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Research Article

A New Extended-F Family: Properties and Applications to Lifetime Data

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In this article, a new approach is used to introduce an additional parameter to a continuous class of distributions. The new class is referred to as a new extended-F family of distributions. The new extended-Weibull distribution, as a special submodel of this family, is discussed. General expressions for some mathematical properties of the proposed family are derived, and maximum likelihood estimators of the model parameters are obtained. Furthermore, a simulation study is provided to evaluate the validity of the maximum likelihood estimators. Finally, the flexibility of the proposed method is illustrated via two applications to real data, and the comparison is made with the Weibull and some of its well-known extensions such as Marshall–Olkin Weibull, alpha power-transformed Weibull, and Kumaraswamy Weibull distributions.

1. Introduction

In many practical situations, classical distributions do not provide adequate fits to real data. For example, when modeling data with a monotonic hazard rate function (hrf), one frequently uses the Rayleigh, exponential, or Weibull distributions. Among these models, the Weibull distribution is the most prominent one for modeling real phenomena of nature. Unfortunately, the Weibull model is inappropriate in modeling data having nonmonotonic hrf such as unimodal, modified unimodal, or bathtub-shaped.

To address the abovementioned problems, the researchers have shown an increased interest in developing more flexible distributions. This has been performed via extending the classical distributions by introducing additional parameter(s) to the baseline model. In this regard, numerous generalized families of distributions have been proposed and studied over the last three decades for modeling data in many applied areas such as economics, engineering, biological studies, environmental sciences, medical sciences, and finance. Some well-known families include odd exponentiated half-logistic-G [1], Topp Leone odd Lindley-G [2], Marshall–Olkin alpha power-G [3], transmuted transmuted-G [4], generalized Burr XII power series class [5], Weibull Marshall–Olkin [6], new exponentiated TX [7], Weibull-G Poisson [8], odd Dagum-G [9], arcsine exponentiated-X [10], and odd log-logistic Lindley-G [11] families, among many others.

Recently, Ahmad et al. [12] proposed a new method of introducing an additional parameter to extend the existing distributions, called the extended alpha power-transformed (EAPT) family of distributions. The cumulative distribution function (cdf) of the EAPT family is defined by

$$G(x;\alpha,\boldsymbol{\xi}) = \frac{\alpha^{F(x;\boldsymbol{\xi})} - e^{F(x;\boldsymbol{\xi})}}{\alpha - e}, \quad \alpha > 0, \alpha \neq e, x \in \mathbb{R}, \tag{1}$$

where α is an additional shape parameter and $F(x;\xi)$ is the cdf of the baseline model depending on the vector of parameters $\xi \in \mathbb{R}$.

Ahmad et al. [13] proposed another new method to define new lifetime distributions, called new extended alpha power-transformed (NEAPT) family that is defined by the cdf:

$$G(x;\alpha,\boldsymbol{\xi}) = \frac{\alpha^{F(x;\boldsymbol{\xi})} - e^{\alpha F(x;\boldsymbol{\xi})}}{\alpha - e^{\alpha}}, \quad \alpha > 0, \alpha \neq e, x \in \mathbb{R}. \tag{2}$$

In this article, we further propose a new method to provide flexible lifetime distributions called a new extended-F (NE-F)family with additional shape parameter θ. The proposed NE-F family is specified by the following cdf:

$$G(x;\theta,\boldsymbol{\xi}) = F(x;\boldsymbol{\xi})e^{\overline{\theta F}(x;\boldsymbol{\xi})}, \quad \theta,\boldsymbol{\xi} > 0, x \in \mathbb{R}, \tag{3}$$

where $\overline{\theta} = 1-\theta$ and $\overline{F}(x;\xi) = 1 - F(x;\xi)$. Here, in (3), the baseline cdf is weighted by the quantity $e^{\overline{\theta F}(x;\xi)}$. Clearly, when $\theta = 1$, the cdf (3) reduces to the baseline model. The probability density function (pdf) and hrf corresponding to (3) are specified by

$$\begin{aligned} g(x;\theta,\boldsymbol{\xi}) &= f(x;\boldsymbol{\xi})e^{\overline{\theta F}(x;\boldsymbol{\xi})}\left\{1 - \overline{\theta}F(x;\boldsymbol{\xi})\right\}, \quad x \in \mathbb{R}, \\ h(x;\theta,\boldsymbol{\xi}) &= \frac{f(x;\boldsymbol{\xi})e^{\overline{\theta F}(x;\boldsymbol{\xi})}\left\{1 - \overline{\theta}F(x;\boldsymbol{\xi})\right\}}{1 - F(x;\boldsymbol{\xi})e^{\overline{\theta F}(x;\boldsymbol{\xi})}}, \quad x \in \mathbb{R}. \end{aligned} \tag{4}$$

The key motivations for using the NE-F family of distributions in practice are as follows:

(i) A very simple and convenient method of adding an additional parameter to modify the existing distributions

(ii) To improve the characteristics and flexibility of the existing distributions

(iii) To introduce the extended version of the baseline distribution whose cdf, survival function (sf), and hrf have closed forms

(iv) To compare the goodness of fit with other well-known models having the same, as well as higher, number of parameters

(v) To provide better fits than the other competing modified models

This paper is unfolded as follows. Section 2 offers a special submodel of the new family called the new extended-Weibull (NE-W) distribution. Mathematical properties of the NE-F family are derived in Section 3. The maximum likelihood estimators (MLEs) of the model parameters are obtained in Section 4. A Monte Carlo simulation study is provided in the same section. Two practical applications are discussed in Section 5. Finally, Section 6 concludes the article.

2. The NE-W Distribution

Consider the distribution and density functions of the Weibull random variable given by $F(x) = 1 - e^{-\gamma x^{\alpha}}$, $x \geq 0, \alpha, \gamma > 0$ and $f(x) = \alpha\gamma x^{\alpha-1}e^{-\gamma x^{\alpha}}$. Then, the cdf of the NE-W distribution is given by

$$G(x) = \left(1 - e^{-\gamma x^{\alpha}}\right)e^{\overline{\theta}\left(e^{-\gamma x^{\alpha}}\right)}, \quad x \geq 0, \alpha, \theta, \gamma > 0. \tag{5}$$

The pdf corresponding to (5) is given by

$$g(x) = \alpha\gamma x^{\alpha-1}e^{-\gamma x^{\alpha}}e^{\overline{\theta}e^{-\gamma x^{\alpha}}}\left\{1 - \overline{\theta}\left(1 - e^{-\gamma x^{\alpha}}\right)\right\}, \quad x > 0. \tag{6}$$

Plots for the pdf and hrf of the NE-W for selected parametric values are shown in Figures 1 and 2, respectively.

3. Properties of the NE-F Family

In this section, we derive some general properties of the NE-F family including the linear representation, moments, moment-generating function (mgf), and order statistics.

3.1. Linear Representation. Using the exponential series, the cdf of the NE-F family reduces to

$$G(x;\theta,\boldsymbol{\xi}) = F(x;\boldsymbol{\xi})\sum_{j=0}^{\infty}\frac{\overline{\theta}^{j}}{j!}[1 - F(x;\boldsymbol{\xi})]^{j}. \tag{7}$$

Applying the binomial expasion, we have

$$G(x;\theta,\boldsymbol{\xi}) = \sum_{j,k=0}^{\infty}\frac{\overline{\theta}^{j}(-1)^{k}}{j!}\binom{j}{k}F(x;\boldsymbol{\xi})^{k+1}. \tag{8}$$

By differentiating (8), the pdf of the NE-F family reduces to

$$\begin{aligned} g(x;\theta,\boldsymbol{\xi}) &= \sum_{j,k=0}^{\infty}\frac{\overline{\theta}^{j}(-1)^{k}(k+1)}{j!}\binom{j}{k}f(x;\boldsymbol{\xi})F(x;\boldsymbol{\xi})^{k}, \\ g(x;\theta,\boldsymbol{\xi}) &= \sum_{k=0}^{\infty}\delta_{k}h_{k+1}(x), \end{aligned} \tag{9}$$

where

$$\delta_{k} = \sum_{j=0}^{\infty}\frac{\overline{\theta}^{j}(-1)^{k}}{j!}\binom{j}{k} \tag{10}$$

and $h_{k+1}(x) = (k+1)f(x;\xi)F(x;\xi)^{k}$ refers to the exponentiated-F (Ex-F) family pdf with power parameter $(k+1) > 0$. Hence, the pdf of the NE-F family is expressed as a linear combination of Ex-F densities. Equation (9) can be used to obtain several mathematical properties of the NE-F family from those properties of the Ex-F class.

Henceforth, let Y_{k+1} refer to a random variable having the Ex-F distribution with parameter $(k+1)$. Some mathematical properties of X can be expressed from those of Y_{k+1}.

3.2. Moments and Generating Function. The r th moment of X follows simply from (9) as

$$\mu_{r}' = E(X^{r}) = \sum_{k=0}^{\infty}\delta_{k}E\left(Y_{k+1}^{r}\right). \tag{11}$$

The s th incomplete moment of X is expressed from (9) as

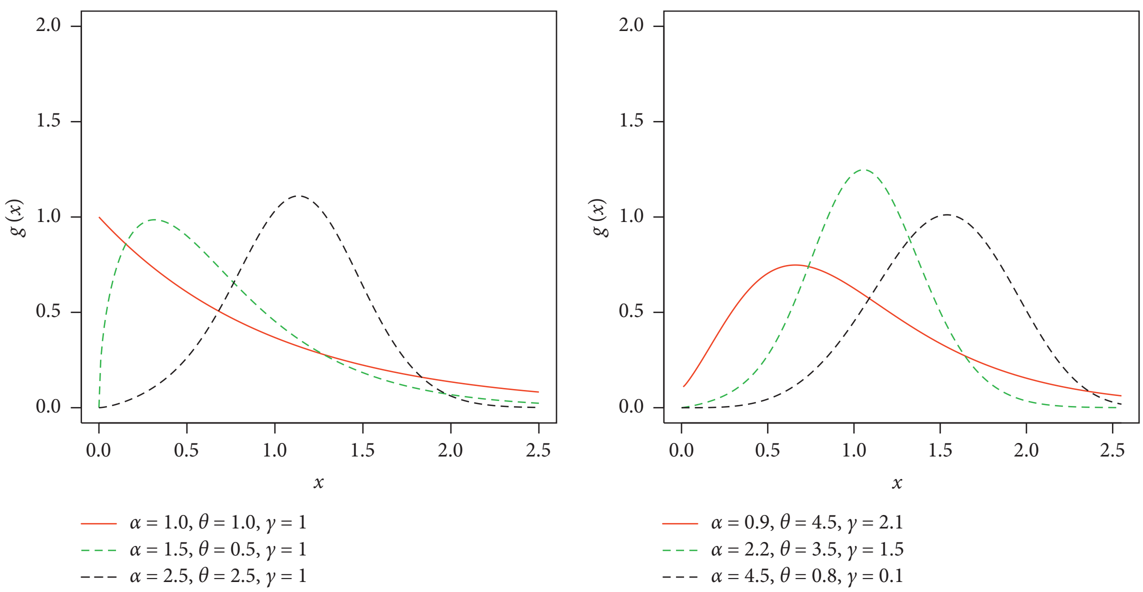

Figure 1: Plots for the density function of the NE-W distribution.

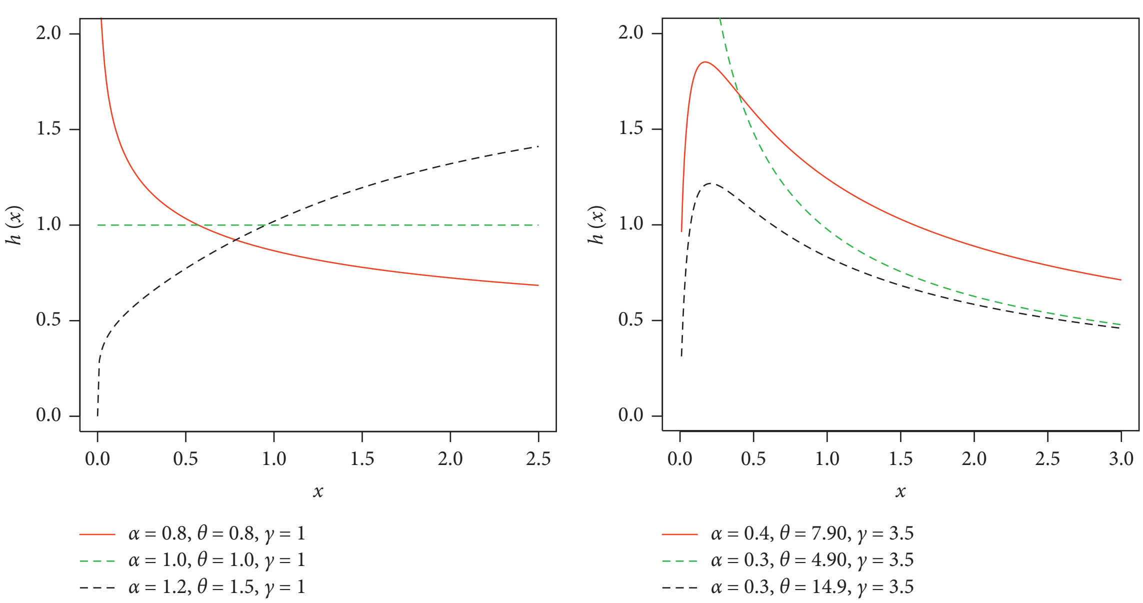

Figure 2: Plots for the hrf of the NE-W distribution.

$$m_s(t) = \int_{-\infty}^{t} x^s f(x)\mathrm{d}x = \sum_{k=0}^{\infty} \delta_k \int_{-\infty}^{t} x^s h_{k+1}(x)\mathrm{d}x. \quad (12)$$

The first incomplete moment of X can be obtained from (12) as

$$m_1(z) = \sum_{k=0}^{\infty} \delta_k J_{k+1}(t), \quad (13)$$

where $J_{k+1}(t) = \int_{-\infty}^{t} x h_{k+1}(x)\mathrm{d}x$ is the first incomplete moment of the Ex-F class.

The mgf of X can be derived from equation (9) as

$$M(t) = E\left(e^{tX}\right) = \sum_{k=0}^{\infty} \delta_k M_{k+1}(t) = \sum_{k=0}^{\infty} (k+1)\delta_k \tau(t,k), \quad (14)$$

where $M_{k+1}(t)$ is the mgf of Y_{k+1} and $\tau(t,k) = \int_0^1 \exp[tQ_G(u)]u^k \mathrm{d}u$. Hence, $M(t)$ follows from the Ex-F generating function.

3.3. Order Statistics. The order statistics are very important in many fields of statistical theory and its practice. Let $X_1, \ldots, X_n$ be a random sample from the NE-F family. The pdf of rth order statistic, denoted by $X_{r:\,n}$, $r = 1, \ldots, n$, takes the following form:

$$g_{r:\,n}(x;\boldsymbol{\xi}) = \frac{n!g(x;\boldsymbol{\xi})}{(r-1)!(n-r)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}G(x;\boldsymbol{\xi})^{r+i-1}. \tag{15}$$

Using the pdf and cdf of the NE-F family, we can write

$$g(x;\boldsymbol{\xi})G(x;\boldsymbol{\xi})^{r+i-1} = f(x;\boldsymbol{\xi})F(x;\boldsymbol{\xi})^{r+i-1}e^{(r+i)\overline{\theta}\overline{F}(x;\boldsymbol{\xi})}[1-\overline{\theta}F(x;\boldsymbol{\xi})]. \tag{16}$$

Applying both exponential and binomial series, we get

$$g(x;\boldsymbol{\xi})G(x;\boldsymbol{\xi})^{r+i-1} = \sum_{j,k=0}^{\infty}\frac{(-1)^k\overline{\theta}^j}{j!(r+i)^{-j}} \cdot \binom{j}{k}f(x;\boldsymbol{\xi})F(x;\boldsymbol{\xi})^{k+r+i-1}[1-\overline{\theta}F(x;\boldsymbol{\xi})]. \tag{17}$$

Combining (16) and (17), the pdf of the r th order statistic reduces to

$$g_{r:\,n}(x;\boldsymbol{\xi}) = \sum_{k=0}^{\infty}\sum_{i=0}^{n-r}m_{k,i}\{qh_{k+r+i}(x) - q^*h_{k+r+i+1}(x)\}, \tag{18}$$

where, as before, $h_s(x) = sf(x;\xi)F(x;\xi)^{s-1}$, for $s = k+r+i$, $k+r+i+1$,

$$\begin{aligned} m_{k,i} &= \frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{\infty}(-1)^i\binom{n-r}{i}\frac{(-1)^k\overline{\theta}^j}{j!(r+i)^{-j}}\binom{j}{k}, \\ q &= \frac{1}{k+r+i}, \\ q^* &= \frac{\overline{\theta}}{k+r+i+1}. \end{aligned} \tag{19}$$

Equation (18) refers to the pdf of the NE-F order statistics as a linear mixture of Ex-F densities and can be used to derive some mathematical properties of the rth order statistic from the properties of Y_s.

4. Maximum Likelihood Estimation and Monte Carlo Simulations

This section covers the estimation of the NE-W parameters. Section 4.1 offers estimation of the NE-W parameters using the maximum likelihood method. Section 4.2 explores the performance of the maximum likelihood estimators (MLEs) in terms of biases and mean squared errors (MSEs) by means of a Monte Carlo simulation study.

4.1. Maximum Likelihood Estimation. Here, we consider the estimation of the unknown parameters of the NE-W model from complete samples via the maximum likelihood approach. Let $x_1, x_2, \ldots, x_n$ be the observed values of a random sample of this distribution with parameter vector $\Theta = (\alpha, \theta, \gamma)^{\mathrm{T}}$. The log-likelihood function for Θ, say $\ell = \ell(\Theta)$, takes the following form:

$$\begin{aligned} \ell &= n\log(\alpha) + n\log(\gamma) + (\alpha-1)\sum_{i=0}^{n}\log(x_i) + \sum_{i=0}^{n}\overline{\theta}\left(e^{-\gamma x_i^{\alpha}}\right) \\ &\quad + \sum_{i=0}^{n}\log\left[1-\overline{\theta}\left(1-e^{-\gamma x_i^{\alpha}}\right)\right]. \end{aligned} \tag{20}$$

The log-likelihood function can be maximized either directly by using the R (AdequecyModel package), SAS (PROC NLMIXED), or the Ox program (subroutine MaxBFGS) or by solving the nonlinear likelihood equations. The partial derivatives of the log-likelihood function are given by

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha} &= \frac{n}{\alpha} + \sum_{i=0}^{n}\log(x_i) - \gamma\overline{\theta}\sum_{i=0}^{n}e^{-\gamma x_i^{\alpha}}\log(x_i)x_i^{\alpha} \\ &\quad - \gamma\overline{\theta}\sum_{i=0}^{n}\frac{e^{-\gamma x_i^{\alpha}}\log(x_i)x_i^{\alpha}}{\left[1-\overline{\theta}\left(1-e^{-\gamma x_i^{\alpha}}\right)\right]}, \\ \frac{\partial\ell}{\partial\gamma} &= \frac{n}{\gamma} - \overline{\theta}\sum_{i=0}^{n}e^{-\gamma x_i^{\alpha}}x_i^{\alpha} - \overline{\theta}\sum_{i=0}^{n}\frac{e^{-\gamma x_i^{\alpha}}x_i^{\alpha}}{\left[1-\overline{\theta}\left(1-e^{-\gamma x_i^{\alpha}}\right)\right]}, \\ \frac{\partial\ell}{\partial\theta} &= -\sum_{i=0}^{n}e^{-\gamma x_i^{\alpha}} + \sum_{i=0}^{n}\frac{\left(1-e^{-\gamma x_i^{\alpha}}\right)}{\left[1-\overline{\theta}\left(1-e^{-\gamma x_i^{\alpha}}\right)\right]}. \end{aligned} \tag{21}$$

4.2. Monte Carlo Simulations. This section provides a comprehensive simulation study to explore the behavior of the MLEs. The NE-F family is easily simulated by inverting (3) as follows: If U has a uniform U (0,1) distribution, then the nonlinear equation is as follows:

$$x = Q(u) = G^{-1}(u) = F^{-1}(t), \tag{22}$$

where t is the solution of $\log t + \overline{\theta}(1-t) - \log(u) = 0$. Expression (22) can be used to simulate any special subcase of the NE-F family. Particularly, the quantile function of the NE-W distribution has the following form:

$$x_p = \left(\frac{1}{\gamma}\log\left\{\frac{\theta-1}{\theta-1-W\left[p(\theta-1)e^{\theta-1}\right]}\right\}\right)^{1/\alpha}, \tag{23}$$

where $W(s)$ refers to the negative branch of the Lambert W-function that is known as a product log function in Mathematica software and Wolfram Language.

Now, we consider the NE-W distribution to assess the behavior of the MLEs in estimating the NE-W parameters. We simulate the NE-W distribution for two combinations of parameters using Monte Carlo simulations with 1,000 replications. We calculate the MSEs and the biases of the parameter estimates using the `optim()` *R*-function with an argument `method = "L-BFGS-B."` We generate $N = 1000$ samples of sizes $n = 25, 50, 75, \ldots, 1000$ from the NE-W distribution with set 1: $\alpha = 1.4$, $\theta = 0.8$, $\gamma = 1.2$ and set 2: $\alpha = 0.9$, $\theta = 1.2$, $\gamma = 1.5$. The reason for choosing the initial values of the model parameters is to check the performance of the MLEs, for example, to see whether (i) the simulated values of the model parameters approach the true value or not, (ii) the MSEs decrease or not, and (iii) the biases approach zero or not. The estimated biases and MSEs of $\widehat{\Theta} = (\alpha, \theta, \gamma)^{T}$ are defined, respectively, by

$$\begin{aligned} \operatorname{Bias}(\widehat{\Theta}) &= \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\mathbf{\Theta}}_i - \mathbf{\Theta}\right), \\ \operatorname{MSE}(\widehat{\Theta}) &= \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\mathbf{\Theta}}_i - \mathbf{\Theta}\right)^2. \end{aligned} \tag{24}$$

The numerical results for the abovementioned measures are displayed in Tables 1 and 2. It is noted, from these tables, that the estimated biases decrease when the sample size n increases. Furthermore, the estimated MSEs decay toward zero as n increases. This fact reveals the consistency property of the MLEs.

Table 1: Simulation results of NE-W distribution.

	Set 1: $\alpha = 1.4$, $\theta = 0.8$, $\gamma = 1.2$			
n	Parameters	MLE	MSEs	Biases
25	$\widehat{\alpha}$	1.8475	1.4867	0.9865
	$\widehat{\theta}$	1.4865	1.8643	1.4987
	$\widehat{\gamma}$	1.9756	0.8643	0.9654
100	$\widehat{\alpha}$	1.7759	1.2987	0.8643
	$\widehat{\theta}$	1.3809	1.5978	1.2850
	$\widehat{\gamma}$	1.7690	0.8209	0.8609
200	$\widehat{\alpha}$	1.6965	1.1908	0.7689
	$\widehat{\theta}$	1.2865	1.1298	1.1095
	$\widehat{\gamma}$	1.6298	0.7609	0.6094
400	$\widehat{\alpha}$	1.6498	0.9075	0.5534
	$\widehat{\theta}$	1.1987	1.0795	0.8378
	$\widehat{\gamma}$	1.5987	0.5609	0.4732
600	$\widehat{\alpha}$	1.5864	0.6897	0.4398
	$\widehat{\theta}$	1.0073	0.7865	0.5563
	$\widehat{\gamma}$	1.4075	0.43987	0.3297
800	$\widehat{\alpha}$	1.4954	0.2875	0.1598
	$\widehat{\theta}$	0.9354	0.4086	0.2476
	$\widehat{\gamma}$	1.3278	0.3198	0.2388
1000	$\widehat{\alpha}$	1.4386	0.0965	0.0128
	$\widehat{\theta}$	0.8490	0.1043	0.1056
	$\widehat{\gamma}$	1.2487	0.1187	0.1006

Table 2: Simulation results of NE-W distribution.

	Set 1: $\alpha = 0.9$, $\theta = 1.2$, $\gamma = 1.5$			
n	Parameters	MLE	MSEs	Biases
25	$\widehat{\alpha}$	0.7596	0.8987	0.7956
	$\widehat{\theta}$	2.5523	1.5964	1.3578
	$\widehat{\gamma}$	2.0967	0.9552	0.5488
100	$\widehat{\alpha}$	0.7856	0.8057	0.7198
	$\widehat{\theta}$	1.6847	1.2971	0.9574
	$\widehat{\gamma}$	1.7847	0.8453	0.4702
200	$\widehat{\alpha}$	0.8044	0.7532	0.6056
	$\widehat{\theta}$	1.4965	1.1569	0.7436
	$\widehat{\gamma}$	1.7467	0.7873	0.3764
400	$\widehat{\alpha}$	0.8285	0.5864	0.4534
	$\widehat{\theta}$	1.4197	0.7745	0.5148
	$\widehat{\gamma}$	1.7159	0.4863	0.3189
600	$\widehat{\alpha}$	0.8307	0.5042	0.3583
	$\widehat{\theta}$	1.3843	0.6042	0.3178
	$\widehat{\gamma}$	1.6552	0.4198	0.2052
800	$\widehat{\alpha}$	0.8485	0.3208	0.1267
	$\widehat{\theta}$	1.3257	0.3758	0.1482
	$\widehat{\gamma}$	1.5962	0.2953	0.1065
1000	$\widehat{\alpha}$	0.8874	0.1065	0.0136
	$\widehat{\theta}$	1.2367	0.1207	0.0608
	$\widehat{\gamma}$	1.5286	0.0974	0.0299

5. The Applicability of the NE-W Distribution

This section explores the applicability of the proposed NE-W model in biological sciences as compared with other competitive distributions including Weibull, Marshall–Olkin Weibull (MOW), alpha power-transformed Weibull (APTW), and Kumaraswamy Weibull (Ku-W) distributions by analyzing two real-life applications. The distribution functions of the competitive models are as follows:

(1) Weibull distribution:

$$G(x;\alpha,\gamma) = 1 - e^{-\gamma x^{\alpha}}, \quad x \geq 0, \alpha, \gamma > 0. \tag{25}$$

(2) MOW distribution [14]:

$$G(x;\alpha,\gamma,\theta) = \frac{1 - e^{-\gamma x^{\alpha}}}{1 - (1-\theta)e^{-\gamma x^{\alpha}}}, \quad x \geq 0, \alpha, \gamma, \theta > 0. \tag{26}$$

(3) Ku-W distribution [15]:

$$G(x;\alpha,\gamma,a,b) = 1 - \left[1 - \left(1 - e^{-\gamma x^{\alpha}}\right)^{a}\right]^{b}, \quad x \geq 0, \alpha, \gamma, a, b > 0. \tag{27}$$

(4) APTW distribution [16]:

$$G(x,\alpha_1,a,\gamma) = \frac{\alpha_1^{\left(1-e^{-\gamma x^{\alpha}}\right)} - 1}{\alpha_1 - 1}, \quad x \geq 0, \alpha_1 \neq 1, \alpha_1, \gamma, a > 0. \tag{28}$$

First, we check whether the considered data set actually comes from the NE-W model or not using the goodness of fit test, based on the Anderson–Darling (AD) test statistic, Cramer–von-Mises (CM) test statistic, and Kolmogorov–Smirnov (KS) statistic with its corresponding p value. Note that the AD, CM, and KS statistics are to be used only to verify the goodness of fit and not as discrimination criteria. Therefore, we consider four discrimination criteria, based on the log-likelihood function evaluated at the maximum likelihood estimates, including the Akaike information (AIC), Bayesian information (BIC), Hannan–Quinn

Table 3: Maximum likelihood estimates with standard errors (in parenthesis) of the fitted distributions for data 1.

Dist.	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}_1$
NE-W	0.873 (0.1334)	7.107 (2.563)	0.028 (0.0231)			
Weibull	1.212 (0.8657)		0.003 (0.5634)			
MOW	0.6910 (0.1630)	53.657 (55.411)	0.121 (0.1300)			
APTW	1.154 (0.0459)		0.0043 (0.0011)			15.315 (7.5669)
Ku-W	0.843 (0.1908)		0.018 (0.0146)	4.061 (1.2527)	2.066 (2.3987)	

Table 4: The analytical measures of the fitted models for data 1.

Dist.	AIC	BIC	CAIC	HQIC
NE-W	855.514	862.344	855.867	858.233
Weibull	863.577	870.408	863.923	866.297
MOW	859.554	866.384	859.907	862.273
APTW	860.675	867.505	861.028	863.394
Ku-W	859.378	868.485	859.975	863.003

Table 5: Goodness of fit measures of the fitted models for data 1.

Dist.	CM	AD	KS	p value
NE-W	0.072	0.436	0.077	0.782
Weibull	0.191	1.113	0.123	0.322
MOW	0.149	0.848	0.105	0.405
APTW	0.130	0.763	0.126	0.444
Ku-W	0.084	0.531	0.089	0.605

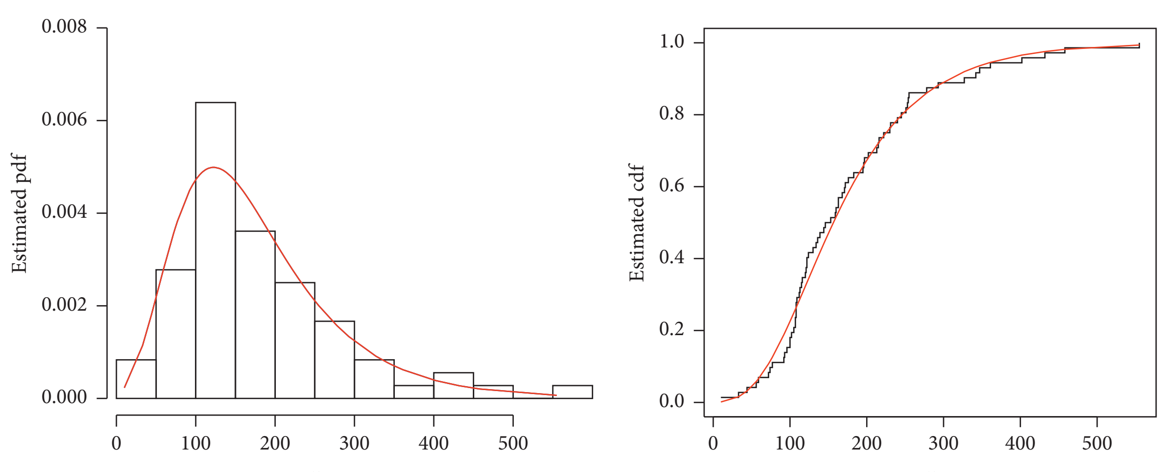

Figure 3: Plots for the estimated pdf and cdf of the NE-W distribution for data 1.

information (HQIC), and corrected Akaike information (CAIC) criteria.

All the required computations have been performed via the `optim()` R-function with an argument `method` = "BFGS." In general, a model with smaller values of these measures indicates better fit to the data. Based on the considered analytical measures, one can observe that the proposed NE-W model provides the best fit to the two analyzed real-life data sets.

5.1. Data 1: Infected Guinea Pigs Data. The first data set consists of 72 observations, and it reported in Bjerkedal [17]. The data represent the Guinea pigs infected with virulent tubercle bacilli. The NE-W and other competitors are applied to this data set. It is observed that the proposed model provides better fit than other competitors. The values of the model parameters are presented in Table 3. The discrimination measures of the fitted models are provided in Table 4. The analytical measures of the NE-W model and other

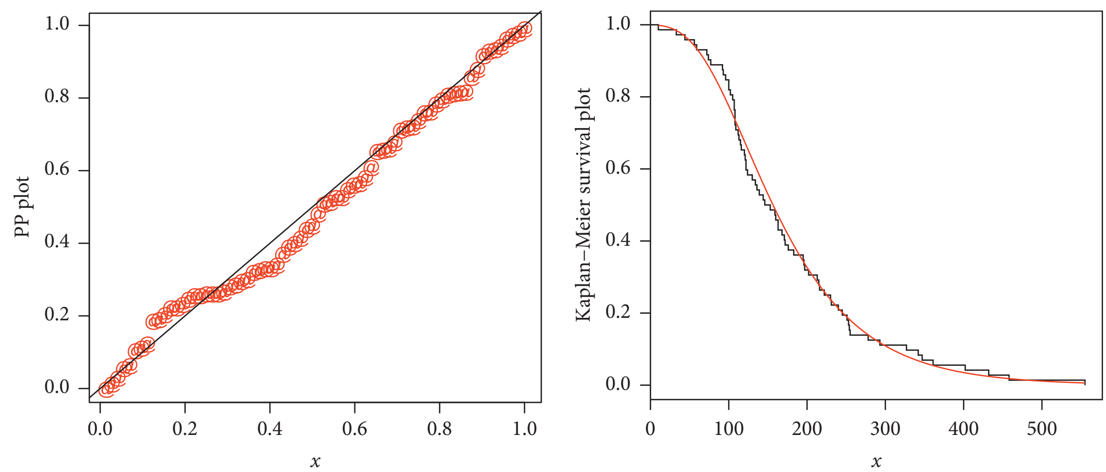

Figure 4: PP and Kaplan–Meier survival plots of the NE-W distribution for data 1.

Table 6: Maximum likelihood estimates with standard errors (in parenthesis) of the fitted distributions for data 2.

Dist.	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\gamma}$	$\widehat{a}$	$\widehat{b}$	$\widehat{\alpha}_1$
NE-W	0.284 (0.1158)	35.556 (53.809)	1.001 (0.9512)			
Weibull	0.948 (0.0772)		0.0061 (0.0028)			
MOW	0.589 (0.1097)	5.8871 (5.5984)	0.103 (0.0942)			
APTW	1.016 (0.0568)		0.003 (0.0019)			0.577 (1.0330)
Ku-W	0.481 (0.2970)		0.379 (0.4360)	11.607 (12.8913)	0.449 (0.700)	

Table 7: The analytical measures of the fitted models for data 2.

Dist.	AIC	BIC	CAIC	HQIC
NE-W	560.722	566.075	561.322	562.707
Weibull	567.693	571.261	567.986	569.016
MOW	571.706	577.059	572.306	573.691
APTW	568.802	574.155	569.402	570.787
Ku-W	562.737	569.874	563.763	565.384

Table 8: Goodness of fit measures of the fitted models for data 2.

Dist.	CM	AD	KS	p value
NE-W	0.018	0.120	0.058	0.996
Weibull	0.143	0.833	0.128	0.422
MOW	0.167	0.972	0.118	0.531
APTW	0.130	0.763	0.126	0.444
Ku-W	0.022	0.139	0.067	0.981

competitive models are provided in Table 5. The estimated pdf and cdf are sketched in Figure 3, whereas the probability-probability (pp) plot and Kaplan–Meier survival plot are provided in Figure 4. Figures 3 and 4 reveal that the NE-W distribution provides the superior fit to the Guinea pigs infected data.

5.2. Data 2: Survival Times of Head and Neck Cancer Patients. The second data set consists of 44 observations as reported in [18], and it represents the survival times of a group of patients suffering from head and neck cancer who are treated using a combination of radiotherapy. The NE-W and other selected distributions are applied to analyze this data set. The values of the model parameters are presented in Table 6. The discrimination measures of the fitted models are provided in Table 7, whereas the analytical measures of the proposed NE-W model and other competitive models are provided in Table 8. The estimated pdf and cdf are sketched in Figure 5, which shows that the proposed distribution fits the estimated pdf and cdf plots very closely. The PP plot and Kaplan–Meier survival plot are presented in Figure 6. One can see, from Tables 7 and 8 and Figures 5 and 6, that the proposed model outclasses other competitors.

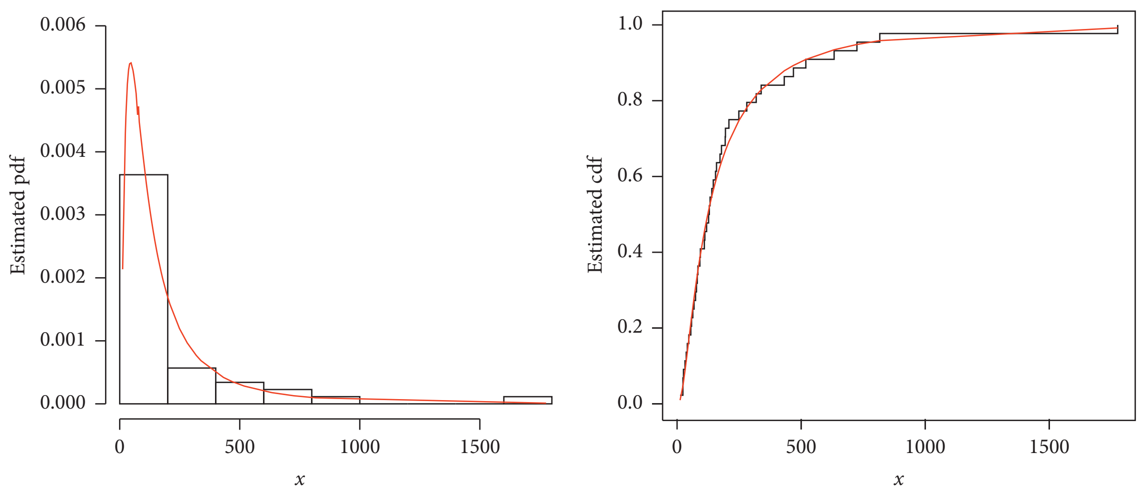

FIGURE 5: Plots for the estimated pdf and cdf of the NE-W distribution for data 2.

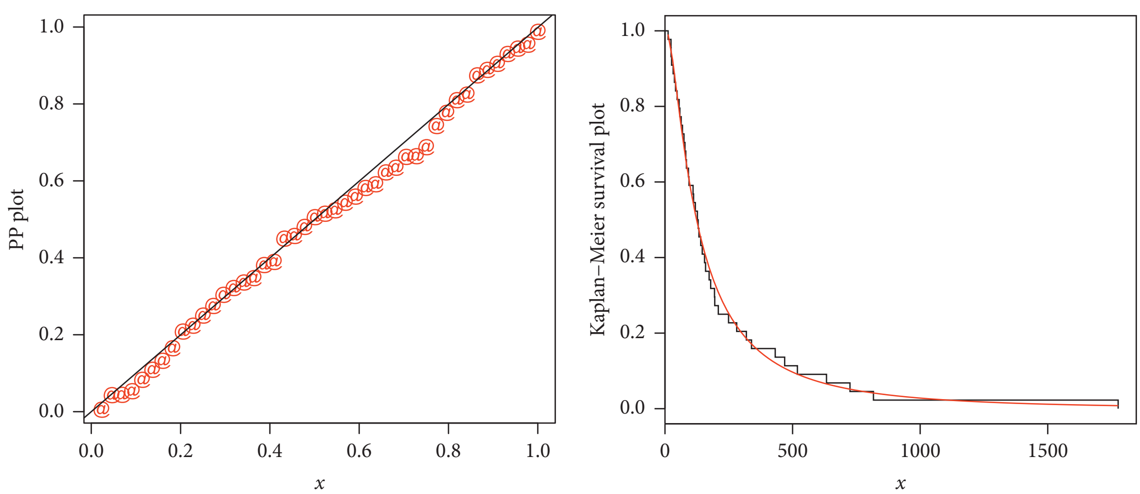

FIGURE 6: PP and Kaplan–Meier survival plots of the NE-W distribution for data 2.

6. Concluding Remarks

This article proposed a new method for generating flexible models, called a new extended-F (NE-F) family. One special submodel of the NE-F family, called a new extended-Weibull (NE-W) distribution, is considered. Some general properties are derived for the NE-F family. The model parameters are estimated via the maximum likelihood along with simulation results to explore the performance of these estimators. Two applications to biological science data are analyzed to illustrate, empirically, the flexibility of the proposed NE-W model. The comparison of the NE-W distribution is conducted with some well-known lifetime distributions such as Weibull, Marshall–Olkin Weibull, alpha power-transformed Weibull, and Kumaraswamy Weibull distributions. The practical applications of the proposed model reveal better fit to both analyzed data sets than other competing models.

It is worth mentioning that the results in this paper can be extended in some ways. For example, extreme stability, characterization properties, entropies, and stochastic orders could be considered. Exponentiated and transmuted versions may be studied, several fundamental properties could be explored, and a bivariate extended-F family may also be established.

Data Availability

The references of the data sets used in this paper are provided within the main body of the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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