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Research Article

Existence and Uniqueness of Mild Solutions to Impulsive Nonlocal Cauchy Problems

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In this paper, a class of nonlocal impulsive differential equation with conformable fractional derivative is studied. By utilizing the theory of operators semigroup and fractional derivative, a new concept on a solution for our problem is introduced. We used some fixed point theorems such as Banach contraction mapping principle, Schauder's fixed point theorem, Schaefer's fixed point theorem, and Krasnoselskii's fixed point theorem, and we derive many existence and uniqueness results concerning the solution for impulsive nonlocal Cauchy problems. Some concrete applications to partial differential equations are considered. Some concrete applications to partial differential equations are considered.

1. Introduction

Fractional differential equations have gained popularity due to their applications in many domains of science and engineering [1–3]. In consequence, many researchers pay attention to form a simple and best definition of fractional derivative. Recently, a new definition of fractional derivative named conformable fractional derivative has been introduced in [4]. This novel fractional derivative is very easy and satisfies all the properties of the standard one. In short time, many studies and discussion related to conformable fractional derivative have appeared in several areas of applications [1–10].

Motivated by the abovementioned works, we consider the following impulsive differential equation with conformable fractional derivative:

$$\begin{cases} \dfrac{d^\alpha x(t)}{dt^\alpha} = \mathrm{Ax}(t) + f(t, x(t)), & \alpha \in (0,1], t \in J = [0,b], t \neq t_k, \\ x(t_k^+) = x(t_k^-) + y_k, & k = 1, 2, \ldots, n, \\ x(0) = x_0 \in X, \end{cases} \tag{1}$$

where $(d^\alpha(.)/dt^\alpha)$ is the so-called fractional conformable derivative [4]. This novel fractional derivative attracts the attention of many authors in various domains of science [1–10]. $A\colon D(A) \subseteq X \longrightarrow X$ is the generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on a Banach space X, $f\colon J \times X \longrightarrow X$ is continuous, x_0, y_k are the element of X, and $0 = t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} = b$, $x(t_k^+) = \lim_{h \longrightarrow 0^+} x(t_k + h)$ and $x(t_k^-) = x(t_k)$ represent respectively the right and left limits of $x(t)$ at $t = t_k$.

One of the main novelties of this paper is the concept on a mild solution for system (1). Then, using some fixed point theorems such as Banach contraction mapping principle and Schauder's fixed point theorem, we derive many existence and uniqueness results concerning the mild solution for system (1) under the different assumptions on the nonlinear terms.

As a second problem, we discuss in Section 4, a nonlocal impulsive differential equation with conformable fractional derivative

$$\begin{cases} \dfrac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x(t)), & \alpha \in (0,1], t \in J = [0,b], t \neq t_k, \\ x(t_k^+) = x(t_k^-) + y_k, & k = 1, 2, \ldots, n, \\ x(0) = x_0 + g(x), \end{cases} \tag{2}$$

where A, f, y_k are defined as above, g is a given function and constitutes a Cauchy problem. The condition $x(0) = x_0 + g(x)$ represents the nonlocal condition [11]. For good effect of this condition, we refer to [12, 13]. We adopt the ideas given in [14–16] and obtained some new existence and uniqueness results for system (2) under the different assumptions on the nonlocal terms.

The content of this paper is organized as follows. In Section 2, we recall some preliminary facts on the conformable fractional calculus. Sections 3 and 4 are devoted to prove the main result. At last, some interesting examples are presented to illustrate the theory.

2. Preliminary

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper, and we recall some concepts on conformable fractional calculus.

Let $L_b(X)$ be the Banach space of all linear and bounded operators on X. For a C_0-semigroup $\{T(t), t \geq 0\}$ on X, we set $M = \sup_{t\in J}\|T(t^\alpha/\alpha)\|_{L_b(X)}$. Let $\mathscr{C}(J, X)$ be the Banach space of all X-valued continuous functions from $J = [0, b]$ into X, endowed with the norm $\|x\|_{\mathscr{C}} = \sup_{t\in J}\|x(t)\|$. We also introduce the set of functions

$$\begin{aligned}\mathscr{PC}(J, X) = \{x\colon J \longrightarrow X | x \text{ is continuous at } t \in J\backslash\{t_1, t_2, \ldots, t_n\},\\ x \text{ is continuous from left and has right hand limits at } t \in \{t_1, t_2, \ldots, t_n\}\},\end{aligned} \tag{3}$$

endowed with the norm

$$\|x\|_{\mathscr{PC}} = \max\left\{\sup_{t\in J}\|x(t+0)\|, \quad \sup_{t\in J}\|x(t-0)\|\right\}, \tag{4}$$

where it is easy to see that $(\mathscr{PC}(J, X), \|.\|_{\mathscr{PC}})$ is a Banach space.

Theorem 1 (Krasnoselskii's fixed point theorem). *Assume that K is a closed bounded convex subset of a Banach space X. Furthermore, assume that Γ_1 and Γ_2 are mappings from K into X such that*

(1) $\Gamma_1(u) + \Gamma_2(v)$, for all $u, v \in K$

(2) Γ_1 is a contraction

(3) Γ_2 is continuous and compact

Then, $\Gamma_1 + \Gamma_2$ has a fixed point in K.

Definition 1 (see [4]). Let $\alpha \in]0, 1]$. The conformable fractional derivative of order α of a function $x(.)$ for $t > 0$ is defined as

$$\frac{d^\alpha x(t)}{dt^\alpha} = \lim_{\varepsilon\longrightarrow 0}\frac{x\left(t + \varepsilon t^{1-\alpha}\right) - x(t)}{\varepsilon}. \tag{5}$$

For $t = 0$, we adapt the following definition:

$$\frac{d^\alpha x(0)}{dt^\alpha} = \lim_{t\longrightarrow 0^+}\frac{d^\alpha x(t)}{dt^\alpha}. \tag{6}$$

The fractional integral $I^\alpha(.)$ associated with the conformable fractional derivative is defined by

$$I^\alpha(x)(t) = \int_0^t s^{\alpha-1}x(s)ds. \tag{7}$$

Theorem 2 (see [4]). *If $x(.)$ is a continuous function in the domain of $I^\alpha(.)$, then we have*

$$\frac{d^\alpha(I^\alpha(x)(t))}{dt^\alpha} = x(t). \tag{8}$$

Definition 2 (see [2]). The Laplace transform of a function $x(.)$ is defined by

$$\mathscr{L}(f(t))(\lambda) := \int_0^{+\infty} e^{-\lambda t} f(t)dt, \quad \lambda > 0. \tag{9}$$

It is remarkable that the above transform is not compatible with the conformable fractional derivative. For this, the adapted transform is given by the following definition.

Definition 3 (see [5]). The fractional Laplace transform of order $\alpha \in]0, 1]$ of a function $x(.)$ is defined by

$$\mathscr{L}_\alpha(x(t))(\lambda) := \int_0^{+\infty} t^{\alpha-1}e^{-\lambda(t^\alpha/\alpha)}x(t)dt, \quad \lambda > 0. \tag{10}$$

The following proposition gives us the actions of the fractional integral and the fraction Laplace transform on the conformable fractional derivative, respectively.

Proposition 1 (see [5]). *If $x(.)$ is a differentiable function, then we have the following results:*

$$\begin{aligned} I^\alpha\left(\frac{d^\alpha x(.)}{dt^\alpha}\right)(t) &= x(t) - x(0),\\ \mathscr{L}_\alpha\left(\frac{d^\alpha x(t)}{dt^\alpha}\right)(\lambda) &= \lambda\mathscr{L}_\alpha(x(t))(\lambda) - x(0). \end{aligned} \tag{11}$$

According to [6], we have the following remark.

Remark 1. For two functions $x(.)$ and $y(.)$, we have

$$\begin{aligned} \mathscr{L}_\alpha\left(x\left(\frac{t^\alpha}{\alpha}\right)\right)(\lambda) &= \mathscr{L}(x(t))(\lambda),\\ \mathscr{L}_\alpha\left(\int_0^t s^{\alpha-1}x\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)y(s)ds\right)(\lambda) &= \mathscr{L}(x(t))(\lambda)\mathscr{L}_\alpha(y(t))(\lambda). \end{aligned} \tag{12}$$

Lemma 1. *A measurable function* $f: J \longrightarrow X$ *is Bochner integrable, if* $\|f\|$ *is Lebesuge integrable.*

Lemma 2. *For* $\sigma \in (0,1]$ *and* $0 < a \leq b$, *we have* $|a-b|^{\sigma} \leq (b-a)^{\sigma}$.

3. Main Results

Now, we give the main contribution results.

To obtain the uniqueness of mild solution, we will need the following assumption.

Case 1. We suppose that f is Lipschitz.

Let us list the following hypotheses:

(HA): A is the infinitesimal generator of a compact semigroup $\{T(t), t \geq 0\}$ in X.

(HF1): $f: J \times X \longrightarrow X$ is continuous and there exists a constant $q_1 \in (0, \alpha)$ and a real-valued function $L_{f(t)} \in L^{(1/q_1)}(J, R^+)$ such that

$$\|f(t,x) - f(t,y)\| \leq L_{f(t)}\|x-y\|, \quad t \in J, \ x, \ y \in X. \tag{13}$$

For brevity, let us take

$$T^* = \left[\left(\frac{\alpha-1}{1-q_1}\right) t_{k+1}^{(\alpha-1/1-q_1)}\right]^{1-q_1} \left\|L_f\right\|_{L^{(1/q_1)}([t_k,t_{k+1}],R^+)}. \tag{14}$$

By using the following Duhamel formula (see [7]), we can introduce the following definition of the mild solution for system (1).

Definition 4. We say that a function $x \in \mathscr{PC}([0,b], X)$ is called a mild solution of Cauchy problem,

$$\begin{cases} \dfrac{d^{\alpha}x(t)}{dt^{\alpha}} = Ax(t) + f(t, x(t)), & \alpha \in (0,1], \ t \in J = [0,b], \ t \neq t_k, \\ x(t_k^+) = x(t_k^-) + y_k, & k = 1, 2, \ldots, n, \\ x(0) = x_0, \end{cases} \tag{15}$$

if x satisfies

$$x(t) = \begin{cases} T\left(\dfrac{t^{\alpha}}{\alpha}\right)x_0 + \displaystyle\int_0^t s^{\alpha-1} T\left(\dfrac{t^{\alpha}-s^{\alpha}}{\alpha}\right) f(s, x(s))ds, & t \in [0, t_1], \\ T\left(\dfrac{t^{\alpha}}{\alpha}\right)x_0 + T\left(\dfrac{t^{\alpha}-t_1^{\alpha}}{\alpha}\right)y_1 + \displaystyle\int_0^t s^{\alpha-1} T\left(\dfrac{t^{\alpha}-s^{\alpha}}{\alpha}\right) f(s, x(s))ds, & t \in (t_1, t_2], \\ \vdots \\ T\left(\dfrac{t^{\alpha}}{\alpha}\right)x_0 + \displaystyle\sum_{i=1}^{n} T\left(\dfrac{t^{\alpha}-t_1^{\alpha}}{\alpha}\right)y_i + \displaystyle\int_0^t s^{\alpha-1} T\left(\dfrac{t^{\alpha}-s^{\alpha}}{\alpha}\right) f(s, x(s))ds, & t \in (t_n, b]. \end{cases} \tag{16}$$

Theorem 3. *If* $(T(t))_{t>0}$ *is compact and* $(HA)-(HF1)$ *are satisfied, then Cauchy problem (1) has a unique mild solution on* J, *provided that*

$$0 < MT^* < 1. \tag{17}$$

Proof. Let $x_0 \in X$ be fixed. Define an operator $\Gamma: \mathscr{PC}(J, X) \longrightarrow \mathscr{PC}(J, X)$ by

$$(\Gamma x)(t) = \begin{cases} T\left(\dfrac{t^{\alpha}}{\alpha}\right)x_0 + \displaystyle\int_0^t s^{\alpha-1} T\left(\dfrac{t^{\alpha}-s^{\alpha}}{\alpha}\right) f(s, x(s))ds, & t \in [0, t_1], \\ T\left(\dfrac{t^{\alpha}}{\alpha}\right)x_0 + T\left(\dfrac{t^{\alpha}-t_1^{\alpha}}{\alpha}\right)y_1 + \displaystyle\int_0^t s^{\alpha-1} T\left(\dfrac{t^{\alpha}-s^{\alpha}}{\alpha}\right) f(s, x(s))ds, & t \in (t_1, t_2], \\ \vdots \\ T\left(\dfrac{t^{\alpha}}{\alpha}\right)x_0 + \displaystyle\sum_{i=1}^{n} T\left(\dfrac{t^{\alpha}-t_1^{\alpha}}{\alpha}\right)y_i + \displaystyle\int_0^t s^{\alpha-1} T\left(\dfrac{t^{\alpha}-s^{\alpha}}{\alpha}\right) f(s, x(s))ds, & t \in (t_n, b]. \end{cases} \tag{18}$$

By our assumptions and Lemma 1, Γ is well defined on $\mathscr{PC}(J, X)$.

Step 1. We prove that $\Gamma x \in \mathscr{PC}(J, X)$ for $x \in \mathscr{PC}(J, X)$.

Claim 1. For $0 \le \tau < t \le t_1$, taking into account the imposed assumptions and applying Lemma 2, we obtain

$$\begin{aligned}
&(\Gamma x)(t) - (\Gamma x)(\tau)\\
&\le \left\| T\left(\frac{t^\alpha}{\alpha}\right)x_0 - T\left(\frac{\tau^\alpha}{\alpha}\right)x_0 \right\| + \int_\tau^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \| f(s, x(s))\| \mathrm{d}s\\
&\quad + \int_0^\tau s^{\alpha-1}\left[T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{\tau^\alpha - s^\alpha}{\alpha}\right)\right] \| f(s, x(s))\| \mathrm{d}s\\
&\le \left\| T\left(\frac{t^\alpha}{\alpha}\right)x_0 - T\left(\frac{\tau^\alpha}{\alpha}\right)x_0 \right\| + \frac{M\|f\|_{\mathscr{PC}}}{\alpha}\left(t^\alpha - \tau^\alpha\right) + \frac{M\|f\|_{\mathscr{PC}}}{\alpha} t_1^\alpha\\
&\le \left\| T\left(\frac{t^\alpha}{\alpha}\right)x_0 - T\left(\frac{\tau^\alpha}{\alpha}\right)x_0 \right\| + \frac{M\|f\|_{\mathscr{PC}}}{\alpha}\left(t^\alpha - \tau^\alpha\right) + \frac{M\|f\|_{\mathscr{PC}}}{\alpha} t_1^\alpha,
\end{aligned} \tag{19}$$

where we use the inequality $t^\alpha - \tau^\alpha \le (t-\tau)^\alpha$. The first and second terms tend to zero as $t \longrightarrow \tau$. Moreover, it is obvious that the last terms tend to zero too as $t \longrightarrow \tau$. Thus, we can deduce that $\Gamma x \in \mathscr{PC}([0, t_1], X)$.

Claim 2. For $t_1 \le \tau < t < t_2$, keeping in mind our assumptions and applying Lemma 2 again, we have

$$\begin{aligned}
&\|(\Gamma x)(t) - (\Gamma x)(\tau)\|\\
&\le \left\| T\left(\frac{\tau^\alpha}{\alpha}\right)x_0 - T\left(\frac{\tau^\alpha}{\alpha}\right)x_0 \right\| + \left\| T\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right) - T\left(\frac{\tau^\alpha - t_1^\alpha}{\alpha}\right)\right\| \|y_1\|\\
&\quad + \int_\tau^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \| f(s, x(s))\| \mathrm{d}s + \int_0^\tau s^{\alpha-1}\left[T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{\tau^\alpha - s^\alpha}{\alpha}\right)\right] \| f(s, x(s))\| \mathrm{d}s\\
&\le \left\| T\left(\frac{\tau^\alpha}{\alpha}\right)x_0 - T\left(\frac{\tau^\alpha}{\alpha}\right)x_0 \right\| + \left\| T\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right) - T\left(\frac{\tau^\alpha - t_1^\alpha}{\alpha}\right)\right\| \|y_1\|\\
&\quad + \frac{M\|f\|_{\mathscr{PC}}}{\alpha}\left(t^\alpha - \tau^\alpha\right) + \frac{M\|f\|_{\mathscr{PC}}}{\alpha} t_2^\alpha.
\end{aligned} \tag{20}$$

As $t \longrightarrow \tau$, the right-hand side of the above inequality tends to zero. Thus, we can deduce that $\Gamma x \in \mathscr{PC}([t_1, t_2], X)$.

Similarly, we can also obtain that $\Gamma x \in \mathscr{PC}([t_1, t_2], X), \ldots, \Gamma x \in \mathscr{PC}([t_n, b], X)$. That is, $\Gamma x \in \mathscr{PC}(J, X)$.

Step 2. We show that Γ is the contraction on $\mathscr{PC}(J, X)$.

Claim 1. For each $t \in [0, t_1]$, it comes from our assumptions that

$$\begin{aligned}
\|(\Gamma x)(t) - (\Gamma y)(t)\| &\le \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \| f(s, x(s)) - f(s, y(s))\| \mathrm{d}s\\
&\le \sup_{t\in[0,\tau]} T\left(\frac{t^\alpha}{\alpha}\right) \int_0^t s^{\alpha-1} L_{f(s)} \|x - y\| \mathrm{d}s\\
&\le \int_0^t s^{\left(\alpha - 1/1-q_1\right)}\Bigg)^{1-q_1} \mathrm{d}s \left\|L_f\right\|_{L^{(1/q_1)}([0,t_1],R^+)} M\|x-y\|_{\mathscr{PC}}\\
&\le \left[\left(\frac{\alpha-1}{1-q_1}\right) t_1^{\left(\alpha-1/1-q_1\right)}\right]^{1-q_1} \left\|L_f\right\|_{L^{(1/q_1)}([0,t_1],R^+)} M\|x-y\|_{\mathscr{PC}}.
\end{aligned} \tag{21}$$

In general, for each $t \in (t_k, t_{k+1}]$, using our assumptions again,

$$\|(\Gamma x)(t) - (\Gamma y)(t)\| \leq M\left[\left(\frac{\alpha - 1}{1 - q_1}\right) t_{k+1}^{(\alpha - 1/1 - q_1)}\right]^{1-q_1} \|L_f\|_{L^{(1/q_1)}([t_k, t_{k+1}], R^+)} \|x - y\|, \tag{22}$$

thus

$$\|\Gamma x - \Gamma y\|_{\mathscr{PC}} \leq T^* M \|x - y\|_{\mathscr{PC}}. \tag{23}$$

Hence, condition (17) allows us to conclude in view of the Banach contraction mapping principle that Γ has a unique fixed point $x \in \mathscr{PC}(J, X)$ which is just the unique mild solution of system (1).

Case 2. f is not Lipschitz.

We make the following assumptions:

(HF2): $f: J \times X \longrightarrow X$ is continuous and maps a bounded set into a bounded set.

(HF3): the function $f(t, .): X \longrightarrow X$ is continuous, and for all $r > 0$, there exists a function $\mu_r \in L^\infty([0, b], \mathbb{R}^+)$ such that $\sup_{\|x\| \leq r} \|f(t, x)\| \leq \mu_r(t)$, for all $t \in [0, b]$.

$(C1)$: for each $x_0 \in X$, there exists a constant $r > 0$ such that

$$M\left[\|x_0\| + \sum_{i=1}^{n} \|y_k\| + \frac{b^\alpha}{\alpha} \sup_{s \in J, \phi \in Y_\Gamma} \|f(s, \phi(s))\|\right] \leq r, \tag{24}$$

where

$$Y_\Gamma = \{\phi \in \mathscr{PC}(J, X) \|\phi\| \leq r, \quad t \in J\}. \tag{25}$$

Theorem 4. *Suppose that* (HA), $(HF2)$, $(HF3)$, *and* $(C1)$ *are satisfied. Then, for every* $x_0 \in X$, *system (1) has at least a mild solution on* J.

Proof. Let $x_0 \in X$ be fixed. We introduce that map $\Gamma: \mathscr{PC}(J, X) \longrightarrow \mathscr{PC}(J, X)$ by

$$(\Gamma x)(t) = (\Gamma_1 x)(t) + (\Gamma_2 x)(t), \tag{26}$$

where

$$\begin{aligned}
(\Gamma_1 x)(t) &= T\left(\frac{t^\alpha}{\alpha}\right) x_0 + \int_0^t s^{\alpha - 1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) \mathrm{d}s, \quad t \in J \backslash \{t_1, \ldots, t_n\} \\
(\Gamma_2 x)(t) &= \begin{cases} 0, t \in [0, t_1] \\ \displaystyle\sum_{i=1}^{k} T\left((t - t_i)^\alpha\right) y_i, t \in (t_k, t_{k+1}], \quad k = 1, 2, \ldots, n. \end{cases}
\end{aligned} \tag{27}$$

For each $t \in [0, t_1]$, $x \in \Upsilon_\Gamma$,

$$\begin{aligned}
(\Gamma x)(t) &\leq \|(\Gamma_1 x)(t)\| + \|(\Gamma_2 x)(t)\| \\
&\leq M\left[\|x_0\|| + \frac{b^\alpha}{\alpha} \sup_{s \in J, \phi \in Y_\Gamma} \|f(s, \phi(s))\|\right].
\end{aligned} \tag{28}$$

For each $t \in (t_k, t_{k+1}]$, $x \in Y_\Gamma$,

$$\begin{aligned}
\|(\Gamma x)(t)\| &\leq \|(\Gamma_1 x)(t)\| + \|(\Gamma_2 x)(t)\| \\
&\leq M\left[\|x_0\| + \sum_{i=1}^{n} \|y_k\| + \frac{b^\alpha}{\alpha} \sup_{s \in J, \phi \in Y_\Gamma} \|f(s, \phi(s))\|\right].
\end{aligned} \tag{29}$$

Noting the condition $(C1)$, we see that $\Gamma: \Upsilon_\Gamma \longrightarrow \Upsilon_\Gamma$.

Step 1. We prove that Γ is a continuous mapping from Υ_Γ to Υ_Γ.

In order to derive the continuity of Γ, we only check that Γ_1 and Γ_2 are all continuous.

For this purpose, we assume that $x_n \longrightarrow x$ in Υ_Γ; it comes from the continuity of f that $s^{\alpha - 1} f(s, x_n(s)) \longrightarrow s^{\alpha - 1} f(s, x(s))$, as $n \longrightarrow \infty$.

Noting that

$$\begin{aligned}
&s^{\alpha - 1} \|f(s, x_n(s)) - f(s, x(s))\| \\
&\quad \leq s^{\alpha - 1} \sup_{s \in J, \phi \in Y_\Gamma} \|f(s, \phi(s))\|, \quad \text{for } s \in [0, t], t \in J,
\end{aligned} \tag{30}$$

by means of Lebesgue dominated convergence theorem, we obtain that $\int_0^t s^{\alpha - 1} \|f(s, x_n(s)) - f(s, x(s))\| \mathrm{d}s \longrightarrow 0$, as $n \longrightarrow \infty$. It is easy to see that, for each $t \in J$,

$$\begin{aligned}
\|(\Gamma_1 x_n)(t) - (\Gamma_1 x)(t)\| &\leq M \int_0^t s^{\alpha - 1} \| f(s, x_n(s)) \\
&\quad - f(s, x(s)) \| \mathrm{d}s \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{31}$$

Thus, Γ_1 is continuous. On the contrary, it is obvious that Γ_2 is continuous. Since Γ_1 and Γ_2 are continuous, Γ is also continuous.

Step 2. We show that Γ is a compact operator, or Γ_1 and Γ_2 are compact operators.

The compactness of Γ_2 is clear since it is a constant map.

Compactness of Γ_1:

Claim 1. We prove that $\{\Gamma_1(x)(t) | x \in B_r\}$ is relatively compact in X.

For some fixed $t \in [0, b[$, let $\varepsilon \in]0, t[$, $x \in B_r$, and define the operator Γ_1^{ε} by

$$\Gamma_1^{\varepsilon}(x)(t) = \int_0^{(t^{\alpha}-\varepsilon^{\alpha})^{(1/\alpha)}} s^{\alpha-1} T\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right) f(s, x(s)) \mathrm{d}s. \quad (32)$$

We can write Γ_1^{ε} as follows:

$$\Gamma_1^{\varepsilon}(x)(t) = T\left(\frac{\varepsilon^{\alpha}}{\alpha}\right) \int_0^{(t^{\alpha}-\varepsilon^{\alpha})^{(1/\alpha)}} s^{\alpha-1} T\left(\frac{t^{\alpha}-s^{\alpha}-\varepsilon^{\alpha}}{\alpha}\right) f(s, x(s)) \mathrm{d}s. \quad (33)$$

According to the compactness of $(T(t))_{t>0}$, the set $\{\Gamma_1^{\varepsilon}(x)(t) | x \in B_r\}$ is relatively compact in X. Using (HF3), we have

$$\left\|\Gamma_1^{\varepsilon}(x)(t) - \Gamma_1(x)(t)\right\| \leq \left|\mu_r\right|_{L^{\infty}([0,b],\mathbb{R}^+)} M\left(\frac{\varepsilon^{\alpha}}{\alpha}\right). \quad (34)$$

Therefore, $\{\Gamma_1(x)(t) | x \in B_r\}$ is relatively compact in X. It is clear that $\{\Gamma_1(x)(0) | x \in B_r\}$ is compact. Finally, $\{\Gamma_1(x)(t) | x \in B_r\}$ is relatively compact in X, for all $t \in [0, b]$.

Claim 2. We prove that $\Gamma_1(B_r)$ is equicontinuous.

Let $t_1, t_2 \in]0, b]$ such that $t_1 < t_2$. We have

$$\begin{aligned}
\Gamma_1(x)(t_2) - \Gamma_1(x)(t_1) &= \int_0^{t_1} s^{\alpha-1}\left[T\left(\frac{t_2^{\alpha}-s^{\alpha}}{\alpha}\right) - T\left(\frac{t_1^{\alpha}-s^{\alpha}}{\alpha}\right)\right] f(s, x(s)) \mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} s^{\alpha-1} T\left(\frac{t_2^{\alpha}-s^{\alpha}}{\alpha}\right) f(s, x(s)) \mathrm{d}s \\
&= \left[T\left(\frac{t_2^{\alpha}-s^{\alpha}}{\alpha}\right) - I\right)\Bigg] \int_0^{t_1} s^{\alpha-1} T\left(\frac{t_1^{\alpha}-s^{\alpha}}{\alpha}\right) f(s, x(s)) \mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} s^{\alpha-1} T\left(\frac{t_2^{\alpha}-s^{\alpha}}{\alpha}\right) f(s, x(s)) \mathrm{d}s.
\end{aligned} \quad (35)$$

Hence,

$$\left\|\Gamma_1(x)(t_2) - \Gamma_1(x)(t_1)\right\| \leq \frac{M\left|\mu_r\right|_{L^{\infty}([0,b],\mathbb{R}^+)}}{\alpha}\left[\left(t_2^{\alpha}-t_1^{\alpha}\right) + \tau^{\alpha}\left|T\left(\frac{t_2^{\alpha}-t_1^{\alpha}}{\alpha}\right) - I\right|\right]. \quad (36)$$

We conclude that $\Gamma_1(x)$, $x \in B_r$ are equicontinuous at $t \in [0, b]$. By using the Arzela–Ascoli theorem, we obtain that Γ_2 is compact. Now, Schauder's fixed point theorem implies that Γ has a fixed point, which gives rise to a mild solution.

4. Existence Results for Impulsive Nonlocal Cauchy Problems

In this section, we extend the results obtained in Section 3 to nonlocal problems for impulsive conformable fractional evolution equations. More precisely, we will prove the existence and uniqueness of the mild solutions for system (2). As we all known, the nonlocal conditions have a better effect on the solution and are more precise for physical measurements than the classical initial condition alone.

Definition 5. By a mild solution of system (2), we mean that a function $x \in \mathscr{PC}(J, X)$, which satisfies the following integral equation:

$$x(t)=\begin{cases} T\left(\dfrac{t^\alpha}{\alpha}\right)(x_0+g(x))+\displaystyle\int_0^t s^{\alpha-1}T\left(\dfrac{t^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))\mathrm{d}s, & t\in[0,t_1],\\ T\left(\dfrac{t^\alpha}{\alpha}\right)(x_0+g(x))+T\left(\dfrac{t^\alpha-t_1^\alpha}{\alpha}\right)y_1+\displaystyle\int_0^t s^{\alpha-1}T\left(\dfrac{t^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))\mathrm{d}s, & t\in(t_1,t_2],\\ \vdots \\ T\left(\dfrac{t^\alpha}{\alpha}\right)(x_0+g(x))+\displaystyle\sum_{i=1}^{n}T\left(\dfrac{t^\alpha-t_i^\alpha}{\alpha}\right)y_i+\displaystyle\int_0^t s^{\alpha-1}T\left(\dfrac{t^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))\mathrm{d}s, & t\in(t_n,b].\end{cases} \tag{37}$$

Case 1. g is Lipschitz.

To end this section, we make the following assumption:

(Hg1): g: $\mathscr{PC}(J,X)\longrightarrow X$ and there exists a constant $L_g>0$ such that

$\|g(x)-g(y)\|\le L_g\|x-y\|_{\mathscr{PC}}$, x, $y\in\mathscr{PC}(J,X)$.

Theorem 5. *Let* (HA), (HF1), *and* (Hg1) *be satisfied. Then, for every* $x_0\in X$, *system (2) has a unique mild solution on* J, *provided that*

$$0<\mu' := M\left(L_g+T^*\right)<1. \tag{38}$$

Proof. Define the operator $\mathscr{F}$: $\mathscr{PC}(J,X)\longrightarrow\mathscr{PC}(J,X)$ by

$$\mathscr{F}x(t)=\begin{cases} T\left(\dfrac{t^\alpha}{\alpha}\right)(x_0+g(x))+\displaystyle\int_0^t s^{\alpha-1}T\left(\dfrac{t^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))\mathrm{d}s, & t\in[0,t_1],\\ T\left(\dfrac{t^\alpha}{\alpha}\right)(x_0+g(x))+T\left(\dfrac{t^\alpha-t_1^\alpha}{\alpha}\right)y_1+\displaystyle\int_0^t s^{\alpha-1}T\left(\dfrac{t^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))\mathrm{d}s, & t\in(t_1,t_2],\\ \vdots \\ T\left(\dfrac{t^\alpha}{\alpha}\right)(x_0+g(x))+\displaystyle\sum_{i=1}^{n}T\left(\dfrac{t^\alpha-t_i^\alpha}{\alpha}\right)y_i+\displaystyle\int_0^t s^{\alpha-1}T\left(\dfrac{t^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))\mathrm{d}s, & t\in(t_n,b].\end{cases} \tag{39}$$

It is obvious that $\mathscr{F}$ is well defined on $\mathscr{PC}(J,X)$.

Step 1. We prove that $\mathscr{F}x\in\mathscr{PC}(J,X)$, for $x\in \mathrm{PC}(J,X)$.

For $0\le\tau<t\le t_1$, by our assumptions,

$$\begin{aligned}&\left\|T\left(\frac{t^\alpha}{\alpha}\right)g(x)-T\left(\frac{\tau^\alpha}{\alpha}\right)g(x)\right\|\\ &\quad\le\left\|T\left(\frac{t^\alpha}{\alpha}\right)-T\left(\frac{\tau^\alpha}{\alpha}\right)\right\|\left(L_g\|x\|_{\mathscr{PC}}+\|g(0)\|\right).\end{aligned} \tag{40}$$

As $t\longrightarrow\tau$, the right-hand side of the above inequality tends to zero. Recalling Step 1 in Theorem 3, we know that $\mathscr{F}x\in\mathscr{PC}(J,X)$.

Step 2. $\mathscr{F}$ is the contraction.

We only take $t\in(t_k,t_{k+1}]$, then we have

$$\|(\mathscr{F}x)(t)-(\mathscr{F}y)(t)\|\le M\left(L_g+T^*\right)\|x-y\|_{\mathscr{PC}}, \tag{41}$$

so we get

$$\|(\mathscr{F}x)-(\mathscr{F}y)\|_{\mathscr{PC}}\le\mu'\|x-y\|_{\mathscr{PC}}, \tag{42}$$

where

$$\mu' := M\left(L_g+T^*\right). \tag{43}$$

Hence, condition (38) allows us to conclude, in view of the Banach contraction mapping principle again, that $\mathscr{F}$ has a unique fixed point $x\in\mathscr{PC}(J,X)$ which is the mild solution of system (2).

Theorem 6. *Suppose that* (HA), (HF3), *and* (Hg1) *are satisfied. If* $ML_g<(1/2)$, *then system (2) has at least a mild solution on* J.

Proof. Choose

$$\sigma\ge 2M\left[\left(\|x_0\|+\|g(0)\|\right)+\sum_{n}^{i=1}\|y_i\|+\frac{b^\alpha}{\alpha}\|\mu_r\|_{L^\infty(J,R^+)}\right]. \tag{44}$$

Consider $B_\sigma=\{x\in\mathscr{PC}(J,X)|\|x\|_{\mathscr{PC}}\le\sigma\}$. Define the operators $\mathscr{N}$ on B_σ by

$$(\mathcal{N}x)(t) = (\mathcal{N}_1 x)(t) + (\mathcal{N}_2 x)(t) + (\mathcal{N}_3 x)(t), \tag{45}$$

where

$$(\mathcal{N}_1 x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)], \quad t \in J,$$
$$(\mathcal{N}_2 x)(t) = \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) \mathrm{d}s, \quad t \in J, \tag{46}$$

and $\mathcal{N}_3$ is the same as the operator Γ_2, defined in Theorem 4.

It suffices to proceed exactly the steps of the proof in Theorem 4, while replacing B_r by B_σ to obtain that $\mathcal{N}_2 + \mathcal{N}_3$ are continuous and compact. We want to use Krasnoselkii's fixed point theorem. Thus, to complete the rest proof of this theorem, it suffices to show that $\mathcal{N}_1$ is a contraction mapping and that if $x, y \in B_\sigma$, then $\mathcal{N}_1 x + (\mathcal{N}_2 + \mathcal{N}_3) y \in B_\sigma$. Indeed, for any $x \in B_\sigma$, we have

$$\|\mathcal{N}_1 x\|_{\mathcal{PC}} + \|\mathcal{N}_2 y\|_{\mathcal{PC}} + \|\mathcal{N}_3 y\|_{\mathcal{PC}}$$
$$\leq M\left(\|x_0\| + \|g(0)\| + L_g \sigma\right) + M \sum_n^{i=1} \|y_i\| + \frac{\mathrm{Mb}^\alpha}{\alpha} \|\mu_r\|_{L^\infty(J,R^+)}. \tag{47}$$

Since $\mathrm{ML}_g < (1/2)$, we can deduce that

$$\|\mathcal{N}_1 x + (\mathcal{N}_2 + \mathcal{N}_3) y\|_{\mathcal{PC}} \leq \sigma. \tag{48}$$

Next, for any $t \in (t_k, t_{k+1}]$, $x, y \in C((t_k, t_{k+1}], X)$,

$$\|\mathcal{N}_1 x - \mathcal{N}_1 y\|_{\mathcal{C}((t_k, t_{k+1}], X)} \leq \mathrm{ML}_g \|x - y\|_{\mathcal{C}((t_k, t_{k+1}], X)}. \tag{49}$$

Therefore, we can deduce that $\mathcal{N}_1$ is the contraction from $\mathrm{ML}_g < 1$. Moreover, $\mathcal{N}_2 + \mathcal{N}_3$ is compact and continuous. Hence, by the well-known Krasnoselskii's fixed point theorem, we can conclude that system (2) has at least one mild solution on J.

Case 2: g is not Lipschitz.

(Hg2): g: $\mathcal{PC}(J, X) \longrightarrow X$ and maps bounded sets into bounded sets.

($C2$): for each $x_0 \in X$, there exists a constant $r' > 0$ such that

$$M\left[\|x_0\| + \sup_{\phi \in \Upsilon_\Gamma'} \|\phi(t)\|\right] + M \sum_n^{i=1} \|y_i\|$$
$$+ \frac{\mathrm{Mb}^\alpha}{\alpha} \sup_{s \in J, \phi \in \Upsilon_\Gamma'} \|f(s, \phi(s))\| \leq r', \tag{50}$$

where

$$\Upsilon_\Gamma' = \{\phi \in \mathcal{PC}(J, X) \mid \|\phi\| \leq r', \quad \text{for } t \in J\}. \tag{51}$$

Theorem 7. *Suppose that* (HA), ($HF2$), ($C2$), *and* ($Hg2$) *are satisfied. Then, for every $x_0 \in X$, system (2) has at least a $\mathcal{PC}$-mild solution on J.*

Proof. Define an operator $\mathcal{F}$ on $\mathcal{PC}(J, X)$ by

$$(\mathcal{F}x)(t) = (\mathcal{F}_1 x)(t) + (\mathcal{F}_2 x)(t), \tag{52}$$

where

$$(\mathcal{F}_1 x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)]$$
$$+ \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) \mathrm{d}s, \quad t \in J. \tag{53}$$

and $\mathcal{F}_2$ is the same as Γ_2 defined in Theorem 4. Thus, we need to check that $\mathcal{F}_1$ is compact. Observing the expression of the $\mathcal{F}_1$, we only check that, for each $t \in J$, the set $\{T(t)[x_0 + g(x)] \mid x \in \Upsilon_\Gamma'\}$ is precompact in X since $T(t), t > 0$ is compact and (Hg2). On the other hand, the equicontinuity of $\{T(t)[x_0 + g(x)] \mid t \in J, x \in \Upsilon_\Gamma'\}$ can be shown using the same idea.

Therefore, $\mathcal{F}$ is also a compact operator. By Schauder's fixed point theorem again, $\mathcal{F}$ has a fixed point, which gives rise to a mild solution.

4.1. Example. In this section, some interesting examples are presented to illustrate the theory. Consider the following impulsive fractional differential equations with nonlocal conditions:

$$\begin{cases} \dfrac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} x(t, y) = \dfrac{\partial^2}{\partial y^2} x(t, y) + f(t, x(t, y)), & \alpha \in (0, 1], y \in (0, \pi), t \in [0, t_1) \cup (t_1, 1], \\ x(t, 0) = x(t, \pi) = 0, & \\ x(t_1^+) = x(t_1^-) + z, t_1 = \dfrac{1}{2}, \quad t \in (0, \pi), & \\ x(0, y) = x_0(y) + g(x(t, y)), \quad t \in [0, 1], y \in (0, \pi). & \end{cases} \tag{54}$$

Let $X = L_2(0,\pi)$. Define $(E1) Ax = -(\partial^2/\partial y^2 x)$, for $x \in D(A)$, where $D(A) = \{x \in X | (\partial x/\partial y), (\partial^2 x/\partial y^2) \in X$ and $x(0) = x(\pi) = 0\}$. Then, A is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ in $L_2(0,\pi)$. Moreover, $T(.)$ is also compact and $\|T(t)\| \leq e^{-t} \leq 1 = M_1$, $t \geq 0$.

Case 1. Define $f: [0,1] \times X \longrightarrow X (E2) f(t, x(t))(y) = (e^{-t}|x(t,y)|)/((\rho + e^t)(1 + |x(t,y)|))$, $t \in [0,t_1) \cup (t_1, 1], \rho >, 1, x \in X, y \in (0,\pi)$,

$(E3) g(x(t))(y) = \sum_{j=1}^{2} \lambda_j |x(s_j, y)|$, $0 < \lambda_1, \lambda_2, 0 < s_1 < s_2 < 1$, $s_1, s_2 \neq t_1, x \in \mathscr{PC}([0,1], X), y \in (0,\pi)$.

Clearly, f: $[0,1] \times X \longrightarrow X$ are continuous functions,

$\|f(t,x) - f(t,y)\| \leq L_f \|x - y\|$, with $L_f = (1/(\rho + 1)) \in L^{(1/q_1)}([0,1], R^+)$, $q_1 \in (0,\alpha)$.

It is obvious that g: $\mathscr{PC}([0,1], X) \longrightarrow X$ satisfies $\|g(x) - g(y)\| \leq L_g \|x - y\|_{\mathscr{PC}}$ with $L_g = \sum_{j=1}^{2} \lambda_j$.

$(E1) + (E2) + (E3)$ makes the assumptions in Theorem 5 satisfied. Therefore, equation (54) has a unique mild solution on $[0,1]$, provided that

$$\sum_{2}^{j=1} \lambda_j + \frac{1}{\rho + 1} \left[\left(\frac{1 - q_1}{\alpha - q_1} \right) \right]^{1 - q_1} < 1. \quad (55)$$

Case 2. Define $(E4) f(t, x(t))(y) = (e^{-t} \sin(x(t, y)))/((1 + t)(e^t + e^{-t}))$ $+ e^{-t}$, $t \in [0,t_1) \cup (t_1, 1], x \in X, y \in (0,\pi)$.

Clearly, $\|f(t,x(t))(y)\| \leq e^{-t}/(e^{-t} + e^t) + e^{-t} = m(t)$, with $m(t) \in L^{\infty}([0,1], R^+)$.

$(E1) + (E3) + (E4)$ makes the assumptions in Theorem 6 satisfied. Therefore, equations (54) has at least one mild solution on $[0,1]$, provided that $\sum_{2}^{j=1} \lambda_j < (1/2)$.

Case 3. Define $(E5) f(t, x(t)) \ (y) = c_1 |\sin(x(t,y))|$, $c_1 > 0$, $t \in [0,t_1) \cup (t_1, 1], x \in X, y \in (0,\pi)$,.

$(E6) g(x(t)) \ (y) = \int_0^1 l(s) \ \ln(1 + |x(s,y)|^{1/2}) \, ds$, $l \in L^1([0,1], R), x \in \mathscr{PC}([0,1], X), y \in (0,\pi)$.

Clearly, f and g are continuous and map a bounded set into a bounded set.

$(E1) + (E5) + (E6)$ makes the assumptions in Theorem 7 satisfied, for large $r' > 0$. Therefore, equation (54) has at least one mild solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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