Research Article

Nonlocal Conformable-Fractional Differential Equations with a Measure of Noncompactness in Banach Spaces

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This paper deals with the existence of mild solutions for the following Cauchy problem: $d^\alpha x(t)/dt^\alpha = Ax(t) + f(t, x(t))$, $x(0) = x_0 + g(x)$, $t \in [0, \tau]$, where $d^\alpha(\cdot)/dt^\alpha$ is the so-called conformable fractional derivative. The linear part A is the infinitesimal generator of a uniformly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space X , f and g are given functions. The main result is proved by using the Darbo-Sadovskii fixed point theorem without assuming the compactness of the family $(T(t))_{t \geq 0}$ and the Lipschitz condition on the nonlocal part g .

1. Introduction

Many dynamical processes in physics, biology, economics, and other areas of applications can be governed by abstract ordinary differential evolution equations of the following form:

$$\dot{x}(t) = Ax(t) + f(t, x(t)). \quad (1)$$

Unfortunately, the classical derivative $\dot{x}(t)$ appearing in equation (1) is local and cannot model the dynamical processes with memory. Hence, in order to avoid this shortcoming of classical derivative, many authors try to replace the classical derivative by a fractional derivative [1–4] because fractional derivatives have been proved that they are a very good way to model many phenomena with memory in various fields of science and engineering [5–9]. In consequence, many researchers pay attention to form the best definition of fractional derivative. Recently, a novel definition named conformable fractional derivative is introduced in [10]. This new fractional derivative quickly becomes the subject of many contributions in several areas of science [11–22]. Motivated by the better effect of the fractional derivative and simple properties of the conformable fractional derivative, we consider model (1) in the

framework of conformable fractional calculus. Precisely, we study the following Cauchy problem:

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x(t)), \quad x(0) = x_0 + g(x), \quad t \in [0, \tau], \quad (2)$$

where $d^\alpha(\cdot)/dt^\alpha$ is the conformable fractional derivative of the order $\alpha \in]0, 1]$. The linear part A is the infinitesimal generator of a uniformly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space $(X, \|\cdot\|)$. For more details about semigroup theory, we refer to [23]. The nonlinear part $f: [0, \tau] \times X \rightarrow X$ is a given function. The initial condition $x(0) = x_0 + g(x)$ means the nonlocal condition [24]. For physical interpretations of this condition, we can see [25, 26]. The nonlocal condition attracts the attention of many authors in several works [27, 28]. The vector x_0 is an element of X and $g: \mathcal{C} \rightarrow X$ is a given function, with $\mathcal{C}$ is the space of continuous functions $x(\cdot)$ defined from $[0, \tau]$ into X . Throughout this paper, we endow the space $\mathcal{C}$ with the norm $|x|_{\mathcal{C}} = \sup_{t \in [0, \tau]} \|x(t)\|$. It is well known that the space $(\mathcal{C}, |\cdot|_{\mathcal{C}})$ is a Banach space. We also denote by $|\cdot|$ the norm in the space $\mathcal{L}(X)$ of bounded operators defined from X into itself.

Our goal in this paper is to prove the existence of mild solutions for the Cauchy problem (2) by means of the Darbo–Sadovskii fixed point theorem without assuming the compactness of the family $(T(t))_{t>0}$ and the Lipschitz condition on the nonlocal part g .

The content of this paper is organized as follows. In section 2, we recall some preliminary facts on the conformable fractional calculus and measure of noncompactness. Section 3 is devoted to prove the main result.

2. Preliminaries

Recalling some preliminary facts on the conformable fractional calculus.

Definition 1 (see [10]). Let $\alpha \in]0, 1]$. The conformable fractional derivative of order α of a function $x(\cdot)$ for $t > 0$ is defined as follows:

$$\frac{d^\alpha x(t)}{dt^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{x(t + \varepsilon t^{1-\alpha}) - x(t)}{\varepsilon}. \quad (3)$$

For $t = 0$, we adopt the following definition:

$$\frac{d^\alpha x(0)}{dt^\alpha} = \lim_{t \rightarrow 0^+} \frac{d^\alpha x(t)}{dt^\alpha}. \quad (4)$$

The fractional integral $I^\alpha(\cdot)$ associated with the conformable fractional derivative is defined by

$$I^\alpha(x)(t) = \int_0^t s^{\alpha-1} x(s) ds. \quad (5)$$

Theorem 1 (see [10]). If $x(\cdot)$ is a continuous function in the domain of $I^\alpha(\cdot)$, then we have

$$\frac{d^\alpha(I^\alpha(x)(t))}{dt^\alpha} = x(t). \quad (6)$$

Definition 2 (see [8]). The Laplace transform of a function $x(\cdot)$ is defined by

$$\mathcal{L}(x(t))(\lambda) := \int_0^{+\infty} e^{-\lambda t} x(t) dt, \quad \lambda > 0. \quad (7)$$

It is remarkable that the above transform is not compatible with the conformable fractional derivative. For this, the adapted transform is given by the following definition.

Definition 3 (see [11]). The fractional Laplace transform of order $\alpha \in]0, 1]$ of a function $x(\cdot)$ is defined by

$$\mathcal{L}_\alpha(x(t))(\lambda) := \int_0^{+\infty} t^{\alpha-1} e^{-\lambda(t^\alpha/\alpha)} x(t) dt, \quad \lambda > 0. \quad (8)$$

The following proposition gives us the actions of the fractional integral and the fractional Laplace transform on the conformable fractional derivative, respectively.

Proposition 1 (see [11]). If $x(\cdot)$ is a differentiable function, then we have the following results:

$$I^\alpha\left(\frac{d^\alpha x(\cdot)}{dt^\alpha}\right)(t) = x(t) - x(0), \quad (9)$$

$$\mathcal{L}_\alpha\left(\frac{d^\alpha x(t)}{dt^\alpha}\right)(\lambda) = \lambda \mathcal{L}_\alpha(x(t))(\lambda) - x(0).$$

According to [15], we have the following remark.

Remark 1. For two functions $x(\cdot)$ and $y(\cdot)$, we have

$$\mathcal{L}_\alpha\left(x\left(\frac{t^\alpha}{\alpha}\right)\right)(\lambda) = \mathcal{L}(x(t))(\lambda),$$

$$\mathcal{L}_\alpha\left(\int_0^t s^{\alpha-1} x\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) y(s) ds\right)(\lambda) = \mathcal{L}(x(t))(\lambda) \mathcal{L}_\alpha(y(t))(\lambda). \quad (10)$$

Now, we recall some concepts on the Hausdorff measure of noncompactness.

Definition 4 (see [29, 30]). For a bounded set B in a Banach space X , the Hausdorff measure of noncompactness σ is defined as

$$\sigma(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls with radii } \varepsilon\}. \quad (11)$$

The following lemma presents some basic properties of the Hausdorff measure of noncompactness.

Lemma 1 (see [29, 30]). Let X be a Banach space and $B, C \subseteq X$ be bounded. Then, the following properties hold.

- (1) B is precompact if and only if $\sigma(B) = 0$;
- (2) $\sigma(B) = \sigma(\overline{B}) = \sigma(\text{conv}(B))$, where $\overline{B}$ and $\text{conv}(B)$ mean the closure and convex hull of B , respectively;
- (3) $\sigma(B) \leq \sigma(C)$, where $B \subseteq C$;
- (4) $\sigma(B + C) \leq \sigma(B) + \sigma(C)$, where $B + C = \{x + y : x \in B, y \in C\}$;
- (5) $\sigma(B \cup C) \leq \max\{\sigma(B), \sigma(C)\}$;
- (6) $\sigma(\lambda B) = |\lambda| \sigma(B)$ for any $\lambda \in \mathbb{R}$, when X be a real Banach space;
- (7) If the operator $Q : D(Q) \subseteq X \longrightarrow Y$ is Lipschitz continuous with constant $k \geq 0$ then we have $\rho(Q(B)) \leq k \sigma(B)$ for any bounded subset $B \subseteq D(Q)$, where Y is another Banach space and ρ represents the Hausdorff measure of noncompactness in Y .

Definition 5 (see [30]). The operator $Q : D(Q) \subseteq X \longrightarrow X$ is said to be a σ -contraction if there exists a positive constant $k < 1$ such that $\sigma(Q(B)) \leq k \sigma(B)$ for any bounded closed subset $B \subseteq D(Q)$.

Lemma 2 (see [29, 30] (Darbo–Sadovskii theorem)). Let $B \subset X$ be a bounded, closed, and convex set. If $Q : B \longrightarrow B$ is a continuous and σ -contraction operator. Then, Q has at least one fixed point in B .

Lemma 3 (see [31, 32]). Let $D \subset X$ be a bounded set, then there exists a countable set $D_0 \subset D$ such that $\sigma(D) \leq 2\sigma(D_0)$.

We denote by σ_c the Hausdorff measure of non-compactness in the space $\mathcal{C}$ of continuous functions $x(\cdot)$ defined from $[0, \tau]$ into X .

Lemma 4 (see [33]). Let $D_0 := \{x_n\} \subset \mathcal{C}$ be a countable set, then

- (1) $\sigma(D_0(t)) := \sigma(\{x_n(t)\})$ is Lebesgue integral on $[0, \tau]$,
- (2) $\sigma(\int_0^\tau D_0(s)ds) \leq 2 \int_0^\tau \sigma(D_0(s))ds$, where
 $\sigma(\int_0^\tau D_0(s)ds) := \sigma(\{\int_0^\tau x_n(s)ds\})$.

Lemma 5 (see [29]). Let $D \subset \mathcal{C}$ be bounded and equi-continuous, then

- (1) $\sigma(D(t))$ is continuous on $[0, \tau]$,
- (2) $\sigma_c(D) = \max_{t \in [0, \tau]} (\sigma(D(t)))$.

3. Main Result

We first give the definition of mild solutions for the Cauchy problem (2). To do so, applying the fractional Laplace transform in equation (2), we obtain

$$\lambda \mathcal{L}_\alpha(x(t))(\lambda) = x_0 + g(x) + A \mathcal{L}_\alpha(x(t))(\lambda) + \mathcal{L}_\alpha(f(t, x(t)))(\lambda). \quad (12)$$

Then, one has

$$\mathcal{L}_\alpha(x(t))(\lambda) = (\lambda - A)^{-1}(x_0 + g(x)) + (\lambda - A)^{-1} \mathcal{L}_\alpha(f(t, x(t)))(\lambda). \quad (13)$$

Using the inverse fractional Laplace transform combined with Remark 1, we obtain

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds. \quad (14)$$

Motivate by the above calculus, we can introduce the following definition.

Definition 6. A function $x \in \mathcal{C}$ is called a mild solution of the Cauchy problem (2) if

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds. \quad (15)$$

To obtain the existence of mild solutions, we will need the following assumptions:

(H_1) The function $f(t, \cdot): X \rightarrow X$ is continuous, and for all $r > 0$ there exists a function $\mu_r \in L^\infty([0, \tau], \mathbb{R}^+)$ such that $\sup_{\|x\| \leq r} \|f(t, x)\| \leq \mu_r(t)$, for all $t \in [0, \tau]$.

(H_2) The function $f(\cdot, x): [0, \tau] \rightarrow X$ is continuous, for all $x \in X$.

(H_3) The function $g: \mathcal{C} \rightarrow X$ is continuous and compact.

(H_4) There exist positive constants a and b such that $\|g(x)\| \leq a\|x\|_c + b$, for all $x \in \mathcal{C}$.

(H_5) There exists a positive constant L such that $\sigma(f(t, D_0)) \leq L\sigma(D_0)$, for any countable set $D_0 \subset X$ and $t \in [0, \tau]$.

Theorem 2. Assume that (H_1) – (H_5) hold, then the Cauchy problem (2) has at least one mild solution provided that

$$\sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \max\left(a, \frac{4L\tau^\alpha}{\alpha}\right) < 1. \quad (16)$$

Proof. In order to use the Darbo–Sadovskii fixed point theorem, we put $B_r := \{x \in \mathcal{C}, \|x\|_c \leq r\}$ for $r > 0$ and define the operator $\Gamma: \mathcal{C} \rightarrow \mathcal{C}$ by

$$\begin{aligned} \Gamma(x)(t) &= T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] \\ &+ \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds. \end{aligned} \quad (17)$$

The proof will be given in four steps.

Step 1. Prove that there exists a radius $\delta > 0$ such that $\Gamma: B_\delta \rightarrow B_\delta$.

Let $x \in \mathcal{C}$, we have

$$\begin{aligned} \|\Gamma(x)(t)\| &\leq \left\| T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] \right\| \\ &+ \int_0^t s^{\alpha-1} \left\| T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) \right\| ds. \end{aligned} \quad (18)$$

Taking the supremum, we obtain

$$\begin{aligned} \|\Gamma(x)\|_c &\leq \sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \left[\|x_0\| + \|g(x)\| \right. \\ &\left. + \int_0^\tau s^{\alpha-1} \|f(s, x(s))\| ds \right]. \end{aligned} \quad (19)$$

Using assumption (H_4), we deduce that

$$\begin{aligned} \|\Gamma(x)\|_c &\leq \sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \left[\|x_0\| + a\|x\|_c \right. \\ &\left. + b + \int_0^\tau s^{\alpha-1} \|f(s, x(s))\| ds \right]. \end{aligned} \quad (20)$$

Hence, it suffices to consider δ as a solution of the following inequality:

$$\sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \left[\|x_0\| + ar + b + \frac{\tau^\alpha}{\alpha} \mu_r|_{L^\infty([0, \tau], \mathbb{R}^+)} \right] \leq r. \quad (21)$$

Precisely, we can choose δ such that

$$\delta \geq \frac{\sup_{t \in [0, \tau]} |T(t^\alpha/\alpha)|}{1 - a \sup_{t \in [0, \tau]} |T(t^\alpha/\alpha)|} \left[\|x_0\| + b + \frac{\tau^\alpha}{\alpha} |\mu_\delta|_{L^\infty([0, \tau], \mathbb{R}^+)} \right]. \quad (22)$$

Step 2. Prove that $\Gamma: B_\delta \rightarrow B_\delta$ is continuous.

Let $(x_n) \subset B_\delta$ such that $x_n \rightarrow x$ in B_δ . We have

$$\begin{aligned} \Gamma(x_n)(t) - \Gamma(x)(t) &= T\left(\frac{t^\alpha}{\alpha}\right) [g(x_n) - g(x)] \\ &\quad + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \\ &\quad \cdot [f(s, x_n(s)) - f(s, x(s))] ds. \end{aligned} \quad (23)$$

Then, by using a direct computation, we obtain

$$\begin{aligned} \|\Gamma(x_n) - \Gamma(x)\|_c &\leq \sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \left[\|g(x_n) - g(x)\| \right. \\ &\quad \left. + \int_0^t s^{\alpha-1} \|f(s, x_n(s)) - f(s, x(s))\| ds \right]. \end{aligned} \quad (24)$$

Using assumption (H_1) , we get $\|s^{\alpha-1} [f(s, x_n(s)) - f(s, x(s))]\| \leq 2\mu_\delta(s)s^{\alpha-1}$ and $f(s, x_n(s)) \rightarrow f(s, x(s))$ as $n \rightarrow +\infty$.

The Lebesgue dominated convergence theorem proves that $\int_0^\tau s^{\alpha-1} \|f(s, x_n(s)) - f(s, x(s))\| ds \rightarrow 0$ as $n \rightarrow +\infty$. According to continuity of the function g , we deduce that $\lim_{n \rightarrow +\infty} \|g(x_n) - g(x)\| = 0$. Hence, Γ is continuous.

Step 3. Prove that $\Gamma(B_\delta)$ is equicontinuous.

For $x \in B_\delta$ and $t_1, t_2 \in [0, \tau]$ such that $t_1 < t_2$. We have

$$\begin{aligned} \Gamma(x)(t_2) - \Gamma(x)(t_1) &= \left[T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I \right] \left[T\left(\frac{t_1^\alpha}{\alpha}\right) (x_0 + g(x)) \right. \\ &\quad \left. + \int_0^{t_1} s^{\alpha-1} T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds \right] \\ &\quad + \int_{t_1}^{t_2} s^{\alpha-1} T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds. \end{aligned} \quad (25)$$

By using assumptions (H_1) and (H_4) , we obtain

$$\begin{aligned} \|\Gamma(x)(t_2) - \Gamma(x)(t_1)\| &\leq \left(\|x_0\| + a\delta + b + \frac{\tau^\alpha}{\alpha} |\mu_\delta|_{L^\infty([0, \tau], \mathbb{R}^+)} \right) \sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \\ &\quad \cdot \left[\left| T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I \right| + \sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| |\mu_\delta|_{L^\infty([0, \tau], \mathbb{R}^+)} \left[\frac{t_2^\alpha - t_1^\alpha}{\alpha} \right] \right]. \end{aligned} \quad (26)$$

The above inequality combined with the uniform continuity of the family $(T(t))_{t \geq 0}$ proves that $\Gamma(B_\delta)$ is equicontinuous on $[0, \tau]$.

Step 4. Prove that $\Gamma: B_\delta \rightarrow B_\delta$ is a σ_c -contraction operator.

Let $D \subset B_\delta$, then by Lemma 3 there exists a countable set D_0 such that $D_0 = \{x_n\} \subset D$. Hence, $\Gamma(D_0)$ becomes a countable subset of $\Gamma(D)$. Thus, Lemma 3 proves that $\sigma_c(\Gamma(D)) \leq 2\sigma_c(\Gamma(D_0))$. Since $\Gamma(D_0)$ is bounded and equicontinuous, then by using Lemma 5, we obtain

$$\sigma_c(\Gamma(D_0)) = \max_{t \in [0, \tau]} (\sigma(\Gamma(D_0)(t))). \quad (27)$$

Then, one has

$$\begin{aligned} \sigma_c(\Gamma(D)) &\leq 2\sigma_c(\Gamma(D_0)) \\ &= 2 \max_{t \in [0, \tau]} (\sigma(\Gamma(D_0)(t))) \\ &= 2 \max_{t \in [0, \tau]} \left(\sigma \left(T\left(\frac{t^\alpha}{\alpha}\right) [x_0 + g(D_0)] \right. \right. \end{aligned} \quad (28)$$

$$\left. \left. + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, D_0(s)) ds \right) \right).$$

By using point (4) of Lemma 1, we deduce that

$$\begin{aligned} \sigma_c(\Gamma(D)) &\leq 2 \max_{t \in [0, \tau]} \left(\sigma \left(T\left(\frac{t^\alpha}{\alpha}\right) [x_0 + g(D_0)] \right) \right. \\ &\quad \left. + \sigma \left(\int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, D_0(s)) ds \right) \right). \end{aligned} \quad (29)$$

Since g is compact, then $T(t^\alpha/\alpha)[x_0 + g(D_0)]$ is relatively compact. Hence, using point (1) of Lemma 1 in the above inequality, we obtain

$$\sigma_c(\Gamma(D)) \leq 2 \max_{t \in [0, \tau]} \left(\sigma \left(\int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, D_0(s)) ds \right) \right). \quad (30)$$

In view of Lemma 4, we get

$$\sigma_c(\Gamma(D)) \leq 4 \max_{t \in [0, \tau]} \left(\int_0^t s^{\alpha-1} \sigma \left(T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, D_0(s)) ds \right) \right). \quad (31)$$

Next, point (7) of Lemma 1 shows that

$$\sigma_c(\Gamma(D)) \leq 4 \sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \max_{t \in [0, \tau]} \left(\int_0^t s^{\alpha-1} \sigma(f(s, D_0(s))) ds \right). \quad (32)$$

By using assumption (H_5) , we obtain

$$\sigma_c(\Gamma(D)) \leq 4L \sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \max_{t \in [0, \tau]} \left(\int_0^t s^{\alpha-1} \sigma(D_0(s)) ds \right). \quad (33)$$

Hence, by using a direct computation combined with point (2) of Lemma 5, we obtain

$$\begin{aligned}\sigma_c(\Gamma(D)) &\leq 4L \sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \sigma_c(D) \int_0^\tau s^{\alpha-1} ds \\ &= \frac{4L\tau^\alpha}{\alpha} \sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \sigma_c(D).\end{aligned}\quad (34)$$

In consequence, we have

$$\sigma_c(\Gamma(D)) \leq \frac{4L\tau^\alpha}{\alpha} \sup_{t \in [0, \tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \sigma_c(D). \quad (35)$$

Since $4L\tau^\alpha/\alpha \sup_{t \in [0, \tau]} |T(t^\alpha/\alpha)| < 1$, then Γ is a σ_c -contraction operator.

In conclusion, Lemma 2 shows that Γ has at least one fixed point, which is a mild solution of the Cauchy problem (2). $\square$

Remark 2. We note that Theorem 2 improves Theorem 3 in [18] because in Theorem 2 we have not imposed the compactness of the family $(T(t))_{t>0}$ and the Lipschitz condition on the nonlocal part g .

4. Conclusion

Without imposing the compactness condition on the semigroup family and the Lipschitz condition on the nonlocal condition, we have proved the existence of mild solutions for a class of conformable-fractional differential equations with nonlocal conditions in a Banach space. The main result is obtained by means of semigroup theory combined with the Darbo-Sadovskii fixed point theorem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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