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Research Article

The Numerical Invariants concerning the Total Domination for Generalized Petersen Graphs

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A subset S of $V(G)$ is called a total dominating set of a graph G if every vertex in $V(G)$ is adjacent to a vertex in S. The total domination number of a graph G denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G. The maximum order of a partition of $V(G)$ into total dominating sets of G is called the total domatic number of G and is denoted by $d_t(G)$. Domination in graphs has applications to several fields. Domination arises in facility location problems, where the number of facilities (e.g., hospitals and fire stations) is fixed, and one attempts to minimize the distance that a person needs to travel to get to the closest facility. In this paper, the numerical invariants concerning the total domination are studied for generalized Petersen graphs.

1. Introduction

Graph theory has been used to study various concepts of partition of the vertex set for the graph G [1–3]. The domination (total domination) problem for the graph was studied from 1950s onward, and domination (total domination) in graphs is said to be NP-complete problem [4]. Domination (total domination) sets are of practical interest in several areas of mathematics and other branches of science [5, 6]. In wireless networks, dominating (total dominating) sets are used to find efficient routes within ad hoc mobile networks. In documents' summarization, domination (total domination) sets are used. Dominating (total dominating) sets are closely related to independent sets. Total dominating set is independent if and only if it is a maximal input set [7].

Moreover, in computer science, weighted graphs are widely used in the developments of data mining, software testing, image processing, communication networks, and information security [8–11].

In this paper, the numerical invariants concerning the total domination are studied for generalized Petersen graphs [12–16].

2. Preliminaries

Let $G = (V, E)$ be a simple, undirected, and finite graph with the vertex set $V(G)$ and the edge set $E(G)$.

The (open) neighborhood of a vertex $u \in V(G)$, denoted by $N(u)$, is the set of all vertices adjacent to u, i.e., $N(u) = \{v \in V(G)\colon uv \in E(G)\}$. The closed neighborhood of u is defined as $N[u] = \{u\} \cup N(u)$ [17–19]. For a set $S \subseteq V(G)$, its open neighborhood is the set $N(S) = \bigcup_{u\in S} N(u)$, and its closed neighborhood is the set $N[S] = N(S) \cup S$. A dominating set of a graph G, abbreviated as *DS*, is a set S of vertices in G such that every vertex in $V(G)\backslash S$ is adjacent to a vertex in S. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A total dominating set of a graph G, abbreviated as TDS, is a set S of vertices in G such that every vertex in $V(G)$ is adjacent to a vertex in S. If no proper subset of S is a TDS of G, then S is a minimal TDS of G. Every graph without isolated vertices has a TDS. The total domination number of a graph G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A TDS of G of minimum cardinality is called a $\gamma_t(G)$-set.

The concept of total domination was introduced by Cockayne et al. [20]. For more information, see [21, 22].

Let $\tau(G)$ be the total number of $\gamma_t(G)$-sets. For each vertex $v \in V(G)$, we define the total domination value of v, denoted by $\mathrm{TDV}_G(v)$, to be the number of $\gamma_t(G)$-sets to which v belongs. Evidently, $0 \le \mathrm{TDV}_G(v) \le \tau(G)$ for any G and any $v \in V(G)$.

Generalized Petersen graphs are important classes of commonly used interconnection networks and have been intensively studied. The generalized Petersen graph $P(n,m)$, $n \ge 3$ and $1 \le m \le \lfloor (n-1)/2 \rfloor$, consists of an outer n-cycle $u_1, u_2, \ldots, u_n$, a set of n spokes $u_i v_i$, $1 \le i \le n$, and n edges $v_i v_{i+m}$, $1 \le i \le n$, with indices taken as modulo n, see Figure 1. Thus, its vertex set is the union of $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. We say that the first set consists of u vertices and the second one of v vertices. By a u-path in $P(n,m)$, we mean a path whose vertices consist of just u-vertices. A v-path is defined similarly. The generalized Petersen graph $P(n,m)$ is shown in Figure 1.

Investigation of the total number of total domatic sets and the total number of total dominating sets of the generalized Petersen graph is the sole aim and interest of writing this paper. Cao et al. in [23] studied the total domination number of generalized Petersen graphs and obtained the exact value of $\gamma_t(P(n,2))$. For $k \ge 2$, they proved

$$\gamma_t(P(n,2)) = \begin{cases} 2k, & \text{for } n = 3k, \\ 2k+2, & \text{for } n = 3k+1, \\ 2k+2, & \text{for } n = 3k+2. \end{cases} \tag{1}$$

In this paper, we determine the exact values for $\gamma_t(P(n,m))$ when $n \equiv 0 \pmod 3$ and $m \not\equiv 0 \pmod 3$ and when $m = 4$, 5, and 7. We also investigate total number of total dominating sets for these cases.

3. Main Results

The following result provides a trivial lower bound on the total domination number of a graph in terms of the maximum degree of the graph.

Theorem 1 (see [24]). *If G is a graph with no isolated vertex and maximum degree $\Delta(G)$, then*

$$\gamma_t(G) \ge \left\lceil \frac{|V(G)|}{\Delta(G)} \right\rceil. \tag{2}$$

Theorem 2. *Let $P(n,m)$ be a generalized Petersen graph with $n = 3k \ge 6$, $k \ge 2$. If $m \not\equiv 0 \pmod 3$, then*

$$\gamma_t(P(n,m)) = \left\lceil \frac{2n}{3} \right\rceil = 2k. \tag{3}$$

Proof. Let $n = 3k \ge 6$. According to Theorem 1, we get

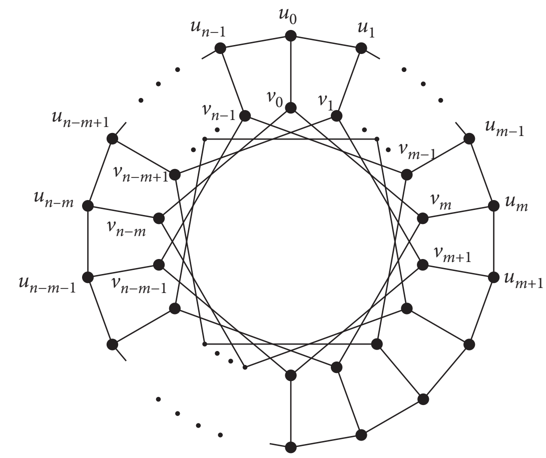

Figure 1: Generalized Petersen graph $P(n,m)$.

$$\gamma_t(P(n,m)) \ge \left\lceil \frac{|V(P(3k,m))|}{3} \right\rceil = \left\lceil \frac{6k}{3} \right\rceil = 2k. \tag{4}$$

Thus, to prove the equality, we just need to find a total dominating set of cardinality $2k$. Let us consider a set

$$S = \{u_{3j+1}, v_{3j+1} \colon j = 0, 1, \ldots, k-1\}. \tag{5}$$

The vertex u_{3j+1}, $j = 0, 1, \ldots, k-1$, dominates vertices v_{3j+1}, u_{3j}, and u_{3j+2}, and the vertex v_{3j+1}, $j = 0, 1, \ldots, k-1$, dominates vertices u_{3j+1}, v_{3j+1+m}, and v_{3j+1-m} with indices taken as modulo n. It is easy to see that, as $m \not\equiv 0 \pmod 3$, all vertices in $P(3k,m)$ are dominated. Thus, S is a total dominating set, and $|S| = 2k$. This concludes the proof.

Now, we proceed for $m = 4$. □

Theorem 3. *Let $P(n,4)$ be a generalized Petersen graph; then,*

$$\gamma_t(P(n,4)) = \begin{cases} 2k, & \text{for } n = 3k, k \ge 2, \\ 2k+4, & \text{for } n = 3k+1, k \ge 7, \\ 2k+4, & \text{for } n = 3k+2, k \ge 7. \end{cases} \tag{6}$$

Proof. Let us consider three cases:

Case 1: when $n = 3k$, $k \ge 2$, the result follows from Theorem 2.

Case 2: when $n = 3k+1$, $k \ge 7$, without loss of generality, consider a set $S_0 = \{u_{3j+1}, v_{3j+1} \colon j = 0, 1, \ldots, k\}$. The vertex u_{3j+1}, $j = 0, 1, \ldots, q$, dominates vertices v_{3j+1}, u_{3j}, and u_{3j+2}, and the vertex v_{3j+1}, $j = 0, 1, \ldots, q$, dominates vertices u_{3j+1}, v_{3j+4}, and v_{3j-3} with indices taken as modulo n. Thus, the vertices from this set S_0 dominate all but two vertices in G. The two vertices which are not dominated are v_2 and v_{n-1}. To totally dominate v_2 and v_{n-1}, we add their neighbors u_2 and u_{n-2} to S_0 because these two vertices are already adjacent vertices in S_0. This means that the set $S = S_0 \cup \{u_2, u_{n-2}\}$ totally dominates all vertices in G. Thus, $\gamma_t(P(3k+1,4)) \le |S| = 2k+4$.

For necessity, we need to prove the reverse inequality, i.e., $\gamma_t(P(3k+1,4)) \geq 2k+4$. The proof will be by mathematical induction. For $n = 22$, the result is obvious.

Since the result is true for all $P(n,4)$ of the number of vertices less than n when $m \equiv 0 \pmod 3$, for $n \equiv 1 \pmod 3$, let D be any total dominating set of $P(n,4)$. Then, $D = D_1 \cup \{u_2\} \cup \{u_{n-2}\} \cup \{u_n\} \cup \{v_n\}$, where D_1 is the total dominating set of $P(n,4)$ if only $n \equiv 0 \pmod 3$, and we get $|D_1| \geq \lceil (2(n-2))/3 \rceil = 2k$. Thus, we proved our required assertion.

Case 3: when $n = 3k+2$, $k \geq 4$, without loss of generality, consider a set $S_0 = \{u_{3j+1}, v_{3j+1}\colon j = 0, 1, \ldots, k\}$. The vertex u_{3j+1}, $j = 0, 1, \ldots, q$, dominates vertices v_{3j+1}, u_{3j}, and u_{3j+2}, and the vertex v_{3j+1}, $j = 0, 1, \ldots, q$, dominates vertices u_{3j+1}, v_{3j+4}, and v_{3j-3} with indices taken as modulo n. Thus, the vertices from this set S_0 dominate all but two vertices in G. The two vertices which are not dominated are v_2 and v_{n-2}. To totally dominate v_2 and v_{n-2}, we add these two vertices to S_0 because these two vertices are adjacent. This means that the set $S = S_0 \cup \{v_2, v_{n-2}\}$ totally dominates all vertices in G. Thus, $\gamma_t(P(3k+2,4)) \leq |S| = 2k+4$.

For necessity, we need to prove the reverse inequality, i.e., $\gamma_t(P(3k+2,4)) \geq 2k+4$. The proof will be by mathematical induction.

Assume the result is true for all $P(n,4)$ of the number of vertices less than n. For $n \equiv 2$ (mod3), let D be any total dominating set of $P(n,4)$. Then, $D = D_1 \cup \{u_2\} \cup \{u_{n-2}\} \cup \{u_{n-1}\} \cup \{v_{n-1}\}$, where D_1 is the total dominating set of $P(n,4)$ if only $n \equiv 0$ (mod3), and then $|D_1| \geq \lceil (2(n-2))/3 \rceil = 2k$, i.e., $|D| \geq 2k+4$. □

Theorem 4. *Let $P(n,m)$ be a generalized Petersen graph. Then, $d_t(P(n,m)) = 3$ if $n \equiv 0$ (mod 3) and $m \neq 3q$.*

Proof. Let $\{P_1, P_2, P_3\}$ be the total domatic partitions since no vertex of any partition is contiguous to any other partitions and all the three domatic partitions totally dominate $G(P(n,m))$. Also, we know that $\mathrm{TDV}(u_i) = \mathrm{TDV}(v_i) = 1$. We proved in Theorem 6 in the following that $\tau(P(n,m)) = 3$ if only $n \equiv 0$ (mod 3). It is observed that there is no recurrence in the total number of total dominating sets. Hence, we can write $d_t(P(n,m)) = \tau(P(n,m)) = 3$ if only $n \equiv 0$ (mod 3).

Henning and Yeo [24] proved the following. □

Theorem 5 (see [24]). *For any connected graph G,*

$$\sum_{v \in V(G)} \mathrm{TDV}_G(v) = \tau(G) \cdot \gamma_t(G). \tag{7}$$

Theorem 6. *Let $n \neq 10$ and $m \geq 4$ be positive integers. If $m \neq 0 \pmod 3$, then*

$$\tau(P(n,m)) = \begin{cases} 3, & \text{for } n \equiv 0 \pmod 3, \\ n, & \text{for } n \equiv 0 \pmod 3, \end{cases} \tag{8}$$

where for $m = 5$, n must be odd.

Proof. Let $m \geq 4$ be a positive integer, $m \neq 0 \pmod 3$.

Case 1: when $n = 3k$, $k \geq 2$, then $\gamma_t(P(n,m)) = 2k$, and a $\gamma_t(P(n,m))$-set Γ comprises kP_2's, and Γ is fixed by the choice of the first P_2. There is exactly one $\gamma_t(P(n,m))$-set comprising vertices u_1 and v_1. Thus, $\mathrm{TDV}(u_1) = \mathrm{TDV}(v_1) = 1$. Similarly, $\mathrm{TDV}(u_2) = \mathrm{TDV}(v_2) = 1, \ldots, \mathrm{TDV}(u_n) = \mathrm{TDV}(v_n) = 1$. Using Theorem 5, we get

$$\begin{aligned} \sum_{i=1}^{2n} 1 &= 2k \cdot \tau(P(n,m)), \\ 2n \cdot 1 &= 2k \cdot \tau(P(n,m)). \end{aligned} \tag{9}$$

Hence, $\tau(P(n,m)) = 3$ in this case.

Case 2: when $n = 3k+1$ or $n = 3k+2$, then a $\gamma_t(P(n,m))$-set Γ is composed of in exactly one of the following ways:

(1) Γ comprises kP_2's and one P_4
(2) Γ comprises $(k-1)P_2$'s and two P_3's
(3) Γ comprises $(k-1)P_2$'s and one P_4
(4) Γ comprises $(k-1)P_2$'s and two P_4's
(5) Γ comprises $(k+1)P_2$'s

Now, we discuss these subcases.

(i) Subcase 1: when $\langle \Gamma \rangle \equiv kP_2 \cup P_4$, note that Γ is fixed by the choice of single P_4. Selecting P_4 is the same as selecting its initial vertex in the counterclockwise order; thus, $\tau(P(n,m)) = n$.

(ii) Subcase 2: when $\langle \Gamma \rangle \equiv (k-1)P_2 \cup 2P_3$, the set Γ is fixed by the choice of two P_3's, and there are n options for choosing the first two P_3's; thus, $\tau(P(n,m)) = n$.

(iii) Subcase 3: when $\langle \Gamma \rangle \equiv (k+1)P_2 \cup P_4$, there is only one chance for choosing single P_4 and $(k+1)$ chances for choosing P_2. Each vertex has equal number of chances. Each vertex is equally repeated in the total number of total dominating sets three times; thus, the total number of total dominating sets will be n.

(iv) Subcase 4: when $\langle \Gamma \rangle \equiv (k-1)P_2 \cup 2P_4$, then Γ is fixed by the choice of double P_4. Each vertex is repeated three times in the total number of dominating sets, and for each vertex, there are equal number of chances for total domination; hence, each TDS will be repeated n times.

(v) Subcase 5: when $\langle \Gamma \rangle \equiv (k+1)P_2$, there are exactly two vertices in each P_2. Each vertex of P_2 dominates three vertices including the vertex itself because this is a 3-regular graph. Thus, there are n number of choices for the total number of dominating sets; thus, in all cases, $\tau(P(n,m)) = n$. □

Corollary 1. *Let $P(n,5)$ be a generalized Petersen graph; then,*

$$\gamma_t(P(n,5)) = \begin{cases} 4k+4, & \text{for } n = 6k+1, k \geq 2, \\ 4k+4, & \text{for } n = 6k+5, k \geq 1. \end{cases} \quad (10)$$

Proof. The proof is similar to the proof of Theorem 3. □

Corollary 2. *Let $P(n,7)$ be a generalized Petersen graph; then,*

$$\gamma_t(P(n,7)) = \begin{cases} 2k+6, & \text{for } n = 3k+1, k \geq 5, \\ 2k+6, & \text{for } n = 3k+2, k \geq 5. \end{cases} \quad (11)$$

Proof. The proof is similar to the proof of Theorem 3. □

4. Total Domatic Number of a Graph

The total domatic number of a graph G is denoted by $d_t(G)$ and is defined as $d_t(G) = s$, where $V(G) = \cup_{i=1}^{s} D_i$ and $D_i's$ are the total dominating sets and are the maximum disjoint partition of the vertex set of the graph.

Proposition 1. *For any simple graph G with minimum degree $\delta(G)$, the total domatic number $d_t(G) \leq \delta(G)$.*

In the following result, our proof is simple from the proof of [25].

Theorem 7 (see [25]). *For a graph G of order n and minimum degree $\delta(G)$, the lower bound for the total domatic number $d_t(G) \geq \lfloor n/(n-\delta(G)+1) \rfloor$, where $n-\delta(G)+1 = d_t(G)$.*

Proof. Let D be the total dominating set of the graph G and G^c be the complement of G. If $\delta(G)$ is the minimum degree of G, then $n-\delta(G)-1$ is the maximum degree of G^c, and due to this, any vertex x of the graph G is adjacent to at most $n-\delta(G)-1$ vertices of D in G^c. If the vertex $x \in D$, then a vertex z of the graph exists which is not adjacent to x in G^c but $xz \in E(G)$; this confirms that $|D| \geq n-\delta(G)+1$, and hence, $d_t(G) \geq \lfloor n/(n-\delta(G)+1) \rfloor$. □

Corollary 3. *For $G = P(n,m)$ generalized Petersen graph of order $2n$ and $\delta(G) = 3$, we have $1 \leq d_t(G) \leq 3$.*

5. Open Problems

We conclude the paper with the following open problems for further investigation.

Open Problem 1. Characterize $\gamma_t(P(n,m))$ for all feasible values of n and m.

Open Problem 2. Characterize $\text{TDV}(u)$ for each $u \in V(P(n,m))$.

5.1. Discussion. In this paper, the generalized Petersen graph $(P(m,n))$ is investigated for the total domination number γ_t, total number of dominating sets τ, and total domatic number $d_t(P(n,m))$. Here, we found the total dominating number for some restricted values of $m \in \mathbb{Z}^+$. For detailed applications of our results, refer [26–28].

Data Availability

All data required for this paper are included within this paper.

Conflicts of Interest

The authors do not have any conflicts of interest.

Authors' Contributions

Taiyin Zhao gave applications of results and improved the presentation of the paper, Gohar Ali supervised this work, Nabila Hameed proved the results, Syed Inayat Ali Shah wrote the first version of the paper, and Yu-Ming Chu wrote the final version of the paper and arranged funding for this paper.

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