \right]. \end{aligned} \quad (22)$$

Proof. We indicate the R.H.S. of equation (22) by I_1 , and invoking equation (10), we obtain

Now applying equation (6), we get

$$\begin{aligned} I_1 &= I_{0+,k}^{\tau,\delta,\gamma} \left(t^{(\zeta/k)-1} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma_k(a_i + \varepsilon_i n)}{\prod_{j=1}^q \Gamma_k(b_j + \varsigma_j n)} \frac{(ct^{v/k})^n}{n!} \right) (x) \\ &= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma_k(a_i + \varepsilon_i n)}{\prod_{j=1}^q \Gamma_k(b_j + \varsigma_j n)} \frac{c^n}{n!} I_{0+,k}^{\tau,\delta,\gamma} \left(t^{(\zeta+vn)/k} \right) (x). \end{aligned} \quad (23)$$

$$I_1 = x^{((\zeta-\delta)/k)-1} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma_k(a_i + \varepsilon_i n) \Gamma_k(\zeta + vn) \Gamma_k(\zeta + \gamma - \delta + vn)}{\prod_{j=1}^q \Gamma_k(b_j + \varsigma_j n) \Gamma_k(\zeta - \delta + vn) \Gamma_k(\zeta + \tau + \gamma + vn)} \frac{(kcx^{v/k})^n}{n!}. \quad (24)$$

Now, interpreting definition (10) on the aforementioned equation, we arrive at the desired result (22). $\square$

and $\Re(\zeta + \tau) > \max[-\Re(\delta), -\Re(\gamma)]$, with $\Re(\delta) \neq \Re(\gamma)$. If condition (11) is satisfied and $I_{-,k}^{\tau,\delta,\gamma}$ be the right-sided integral operator of the generalized k -fractional integration associated with k -Wright function, then the following equation holds true:

Theorem 2. Let $\tau, \delta, \gamma \in \mathbb{C}$; $k \in \mathbb{R}^+$, $c, \varepsilon_i, \varsigma_j \in \mathbb{R}$ ($\varepsilon_i, \varsigma_j \neq 0$; $i = 1, 2, \dots, p$; $j = 1, 2, \dots, q$), and $v > 0$ such that $\Re(\tau) > 0$

$$\begin{aligned} & \left(I_{-,k}^{\tau,\delta,\gamma} \left\{ t^{(-\tau-\zeta)/k} {}_p\Psi_q^k \left[\begin{matrix} (a_i, \varepsilon_i)_{1,p}; \\ (b_j, \varsigma_j)_{1,q}; \end{matrix} ; ct^{(-v/k)} \right] \right\} \right) (x) \\ &= x^{(-\tau-\zeta-\delta)/k} {}_{p+2}\Psi_{q+2}^k \left[\begin{matrix} (a_1, \varepsilon_1), \dots, (a_p, \varepsilon_p), (\tau + \zeta + \delta, v), (\tau + \zeta + \gamma, v); \\ (b_1, \varsigma_1), \dots, (b_q, \varsigma_q), (\tau + \zeta, v), (2\tau + \zeta + \delta + \gamma, v); \end{matrix} ; kcx^{(-v/k)} \right]. \end{aligned} \quad (25)$$

Proof. The finding is similar to that of Theorem 1. So, we omit the details. $\square$

Corollary 1. By assuming $k = 1$ in (22) and (25), the result becomes

$$\begin{aligned} & \left(I_{0+}^{\tau,\delta,\gamma} \left\{ t^{\zeta-1} {}_p\Psi_q \left[\begin{matrix} (a_i, \varepsilon_i)_{1,p}; \\ (b_j, \varsigma_j)_{1,q}; \end{matrix} ; ct^v \right] \right\} \right) (x) \\ &= x^{\zeta-\delta-1} {}_{p+2}\Psi_{q+2} \left[\begin{matrix} (a_1, \varepsilon_1), \dots, (a_p, \varepsilon_p), (\zeta, v), (\zeta + \gamma - \delta, v); \\ (b_1, \varsigma_1), \dots, (b_q, \varsigma_q), (\zeta - \delta, v), (\zeta + \tau + \gamma, v); \end{matrix} ; cx^v \right], \end{aligned} \quad (26)$$

and

$$\begin{aligned} & \left(I_{-}^{\tau, \delta, \gamma} \left\{ t^{-\tau-\zeta} {}_p\Psi_q \left[\begin{matrix} (a_i, \varepsilon_i)_{1,p}; \\ (b_j, \varsigma_j)_{1,q}; \end{matrix} ct^{-\nu} \right] \right\} \right) (x) \\ &= x^{-\tau-\zeta-\delta} {}_{p+2}\Psi_{q+2} \left[\begin{matrix} (a_1, \varepsilon_1), \dots, (a_p, \varepsilon_p), (\tau + \zeta + \delta, \nu), (\tau + \zeta + \gamma, \nu); \\ (b_1, \varsigma_1), \dots, (b_q, \varsigma_q), (\tau + \zeta, \nu), (2\tau + \zeta + \delta + \gamma, \nu); \end{matrix} cx^{-\nu} \right]. \end{aligned} \quad (27)$$

3. Saigo k -Fractional Differentiation in terms of k -Wright Function

In this section, we present the composition formulas of k -fractional derivatives (4) and (5), involving the k -Wright function.

Theorem 3. Let $\tau, \delta, \gamma \in \mathbb{C}$; $k \in \mathbb{R}^+$, $c, \varepsilon_i, \varsigma_j \in \mathbb{R}$ ($\varepsilon_i, \varsigma_j \neq 0$; $i = 1, 2, \dots, p$; $j = 1, 2, \dots, q$), and $\nu > 0$ such that $\Re(\tau) > 0$, $\Re(\zeta) > \max[0, \Re(-\tau - \delta - \gamma)]$, $\Re(\zeta + \gamma + \delta) > 0$. If condition (11) is satisfied and $D_{0+,k}^{\tau, \delta, \gamma}$ be the left-sided differential operator of the generalized k -fractional differentiation associated with k -Wright function, then the following equation holds true:

$$\begin{aligned} & \left(D_{0+,k}^{\tau, \delta, \gamma} \left\{ t^{(\zeta/k)-1} {}_p\Psi_q^k \left[\begin{matrix} (a_i, \varepsilon_i)_{1,p}; \\ (b_j, \varsigma_j)_{1,q}; \end{matrix} ct^{\nu/k} \right] \right\} \right) (x) \\ &= x^{((\zeta+\delta)/k)-1} {}_{p+2}\Psi_{q+2}^k \left[\begin{matrix} (a_1, \varepsilon_1), \dots, (a_p, \varepsilon_p), (\zeta, \nu), (\zeta + \delta + \gamma + \tau, \nu); \\ (b_1, \varsigma_1), \dots, (b_q, \varsigma_q), (\zeta + \gamma, \nu), (\zeta + \delta, 1 - k + \nu); \end{matrix} cx^{((\nu+1)/k)-1} \right]. \end{aligned} \quad (28)$$

Proof. For simplicity, let I_2 denote the left side of (28). Using definition (10), we obtain

$$I_2 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma_k(a_i + \varepsilon_i n)}{\prod_{j=1}^q \Gamma_k(b_j + \varsigma_j n)} \frac{c^n}{n!} D_{0+,k}^{\tau, \delta, \gamma} \left(t^{((\zeta+\nu n)/k)-1} \right) (x). \quad (29)$$

Now, applying equation (8), we obtain

$$I_2 = x^{((\zeta+\delta)/k)-1} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma_k(a_i + \varepsilon_i n) \Gamma_k(\zeta + \nu n) \Gamma_k(\zeta + \delta + \gamma + \tau + \nu n)}{\prod_{j=1}^q \Gamma_k(b_j + \varsigma_j n) \Gamma_k(\zeta + \gamma + \nu n) \Gamma_k(\zeta + \delta + n - nk + \nu n) n!} (cx^{((\nu+1)/k)-1})^n. \quad (30)$$

In accordance with (10), the required result is (28). This completes the proof of Theorem 3. $\square$

Theorem 4. Let $\tau, \delta, \gamma \in \mathbb{C}$; $k \in \mathbb{R}^+$, $c, \varepsilon_i, \varsigma_j \in \mathbb{R}$ ($\varepsilon_i, \varsigma_j \neq 0$; $i = 1, 2, \dots, p$; $j = 1, 2, \dots, q$), and $\nu > 0$ such that $\Re(\tau) > 0$,

$\Re(\zeta) > \max[\Re(\tau + \delta) + n - \Re(\gamma)], \Re(\tau + \delta - \gamma) + n \neq 0$, where $n = [\Re(\tau) + 1]$. If condition (11) is satisfied and $D_{-,k}^{\tau, \delta, \gamma}$ be the right-sided differential operator of the generalized k -fractional differentiation associated with k -Wright function, then the following equation holds true:

$$\begin{aligned} & \left(D_{-,k}^{\tau, \delta, \gamma} \left\{ t^{(\tau-\zeta)/k} {}_p\Psi_q^k \left[\begin{matrix} (a_i, \varepsilon_i)_{1,p}; \\ (b_j, \varsigma_j)_{1,q}; \end{matrix} ct^{(-\nu/k)} \right] \right\} \right) (x) \\ &= x^{(\tau-\zeta+\delta)/k} {}_{p+2}\Psi_{q+2}^k \left[\begin{matrix} (a_1, \varepsilon_1), \dots, (a_p, \varepsilon_p), (\zeta - \tau - \delta, \nu + k - 1), (\zeta + \gamma, \nu); \\ (b_1, \varsigma_1), \dots, (b_q, \varsigma_q), (\zeta - \tau, \nu), (\zeta - \tau - \delta + \gamma, \nu); \end{matrix} cx^{((- \nu+1)/k)-1} \right]. \end{aligned} \quad (31)$$

Proof. The proof is parallel to that of Theorem 3. Therefore, we omit the details. $\square$

Corollary 2. By letting $k = 1$ in (28) and (31), the equation becomes

$$\begin{aligned} & \left(D_{0+}^{\tau, \delta, \gamma} \left\{ t^{\zeta-1} {}_p\Psi_q^k \left[\begin{matrix} (a_i, \varepsilon_i)_{1,p}; \\ (b_j, \varsigma_j)_{1,q}; \end{matrix} ct^v \right] \right\} \right) (x) \\ &= x^{\zeta+\delta-1} {}_{p+2}\Psi_{q+2}^k \left[\begin{matrix} (a_1, \varepsilon_1), \dots, (a_p, \varepsilon_p), (\zeta, \nu), (\zeta + \delta + \gamma + \tau, \nu); \\ (b_1, \varsigma_1), \dots, (b_q, \varsigma_q), (\zeta + \gamma, \nu), (\zeta + \delta, \nu); \end{matrix} cx^v \right], \end{aligned} \quad (32)$$

$$\begin{aligned} & \left(D_-^{\tau, \delta, \gamma} \left\{ t^{\tau-\zeta} {}_p\Psi_q^k \left[\begin{matrix} (a_i, \varepsilon_i)_{1,p}; \\ (b_j, \varsigma_j)_{1,q}; \end{matrix} ct^{-\nu} \right] \right\} \right) (x) \\ &= x^{\tau-\zeta+\delta} {}_{p+2}\Psi_{q+2}^k \left[\begin{matrix} (a_1, \varepsilon_1), \dots, (a_p, \varepsilon_p), (\zeta - \tau - \delta, \nu), (\zeta + \gamma, \nu); \\ (b_1, \varsigma_1), \dots, (b_q, \varsigma_q), (\zeta - \tau, \nu), (\zeta - \tau - \delta + \gamma, \nu); \end{matrix} cx^{-\nu} \right]. \end{aligned} \quad (33)$$

4. Special Cases and Concluding Remarks

Being very general, the results given in (22), (25), (28), and (31) can yield a wide number of special cases by assigning some appropriate values to the parameters involved. Now, as shown in the following, we are explaining a few corollaries.

Corollary 3. If we put $p = 1$ and $q = 2$ in Theorems 1 and 2, then we get the following interesting results on the right known as k -Mittag-Leffler function:

$$\begin{aligned} & \left(I_{0+,k}^{\tau, \delta, \gamma} \left\{ t^{(\zeta/k)-1} E_{k, \varepsilon, \varsigma}^{\vartheta, \omega} [ct^{v/k}] \right\} \right) (x) \\ &= x^{((\zeta-\delta)/k)-1} {}_3\Psi_4^k \left[\begin{matrix} (\vartheta, \omega k), (\zeta, \nu), (\zeta + \gamma - \delta, \nu); \\ (\varsigma, \varepsilon), (\vartheta, 0), (\zeta - \delta, \nu), (\zeta + \tau + \gamma, \nu); \end{matrix} kcx^{v/k} \right], \end{aligned} \quad (34)$$

and

$$\begin{aligned} & \left(I_{-,k}^{\tau, \delta, \gamma} \left\{ t^{(-\tau-\zeta)/k} E_{k, \varepsilon, \varsigma}^{\vartheta, \omega} [ct^{(-v/k)}] \right\} \right) (x) \\ &= x^{(-\tau-\zeta-\delta)/k} {}_3\Psi_4^k \left[\begin{matrix} (\vartheta, \omega k), (\tau + \zeta + \delta, \nu), (\tau + \zeta + \gamma, \nu); \\ (\varsigma, \varepsilon), (\vartheta, 0), (\tau + \zeta, \nu), (2\tau + \zeta + \delta + \gamma, \nu); \end{matrix} kcx^{(-v/k)} \right]. \end{aligned} \quad (35)$$

Corollary 4. If we put $p = 1$ and $q = 3$ in Theorems 1 and 2, then we get the following interesting results on the right known as k -Bessel function of the first kind:

$$\begin{aligned} & \left(I_{0+,k}^{\tau, \delta, \gamma} \left\{ t^{(\zeta/k)-1} J_{k, \varsigma}^{\vartheta, \varepsilon} [ct^{v/k}] \right\} \right) (x) \\ &= x^{((\zeta-\delta)/k)-1} {}_3\Psi_5^k \left[\begin{matrix} (\vartheta, k), (\zeta, \nu), (\zeta + \gamma - \delta, \nu); \\ (\varsigma + 1, \varepsilon), (\vartheta, 0), (\zeta - \delta, \nu), (\zeta + \tau + \gamma, \nu), (k, k); \end{matrix} \frac{-ck^2 x^{v/k}}{2} \right], \end{aligned} \quad (36)$$

and

$$\begin{aligned} & \left(I_{-,k}^{\tau, \delta, \gamma} \left\{ t^{(-\tau-\zeta)/k} J_{k, \varsigma}^{\vartheta, \varepsilon} [ct^{(-v/k)}] \right\} \right) (x) \\ &= x^{(-\tau-\zeta-\delta)/k} {}_3\Psi_5^k \left[\begin{matrix} (\vartheta, k), (\tau + \zeta + \delta, \nu), (\tau + \zeta + \gamma, \nu); \\ (\varsigma + 1, \varepsilon), (\vartheta, 0), (\tau + \zeta, \nu), (2\tau + \zeta + \delta + \gamma, \nu), (k, k); \end{matrix} \frac{-ck^2 x^{(-v/k)}}{2} \right]. \end{aligned} \quad (37)$$

Corollary 5. If we put $p = 3$ and $q = 3$ in Theorems 1 and 2, then we get the following results on the right known as k -hypergeometric function:

$$\begin{aligned} & \left(I_{0+,k}^{\tau,\delta,\gamma} \left\{ t^{(\zeta/k)-1} {}_2F_{1,k} \left((\vartheta, k), (\varphi, k); (\varsigma, k); ct^{v/k} \right) \right\} \right) (x) \\ &= x^{((\zeta-\delta)/k)-1} {}_5\Psi_5^k \left[\begin{matrix} (\vartheta, k), (\varphi, k), (\varsigma, 0), (\zeta, v), (\zeta + \gamma - \delta, v); \\ (\varsigma, k), (\varphi, 0), (\vartheta, 0), (\zeta - \delta, v), (\zeta + \tau + \gamma, v); \end{matrix} kcx^{v/k} \right], \end{aligned} \quad (38)$$

$$\begin{aligned} & \left(I_{-,k}^{\tau,\delta,\gamma} \left\{ t^{(-\tau-\zeta)/k} {}_2F_{1,k} \left((\vartheta, k), (\varphi, k); (\varsigma, k); ct^{(-v/k)} \right) \right\} \right) (x) \\ &= x^{(-\tau-\zeta-\delta)/k} {}_5\Psi_5^k \left[\begin{matrix} (\vartheta, k), (\varphi, k), (\varsigma, 0), (\tau + \zeta + \delta, v), (\tau + \zeta + \gamma, v); \\ (\varsigma, k), (\varphi, 0), (\vartheta, 0), (\tau + \zeta, v), (2\tau + \zeta + \delta + \gamma, v); \end{matrix} kcx^{(-v/k)} \right]. \end{aligned} \quad (39)$$

Corollary 6. If we put $p = 1$ and $q = 2$ in Theorems 3 and 4, then we get the following results on the right known as k -Mittag-Leffler function:

$$\begin{aligned} & \left(D_{0+,k}^{\tau,\delta,\gamma} \left\{ t^{(\zeta/k)-1} E_{k,\varepsilon,\varsigma}^{\vartheta,\omega} \left[ct^{v/k} \right] \right\} \right) (x) \\ &= x^{((\zeta+\delta)/k)-1} {}_3\Psi_4^k \left[\begin{matrix} (\vartheta, \omega k), (\zeta, v), (\zeta + \delta + \gamma + \tau, v); \\ (\varsigma, \varepsilon), (\vartheta, 0), (\zeta + \gamma, v), (\zeta + \delta, 1 - k + v); \end{matrix} cx^{((v+1)/k)-1} \right], \end{aligned} \quad (40)$$

$$\begin{aligned} & \left(D_{-,k}^{\tau,\delta,\gamma} \left\{ t^{(\tau-\zeta)/k} E_{k,\varepsilon,\varsigma}^{\vartheta,\omega} \left[ct^{(-v/k)} \right] \right\} \right) (x) \\ &= x^{(\tau-\zeta+\delta)/k} {}_3\Psi_4^k \left[\begin{matrix} (\vartheta, \omega k), (\zeta - \tau - \delta, v + k - 1), (\zeta + \gamma, v); \\ (\varsigma, \varepsilon), (\vartheta, 0), (\zeta - \tau, v), (\zeta - \tau - \delta + \gamma, v); \end{matrix} cx^{((-v+1)/k)-1} \right]. \end{aligned} \quad (41)$$

Corollary 7. If we put $p = 1$ and $q = 3$ in Theorems 3 and 4, then we get the following results on the right known as k -Bessel function of the first kind:

$$\begin{aligned} & \left(D_{0+,k}^{\tau,\delta,\gamma} \left\{ t^{(\zeta/k)-1} J_{k,\varsigma}^{\vartheta,\varepsilon} \left[ct^{v/k} \right] \right\} \right) (x) \\ &= x^{((\zeta+\delta)/k)-1} {}_3\Psi_5^k \left[\begin{matrix} (\vartheta, k), (\zeta, v), (\zeta + \delta + \gamma + \tau, v); \\ (\varsigma + 1, \varepsilon), (\vartheta, 0), (\zeta + \gamma, v), (\zeta + \delta, 1 - k + v), (k, k); \end{matrix} \frac{-ckx^{((v+1)/k)-1}}{2} \right], \end{aligned} \quad (42)$$

$$\begin{aligned} & \left(D_{-,k}^{\tau,\delta,\gamma} \left\{ t^{(\tau-\zeta)/k} J_{k,\varsigma}^{\vartheta,\varepsilon} \left[ct^{(-v/k)} \right] \right\} \right) (x) \\ &= x^{(\tau-\zeta+\delta)/k} {}_3\Psi_5^k \left[\begin{matrix} (\vartheta, k), (\zeta - \tau - \delta, v + k - 1), (\zeta + \gamma, v); \\ (\varsigma + 1, \varepsilon), (\vartheta, 0), (\zeta - \tau, v), (\zeta - \tau - \delta + \gamma, v), (k, k); \end{matrix} \frac{-ckx^{((-v+1)/k)-1}}{2} \right]. \end{aligned} \quad (43)$$

Corollary 8. If we put $p = 3$ and $q = 3$ in Theorems 3 and 4, then we get the following results on the right known as k -hypergeometric function:

$$\begin{aligned} & \left(D_{0+,k}^{\tau,\delta,\gamma} \left\{ t^{((\zeta/k)-1)} {}_2F_{1,k} \left((\vartheta, k), (\varphi, k); (\varsigma, k); ct^{v/k} \right) \right\} \right) (x) \\ &= x^{((\zeta+\delta)/k)-1} {}_5\Psi_5^k \left[\begin{matrix} (\vartheta, k), (\varphi, k), (\varsigma, 0), (\zeta, v), (\zeta + \delta + \gamma + \tau, v); \\ (\varsigma, k), (\vartheta, 0), (\varphi, 0), (\zeta + \gamma, v), (\zeta + \delta, 1 - k + v); \end{matrix} ; cx^{((v+1)/k)-1} \right]. \end{aligned} \quad (44)$$

$$\begin{aligned} & \left(D_{-,k}^{\tau,\delta,\gamma} \left\{ t^{((\tau-\zeta)/k)} {}_2F_{1,k} \left((\vartheta, k), (\varphi, k); (\varsigma, k); ct^{(-v/k)} \right) \right\} \right) (x) \\ &= x^{(\tau-\zeta+\delta)/k} {}_5\Psi_5^k \left[\begin{matrix} (\vartheta, k), (\varphi, k), (\varsigma, 0), (\zeta - \tau - \delta, v + k - 1), (\zeta + \gamma, v); \\ (\varsigma, k), (\vartheta, 0), (\varphi, 0), (\zeta - \tau, v), (\zeta - \tau - \delta + \gamma, v); \end{matrix} ; cx^{((-v+1)/k)-1} \right]. \end{aligned} \quad (45)$$

The advantage of the generalized k -fractional calculus operators, which are also called by many authors as the general operator, is that they generalize Saigo's fractional calculus operators and classical Riemann–Liouville (R-L) operators. For $k \rightarrow 1$, operators (1), (2), (4), and (5) reduce to Saigo's [5] fractional integral and differentiation operators. If we take $\delta = -\tau$, (1), (2), (4), and (5) reduce the operators to k -Riemann–Liouville as follows:

$$\begin{aligned} & \left(I_{0+,k}^{\tau,-\tau,\gamma} f \right) (x) = \left(I_{0+,k}^{\tau} f \right) (x), \\ & \left(I_{-,k}^{\tau,-\tau,\gamma} f \right) (x) = \left(I_{-,k}^{\tau} f \right) (x), \\ & \left(D_{0+,k}^{\tau,-\tau,\gamma} f \right) (x) = \left(D_{0+,k}^{\tau} f \right) (x), \\ & \left(D_{-,k}^{\tau,-\tau,\gamma} f \right) (x) = \left(D_{-,k}^{\tau} f \right) (x). \end{aligned} \quad (46)$$

Due to the most general character of the k -Wright function, numerous other interesting special cases from (22), (25), (28), and (31) can be given in the form of k -Struve function, k -Wright-type function, and many more, but due to lack of space, they are not represented here.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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