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Research Article

Nonplanarity of Iterated Line Graphs

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The 1-crossing index of a graph G is the smallest integer k such that the kth iterated line graph of G has crossing number greater than 1. In this paper, we show that the 1-crossing index of a graph is either infinite or it is at most 5. Moreover, we give a full characterization of all graphs with respect to their 1-crossing index.

1. Introduction

We only consider simple, connected, and undirected graph $G = (V(G), E(G))$ in this paper. For any vertex v in G, we denote the degree of v by $d_G(v)$. A graph G is r-regular if $d_G(v) = r$ for all $v \in V(G)$. Denote by $\Delta(G)$ the maximum degree of vertices of G, i.e., $\Delta(G) = \max\{d_G(v) \mid v \in V(G)\}$. A cut-vertex (resp. cut-edge) of a connected graph G is a vertex (resp. edge) whose deletion results in a disconnected graph. A block of a graph G is a maximal connected subgraph of G that has no cut-vertex. For notations not defined here, the readers are referred to [1].

A graph is planar if it can be drawn in the plane in such a way that no two of its edges intersect. The crossing number of a graph G, denoted by $cr(G)$, is the smallest number of intersections of pairs of edges in any drawing of G in the plane. Obviously, G is planar if and only if $cr(G) = 0$. The crossing number of a graph was introduced by Turán [2], and crossing numbers of certain families of graphs were studied, see [3–8] and the references therein.

The line graph of G, denoted by $L(G)$, is defined as the graph whose vertices are the edges of G, i.e., $V(L(G)) = E(G)$, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident to a common vertex.

Given a graph G, we denote the kth iterated line graph of G by $L^k(G)$, where $L^k(G) = L(L^{k-1}(G))$. In particular, $L^0(G) = G$ is the original graph and $L^1(G) = L(G)$ is the line graph of G. Since iterated line graphs are well-suited for designing of interconnection networks, the investigation of iterated line graphs has recorded a large progress in recent years, see [9–12].

In paper [13], Sedláček obtained the following result which characterized graphs whose line graph is planar.

Lemma 1 (see [13]). *The line graph of a graph G is planar if and only if G is planar, $\Delta(G) \le 4$, and every vertex of degree 4 in G is a cut-vertex.*

Based on the above result, Ghebleh and Khatirinejad [14] began to study the planarity of iterated line graphs. They defined the line index of a graph G, denoted by $\xi(G)$, to be the smallest k such that $L^k(G)$ is nonplanar. If $L^k(G)$ is planar for all $k \ge 0$, then $\xi(G) = \infty$.

Corollary 1 in [15] showed that $cr(G) \le cr(L(G))$. It is noted that a graph has line index 0 if and only if it is nonplanar. For a planar graph G, we have

$$\xi(G) = 1 + \max\{k \mid L^k(G) \text{ is planar}\}. \quad (1)$$

Let $K_{1,3}$ be the complete bipartite graph with partite sets $\{a\}$ and $\{b_1, b_2, b_3\}$. Let t_1, t_2, and t_3 be positive integers; we define Y_{t_1,t_2,t_3} to be the graph obtained from $K_{1,3}$ by subdividing the edge ab_i to a path of length t_i ($i \in \{1, 2, 3\}$). We define T_{t_3} to be the graph obtained by adding two vertices b_4 and b_5 to $Y_{1,1,t_3}$ and joining them to b_3, see Figure 1.

Ghebleh and Khatirinejad [14] showed that the line index of a graph is either infinite or it is at most 4. Moreover, with the graphs H_i ($1 \le i \le 4$) depicted in Figure 2, a full

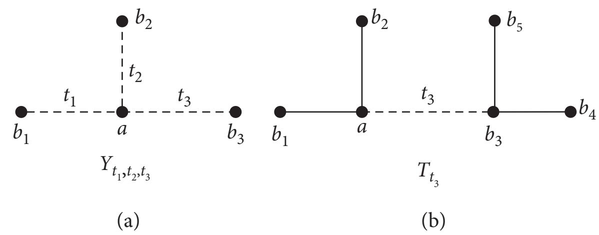

Figure 1: The graphs Y_{t_1,t_2,t_3} and T_{t_3}.

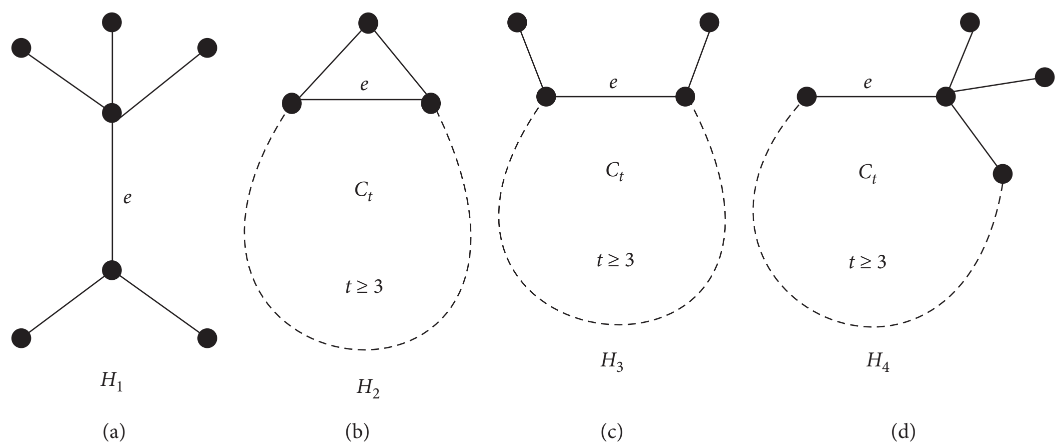

Figure 2: The graphs H_i $(1 \leq i \leq 4)$.

characterization of all graphs with respect to their line index was given.

Lemma 2 (see [14]). *Let G be a connected graph. Then,*

(1) $\xi(G) = 0$ *if and only if G is nonplanar*

(2) $\xi(G) = \infty$ *if and only if G is either a path, a cycle, or* $K_{1,3}$

(3) $\xi(G) = 1$ *if and only if G is planar and either* $\Delta(G) \geq 5$ *or G has a vertex of degree 4 which is not a cut-vertex*

(4) $\xi(G) = 2$ *if and only if* $L(G)$ *is planar and G contains one of the graphs* H_i *in Figure 2 as a subgraph*

(5) $\xi(G) = 4$ *if and only if G is one of the graphs* T_t *or* $Y_{1,1,t}$ *for some* $t \geq 2$

(6) $\xi(G) = 3$, *otherwise*

In 1971, Chartrand et al. [16] obtained a necessary and sufficient condition for graphs whose line graph is outerplanar.

Lemma 3 (see [16]). *The line graph* $L(G)$ *of a graph G is outerplanar if and only if* $\Delta(G) \leq 3$, *and if* $d_G(v) = 3$ *for a vertex v of G, then v is a cut-vertex.*

Using this result, Lin et al. [17] investigated the outerplanarity of line graphs and iterated line graphs; they defined the outerplanar index of a graph G to be the smallest k such that $L^k(G)$ is nonouterplanar. They also showed that the outerplanar index of a graph is either infinite or it is at most 3; furthermore, they completely characterized all graphs with respect to their outerplanar index.

Motivated by these results, we began to consider the problem of nonplanarity of iterated line graphs. Another motivation for this work is a result of Kulli et al. [18] and a result of Jendrol' and Klešč [15], which characterized graphs whose line graphs have crossing number one.

Lemma 4 (see [15, 18]). *The line graph of a planar graph G has crossing number one if and only if* (1) *or* (2) *holds:*

(1) $\Delta(G) = 4$ *and there is a unique noncut-vertex of degree 4*

(2) $\Delta(G) = 5$, *every vertex of degree 4 is a cut-vertex, there is a unique vertex of degree 5, and it has at most 3 incident edges in any block*

Lemma 5 (see [15]). *Let G be a nonplanar graph. Then,* $cr(L(G)) = 1$ *if and only if the following conditions hold:*

(1) $cr(G) = 1$

(2) $\Delta(G) \leq 4$, *and every vertex of degree 4 is a cut-vertex of G*

(3) *There exists a drawing of G in the plane with exactly one crossing in which each crossed edge is incident with a vertex of degree 2*

Here, we give similar results for the 1-crossing index of graphs.

Definition 1. The 1-crossing index of a graph G, denoted by $\xi_1(G)$, is the smallest integer k such that $cr(L^k(G)) > 1$. If $cr(L^k(G)) \leq 1$ for all $k \geq 0$, we define $\xi_1(G) = \infty$.

It is inferred from Definition 1 that $\xi_1(G) = 0$ if and only if $cr(G) > 1$. For a graph G whose crossing number is no more than one, we have

$$\xi_1(G) = 1 + \max\left\{k | cr\left(L^k(G)\right) \leq 1\right\}. \quad (2)$$

In this paper, we show that the 1-crossing index of a graph is either infinite or it is at most 5. Moreover, another purpose of this paper is to characterize all graphs with respect to their 1-crossing index.

This paper is organized as follows. In Section 2, we show that either $\xi_1(G) = \infty$ or $\xi_1(G) \leq 5$ for any graph G; moreover, Theorems 1–3 characterize graphs with $\xi_1(G) = \infty$, $\xi_1(G) = 0$, or $\xi_1(G) = 1$, respectively. Our main efforts are dedicated to characterize graphs whose 1-crossing index is 2, 5, or 4 in Sections 3–5, respectively. Based on these results, graphs with 1-crossing index 3 can also be given since that is the only remaining case.

2. The Bound of $\xi_1(G)$ and Graphs with $\xi_1(G) = \infty$ or $\xi_1(G) \leq 1$

The following fact is clear; however, since it is used several times throughout this paper, we state it formally.

Lemma 6. *If H is a subgraph of G, then $\xi_1(G) \leq \xi_1(H)$.*

Proof. Let $k = \xi_1(H)$. Then, $cr(L^k(H)) > 1$ and $cr(L^k(G)) \geq cr(L^k(H)) > 1$ since $H \subseteq G$, which implies that $\xi_1(G) \leq k$. □

Lemma 7. *If G is a graph with $\Delta(G) \geq 4$, then $\xi_1(G) \leq 3$.*

Proof. If $\Delta(G) \geq 4$, then G has $K_{1,4}$ as a subgraph. It is easily seen that $L^2(K_{1,4})$ is a 4-regular planar graph without cut-vertex, by Lemmas 1 and 4, $cr(L^3(K_{1,4})) > 1$. By Lemma 6, $\xi_1(G) \leq \xi_1(K_{1,4}) = 3$. □

Let P_t and C_t denote, respectively, the path and cycle on t vertices and let P_0 be the empty graph, namely, a graph with no vertices and no edges.

For $t \geq 3$, we define X_t to be the graph obtained by identifying a vertex of a cycle C_t with a vertex of P_2.

Lemma 8. *For $t \geq 3$, we have $\xi_1(X_t) = 4$.*

Proof. It is seen that $L^2(X_t)$ is a planar graph with $\Delta(L^2(X_t)) = 4$ and that $L^2(X_t)$ has a unique noncut-vertex of degree 4, thus $cr(L^3(X_t)) = 1$ by Lemma 4. Furthermore, since $\Delta(L^3(X_t)) = 5$, then we have $cr(L^4(X_t)) > 1$ by Lemma 5. By Definition 1, $\xi_1(X_t) = 4$. □

Theorem 1. *For any graph G, we have $\xi_1(G) \in \{0, 1, 2, 3, 4, 5, \infty\}$. Moreover, $\xi_1(G) = \infty$ if and only if G is a path, a cycle, or $K_{1,3}$.*

Proof. We know $L(P_t) = P_{t-1}$ for $t \geq 1$, $L(P_0) = P_0$, $L(C_t) = C_t$, and $L(K_{1,3}) = C_3$. Therefore, $\xi_1(G) = \infty$ if G is a path, a cycle, or $K_{1,3}$.

Next, assume that G is distinct from a path, a cycle, and $K_{1,3}$. First of all, we know that $\Delta(G) \geq 3$.

Case 1. $\Delta(G) \geq 4$.

Then, it has $\xi_1(G) \leq 3$ according to Lemma 7

Case 2. $\Delta(G) = 3$.

Then, G either has $Y_{1,1,2}$ or X_3 as a subgraph since G is not $K_{1,3}$. It can be verified that $cr(L^4(Y_{1,1,2})) = 1$ and $\Delta(L^4(Y_{1,1,2})) = 5$, thus $cr(L^5(Y_{1,1,2})) > 1$ due to Lemma 5. Therefore, if G has $Y_{1,1,2}$ as a subgraph, then $\xi_1(G) \leq \xi_1(Y_{1,1,2}) = 5$; if G has X_3 as a subgraph, then $\xi_1(G) \leq \xi_1(X_3) = 4$ by Lemma 8. □

In [19], the authors present a structure characterization of graphs with crossing number one, the result lead to an equivalent description of graphs with crossing number at least 2.

Lemma 9 (see [19]). *Let G be a nonplanar graph and let H be a Kuratowski subgraph of G. Then, $cr(G) \geq 2$ if and only if, for every crossing pair $\{e, f\}$ of H, e, f are separated by cycles in G or at least one of $G - e$ and $G - f$ is not planar.*

As a corollary of Lemma 9, we can characterize graphs for which $\xi_1(G) = 0$.

Theorem 2. *For any graph G, $\xi_1(G) = 0$ if and only if the following conditions hold:*

(1) G is nonplanar

(2) Let H be a Kuratowski subgraph of G. For every crossing pair $\{e, f\}$ of H, e, f are separated by cycles in G or at least one of $G - e$ and $G - f$ is not planar

We end this section by characterizing graphs for which $\xi_1(G) = 1$ in terms of Lemmas 4 and 5.

Theorem 3. *For any graph G, $\xi_1(G) = 1$ if and only if one of the following conditions holds:*

(1) G is a planar graph with $\Delta(G) = 4$, and there are at least two noncut-vertices of degree 4

(2) G is a planar graph with $\Delta(G) = 5$, and there are at least two vertices of degree 5

(3) G is a planar graph with $\Delta(G) = 5$; there is a unique vertex of degree 5 and there is at least one noncut-vertex of degree 4

(4) G is a planar graph with $\Delta(G) = 5$, every vertex of degree 4 is a cut-vertex, and there is a unique vertex of degree 5; however, it has at least 4 incident edges in one block

(5) G is a planar graph with $\Delta(G) \geq 6$

(6) $cr(G) = 1$ and $\Delta(G) \geq 5$

(7) $cr(G) = 1$, $\Delta(G) \leq 4$, *and there exists a noncut-vertex of degree 4*

(8) $cr(G) = 1$, $\Delta(G) \leq 4$, *and every vertex of degree 4 is a cut-vertex; however, there does not exist a good drawing of G satisfying condition (3) in Lemma 5*

3. Graphs with $\xi_1(G) = 2$

We use $\overline{G}$ to denote the complement of a graph G. Let G and H be two vertex-disjoint graphs; $G \vee H$ denotes the join of graphs G and H. In the proof of Theorem 2 in [20], we found the following result to be useful.

Proposition 1 (see [20]). *Let G has a vertex v of degree 4 that is not a cut-vertex. Then, G contains a subgraph either homeomorphic to $P_4 \vee K_1$ or to $K_2 \vee \overline{K_3}$.*

To characterize graphs with $\xi_1(G) = 2$, the graphs H_i $(1 \leq i \leq 4)$ depicted in Figure 2 are needed. Moreover, the graph H_5 defined below is also meaningful in the following discussions.

Definition 2. Define H_5 to be the graph satisfying the following four conditions:

(1) $cr(H_5) = 1$

(2) $\Delta(H_5) \leq 4$, and every vertex of degree 4 is a cut-vertex

(3) For any noncut-edge $f = uv \in E(H_5)$, it has $4 \leq d_{H_5}(u) + d_{H_5}(v) \leq 5$

(4) For any drawing of $L(H_5)$ with exactly one crossing, at least one of the crossed edge is not incident with vertices of degree 2

Remark 1. By conditions (1) and (2) of Definition 2, G contains a subgraph homeomorphic to $K_{3,3}$. The third condition of Definition 2 implies that there exists a drawing of H_5 with exactly one crossing in which each crossed edge is incident with a vertex of degree 2. Thus, $cr(L(H_5)) = 1$.

Lemma 10. *Let G be a graph with $cr(L(G)) = 1$ and $\xi_1(G) = 2$. Then, G is either H_5 or has a subgraph isomorphic to one of the graphs $K_{1,5}$, H_1, H_2, H_3, and H_4.*

Proof. Let G be a graph with $cr(L(G)) = 1$ and $\xi_1(G) = 2$. First of all, we have $\Delta(G) \leq 5$. Since $cr(L^2(G)) > 1$, by Lemma 5, either $L(G)$ has a vertex of degree at least 5; or $\Delta(L(G)) = 4$ and there is a noncut-vertex of degree 4 in $L(G)$; or $\Delta(L(G)) \leq 4$, and every vertex of degree 4 is a cut-vertex in $L(G)$; however, there does not exist a good drawing of $L(G)$ in the plane satisfying condition (3) of Lemma 5.

Case 1. $L(G)$ has a vertex, say e, of degree at least 5.
Recall that $\Delta(G) \leq 5$, then G contains either $K_{1,5}$, H_1, H_2, H_3, or H_4 as a subgraph.

Case 2. $\Delta(L(G)) = 4$ and there is a noncut-vertex, say e, of degree 4 in $L(G)$.

Case 2.1. e is a noncut-edge of G.
Hence, G has a cycle containing e. Since e has degree 4 in $L(G)$, G contains either H_2, H_3, or H_4 as a subgraph.

Case 2.2. e is a cut-edge of G.
Since e is a noncut-vertex of $L(G)$, then e must be a pendant edge in G. By the condition that $d_{L(G)}(e) = 4$ and that $\Delta(L(G)) = 4$; thus, G must be $K_{1,5}$.

Case 3. $\Delta(L(G)) \leq 4$, and every vertex of degree 4 in $L(G)$ is a cut-vertex; however, there does not exist a good drawing of $L(G)$ satisfying condition (3) of Lemma 5.

Claim 1. $\Delta(G) \leq 4$.

Proof of Claim 1. Suppose to contrary that $\Delta(G) = 5$. Then, $G = K_{1,5}$ since $\Delta(L(G)) \leq 4$. Thus, $L(G)$ has noncut-vertices of degree 4, a contradiction.

Claim 2. Each vertex of degree 4 in G is a cut-vertex.

Proof of Claim 2. Suppose to contrary that there is a noncut-vertex v of degree 4 in G. W.l.o.g., let e_i $(1 \leq i \leq 4)$ be four edges incident with v in G. By Proposition 1, G contains a subgraph either homeomorphic to $P_4 \vee K_1$ or to $K_2 \vee \overline{K_3}$, thus e_i $(1 \leq i \leq 4)$ is not a bridge of G, which enforces that e_i is a noncut-vertex of degree 4 in $L(G)$, a contradiction.

Claim 3. For any noncut-edge $f = uv \in E(G)$, it has $4 \leq d_G(u) + d_G(v) \leq 5$.

Proof of Claim 3. Suppose to contrary that there is a noncut-edge $f = uv$ with $d_G(u) + d_G(v) \geq 6$. Since $\Delta(L(G)) \leq 4$, then $d_G(u) + d_G(v) = 6$. Thus, f is a noncut-vertex of degree 4 in $L(G)$, a contradiction.

Since $cr(L(G)) = 1$, it follows from Claims 1 and 2 and Lemma 1 that $cr(G) = 1$. Moreover, by the assumption that $\xi_1(G) = 2$, G satisfies all of the conditions in Definition 1. Therefore, G is H_5. □

Lemma 11. *Let G be a graph with $cr(L(G)) = 0$ and $\xi_1(G) = 2$. Then, G satisfies one of the following conditions:*

(1) G has an edge that joins two vertices of degree 4

(2) G has at least two edges, each of whose degree sum of its two end vertices is 7

(3) There is only one edge in G such that the degree sum of its two end vertices is 7, and G has a subgraph isomorphic to H_2, H_3, or H_4

(4) G has at least two noncut-edges whose degree sum of its two end vertices is 6

Proof. Let G be a graph with $cr(L(G)) = 0$ and $\xi_1(G) = 2$. Firstly, Lemma 1 tells that G is a planar graph with $\Delta(G) \leq 4$. Secondly, $cr(L^2(G)) > 1$ because $\xi_1(G) = 2$. By Lemma 4, the following three cases are discussed by considering the maximum degree of $L(G)$.

Case 1. $\Delta(L(G)) \geq 6$.

We have $\Delta(L(G)) = 6$ since $\Delta(G) \leq 4$. W.l.o.g., let f be a vertex of degree 6 in $L(G)$. Thus, each end vertex of f has degree 4 in G and G is the graph depicted in (1).

Case 2. $\Delta(L(G)) = 5$.

Case 2.1. There are at least two vertices of degree 5 in $L(G)$.

W.l.o.g., let $f_1 = u_1v_1$ and $f_2 = u_2v_2$ be two vertices of degree 5 in $L(G)$. Therefore, $d_G(u_i) + d_G(v_i) = 7$, for $i \in \{1, 2\}$ and G is the graph depicted in (2).

Case 2.2. There is a unique vertex of degree 5 in $L(G)$, and $L(G)$ has a noncut-vertex, say e, of degree 4. Then, G has a subgraph isomorphic to H_2, H_3, or H_4. Thus, G is the graph depicted in (3).

Case 2.3. Every vertex of degree 4 in $L(G)$ is a cut-vertex; there is a unique vertex, say e, of degree 5 in $L(G)$; however, it has at least 4 incident edges in one block.

W.l.o.g., assume that $e = uv$. Since $d_{L(G)}(e) = 5$ and $\Delta(G) \leq 4$, we have that $\{d_G(u), d_G(v)\} = \{3, 4\}$. Because e has at least 4 incident edges in a block of $L(G)$, we conclude that e is not a bridge of G. Hence, G has a cycle containing e, which implies that G contains a subgraph isomorphic to H_2, H_3, or H_4. Thus, G is the graph depicted in (3).

Case 3. $\Delta(L(G)) = 4$.

By Lemmas 1 and 4, there are at least two noncut-vertices of degree 4 in $L(G)$ since $cr(L^2(G)) > 1$. W.l.o.g., let $f_1 = u_1v_1$ and $f_2 = u_2v_2$ be two noncut-vertices of degree 4 in $L(G)$. Therefore, for $i \in \{1, 2\}$, $d_G(u_i) + d_G(v_i) = 6$ and f_i is a noncut-edge in G. Thus, G is the graph depicted in (4). □

Theorem 4. *For any graph G, $\xi_1(G) = 2$ if and only if one of the conditions holds:*

(1) *$cr(L(G)) = 1$, and G has a subgraph either isomorphic to $K_{1,5}$ or to H_i $(1 \leq i \leq 4)$*

(2) *G is H_5*

(3) *$L(G)$ is planar, and G has an edge that joins two vertices of degree 4*

(4) *$L(G)$ is planar, and G has at least two edges, each of whose degree sum of its two end vertices is 7*

(5) *$L(G)$ is planar, and there is only one edge in G such that the degree sum of its two end vertices is 7; moreover, G has a subgraph isomorphic to H_2, H_3, or H_4*

(6) *$L(G)$ is planar, and G has at least two noncut-edges whose degree sum of its two end vertices is 6*

Proof. Let G be a graph with $\xi_1(G) = 2$. By the definition, we have $cr(L(G)) \leq 1$ and $cr(L^2(G)) > 1$. According to Lemmas 10 and 11, G is one of the graphs depicted in (1)–(6).

For the converse, we shall show that $\xi_1(G) = 2$ if G is one of the graphs depicted in (1)–(6).

Case 1. G is the graph depicted in (1).

Let H be $K_{1,5}$ or one of the graphs H_i $(1 \leq i \leq 4)$. If $H = K_{1,5}$, then $L(H)$ either has a vertex of degree at least 5 or has a noncut-vertex of degree 4; if $H = H_1$, then $L(H)$ has a vertex of degree at least 5; finally, if $H \in \{H_2, H_3, H_4\}$, then $L(H)$ has a vertex of degree 4 which is a noncut-vertex. Lemma 5 yields that $cr(L^2(H)) > 1$, thus $cr(L^2(G)) \geq cr(L^2(H)) > 1$. Since $cr(L(G)) = 1$, it follows that $\xi_1(G) = 2$.

Case 2. G is the graph depicted in (2).

By Remark 1 and by the forth condition of Definition 2, we have $cr(L(G)) = 1$ and $cr(L^2(G)) > 1$. Therefore, $\xi_1(G) = 2$.

Case 3. G is the graph depicted in (3).

W.l.o.g., let f be an edge of G that joins two vertices of degree 4, then $d_{L(G)}(f) = 6$. By Lemmas 1 and 4, $cr(L^2(G)) > 1$. Since $L(G)$ is planar, it follows that $\xi_1(G) = 2$.

Case 4. G is the graph depicted in (4).

W.l.o.g., let f_1 and f_2 be two edges whose degree sum of its two end vertices is 7 in G, then both f_1 and f_2 are vertices of degree 5 in $L(G)$. By Lemma 1, $L^2(G)$ is a nonplanar graph. Furthermore, we have $cr(L^2(G)) > 1$ by Lemma 4. Therefore, $\xi_1(G) = 2$.

Case 5. G is the graph depicted in (5).

W.l.o.g., let f be the edge whose degree sum of its two end vertices is 7 in G, then $d_{L(G)}(f) = 5$. Moreover, G has a subgraph isomorphic to one of H_2, H_3, or H_4. Note that there is a noncut-edge e in G, see Figure 2.

Now, we consider the degree of e in $L(G)$. If $d_{L(G)}(e) = 4$, then e is a noncut-vertex of degree 4 in $L(G)$ and thus $cr(L^2(G)) > 1$ by Lemma 4. If $d_{L(G)}(e) = 5$, that means $e = f$, then it can be verified that, in $L(G)$, e has at least 4 incident edges in one block, thus $cr(L^2(G)) > 1$ by Lemma 4. Therefore, $\xi_1(G) = 2$.

Case 6. G is the graph depicted in (6).

W.l.o.g., let f_1 and f_2 be two noncut-edges whose degree sum of its two end vertices is 6 in G . Then, both f_1 and f_2 are noncut-vertices of degree 4 in $L(G)$. By Lemma 1, $L^2(G)$ is a nonplanar graph. Furthermore, by Lemma 4, we have $cr(L^2(G)) > 1$ no matter $\Delta(L(G)) = 4$ or $\Delta(L(G)) = 5$. Therefore, $\xi_1(G) = 2$. □

4. Graphs with $\xi_1(G) = 5$

Remind that the graph Y_{t_1,t_2,t_3} defined in Section 1 is obtained by subdividing each edge of $K_{1,3}$ to a path of length t_i $(i \in \{1, 2, 3\})$, respectively.

Lemma 12. *When $t_3 \geq 3$, we have $\xi_1(Y_{1,1,t_3}) = 4$. When $t_2 \geq 2$ and $t_3 \geq 2$, we have $\xi_1(Y_{1,t_2,t_3}) = 4$.*

Proof. Firstly, we prove that $\xi_1(Y_{1,1,t_3}) = 4$ when $t_3 \geq 3$. It is not difficult to see that $L^3(Y_{1,1,t_3})$ is a planar graph with $\Delta(L^3(Y_{1,1,t_3})) = 4$; moreover, there are at least two noncut-vertices of degree 4 in $L^3(Y_{1,1,t_3})$; therefore, $cr(L^4(Y_{1,1,t_3})) > 1$ according to Lemma 4. By Definition 1, $\xi_1(Y_{1,1,t_3}) = 4$ when $t_3 \geq 3$.

Now, we prove that $\xi_1(Y_{1,t_2,t_3}) = 4$ when $t_2 \geq 2$ and $t_3 \geq 2$. One can see that $L^2(Y_{1,t_2,t_3})$ is a planar graph with $\Delta(L^2(Y_{1,t_2,t_3})) = 4$; moreover, $L^2(Y_{1,t_2,t_3})$ has a unique noncut-vertex of degree 4; therefore, $cr(L^3(Y_{1,t_2,t_3})) = 1$ by Lemma 4. Furthermore, $\Delta(L^3(Y_{1,t_2,t_3})) = 5$, thus $cr(L^4(Y_{1,t_2,t_3})) > 1$ by Lemma 5. By the definition, $\xi_1(Y_{1,t_2,t_3}) = 4$. □

Theorem 5. *For any graph G, $\xi_1(G) = 5$ if and only if $G = Y_{1,1,2}$.*

Proof. Assuming that $G = Y_{1,1,2}$, it has been checked in Theorem 1 that $cr(L^4(G)) = 1$ and that $cr(L^5(G)) > 1$, thus $\xi_1(G) = 5$.

For the converse, let G be a graph with $\xi_1(G) = 5$. It follows from Lemma 2 and Theorem 1 that $\Delta(G) = 3$. Assuming that G is not a tree, by Lemmas 6 and 8, we have $\xi_1(G) \leq \xi_1(X_t) = 4$ since G contains X_t as a subgraph. This contradiction enforces that G is a tree. Moreover, Lemmas 6 and 12 indicate that G cannot have $Y_{1,2,2}$ nor $Y_{1,1,3}$ as a subgraph. All of the analyses assert that $G = Y_{1,1,2}$. □

5. Graphs with $\xi_1(G) = 4$

To characterize graphs with $\xi_1(G) = 4$, the following graphs are introduced. Let J_1 be the graph obtained by identifying a vertex of C_t $(t \geq 3)$ with a pendent vertex of P_3. Define J_2 to be the simple graph obtained from C_t by adding a path of arbitrary length to connect two vertices of C_t. Define J_3 to be the graph obtained from C_t by adding two pendant edges, each to a vertex of C_t. These families of graphs are shown in Figure 3.

Let J_4 be a tree with diameter 3 such that two nonleaf vertices a_1 and a_2 both have degree 3. Denote J_5 to be the graph obtained from J_4 firstly by adding two new vertices v_1 and v_2 to J_4 and joining v_i to b_i $(i \in \{1,2\})$ and then by subdividing the edge a_1a_2 to a path of length t $(t \geq 2)$. These families of graphs are depicted in Figure 4.

Lemma 13. *For $1 \leq i \leq 5$, we have $\xi_1(J_i) = 3$.*

Proof. When $i = 1$, it can be seen that $L^2(J_1)$ is a planar graph with at least two noncut-vertices of degree 4, then $cr(L^3(J_1)) > 1$ by Lemma 4. By the definition, $\xi_1(J_1) = 3$.

When $i = 2$, observe that, if the path added to C_t is of length greater than one, then $L^2(J_2)$ is a planar graph with at least two noncut-vertices of degree 4; if the path added to C_t is of length one, then $cr(L^2(J_2)) = 1$ and $\Delta(L^2(J_2)) = 5$. According to Lemmas 5 and 4, we have $cr(L^3(J_2)) > 1$. Thus, $\xi_1(J_2) = 3$.

Using the analogous arguments, we have $\xi_1(J_i) = 3$ for $3 \leq i \leq 5$. □

Define W_{t_1,t_2} to be the graph obtained from J_4 by subdividing the edges a_1a_2 and a_2b_2 to a path of length t_1 and t_2, respectively. Define Z_{t_1,t_2} to be the graph obtained by adding two new vertices v_1 and v_2 to W_{t_1,t_2} and joining them to b_2. These two families of graphs are depicted in Figure 5.

Lemma 14. *When $t_1 \geq 2$ and $t_2 \geq 1$, we have $\xi_1(W_{t_1,t_2}) = 4$. When $t_1 \geq 2$ and $t_2 \geq 2$, we have $\xi_1(Z_{t_1,t_2}) = 4$.*

Proof. Firstly, we show that $\xi_1(W_{t_1,t_2}) = 4$ when $t_1 \geq 2$ and $t_2 \geq 1$. It is seen that, when $t_2 = 1$, $L^3(W_{t_1,t_2})$ is a planar graph with two noncut-vertices of degree 4; when $t_2 \geq 2$, we have $cr(L^3(W_{t_1,t_2})) = 1$ and $\Delta(L^3(W_{t_1,t_2})) = 5$. This observation together with Lemmas 5 and 4 yields $cr(L^4(W_{t_1,t_2})) > 1$. By the definition, $\xi_1(W_{t_1,t_2}) = 4$.

Now, we prove that $\xi_1(Z_{t_1,t_2}) = 4$ when $t_1 \geq 2$ and $t_2 \geq 2$. Note that $L^2(Z_{t_1,t_2})$ is a planar graph with $\Delta(L^2(Z_{t_1,t_2})) = 4$ and that $L^2(Z_{t_1,t_2})$ has a unique noncut-vertex of degree 4, thus $cr(L^3(Z_{t_1,t_2})) = 1$ by Lemma 4. Furthermore, since $\Delta(L^3(Z_{t_1,t_2})) = 5$, it follows from Lemma 5 that $cr(L^4(Z_{t_1,t_2})) > 1$. Consequently, $\xi_1(Z_{t_1,t_2}) = 4$. □

Lemma 15. $\xi_1(Y_{2,2,2}) = 3$.

Proof. Observe that $L^2(Y_{2,2,2})$ is a planar graph with maximum degree 4; moreover, $L^2(Y_{2,2,2})$ has at least two noncut-vertices of degree 4. Thus, by Lemma 4, $cr(L^3(Y_{2,2,2})) > 1$. By the definition, $\xi_1(Y_{2,2,2}) = 3$. □

Theorem 6. *For any graph G, $\xi_1(G) = 4$ if and only if G is one of the graphs X_t $(t \geq 3)$, Z_{t_1,t_2} $(t_1 \geq 2, t_2 \geq 2)$, W_{t_1,t_2} $(t_1 \geq 2, t_2 \geq 1)$, $Y_{1,1,t_3}$ $(t_3 \geq 3)$, or Y_{1,t_2,t_3} $(t_2 \geq 2, t_3 \geq 2)$.*

Proof. By Lemmas 8, 12, and 14, if G is one of the graphs X_t $(t \geq 3)$, Z_{t_1,t_2} $(t_1 \geq 2, t_2 \geq 2)$, W_{t_1,t_2} $(t_1 \geq 2, t_2 \geq 1)$, $Y_{1,1,t_3}$ $(t_3 \geq 3)$, or Y_{1,t_2,t_3} $(t_2 \geq 2, t_3 \geq 2)$, we obtain that $\xi_1(G) = 4$.

For the converse, let G be a graph with $\xi_1(G) = 4$. By Lemma 7 and Theorem 1, $\Delta(G) = 3$.

Claim 4. Each edge of G is contained in at most one cycle; moreover, there is at most one cycle in G.

Proof of Claim 4. If not, G contains a subgraph either isomorphic to J_1 or to J_2. By Lemmas 6 and 13, $\xi_1(G) \leq 3$, a contradiction. □

By Claim 4, we only need to consider the following two cases:

Case 1. G is unicyclic.

Let C_t $(t \geq 3)$ be the unique cycle in G. Since $\Delta(G) = 3$, then at least one vertex of C_t has degree 3 in G. If there are at least two such vertices, then G contains J_3 as a subgraph; hence, $\xi_1(G) \leq \xi_1(J_3) = 3$ by Lemmas 6 and 13. This contradiction implies that, in C_t, there is exactly one vertex of degree 3 in G.

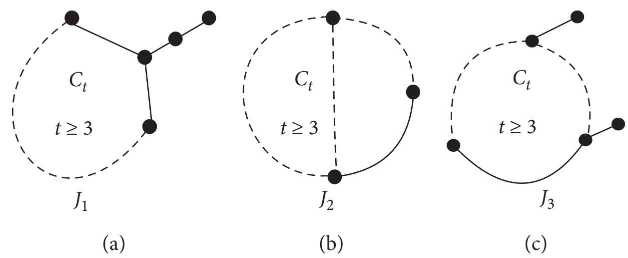

Figure 3: The graphs J_1, J_2, and J_3.

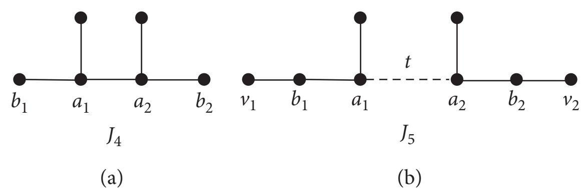

Figure 4: The graphs J_4 and J_5.

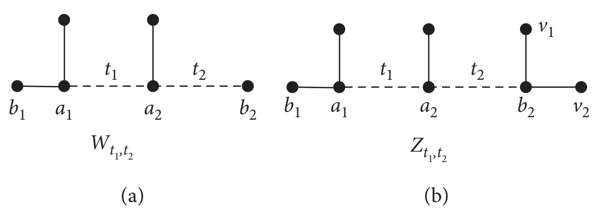

Figure 5: The graphs W_{t_1,t_2} and Z_{t_1,t_2}.

As we mentioned above, G cannot contain J_1 as a subgraph. By Lemma 8, G is the graph X_t $(t \geq 3)$.

Case 2. G is a tree.

By Lemmas 6, 13, and 15, we claim the following.

Claim 5. G cannot contain a subgraph isomorphic to J_4, J_5, or to $Y_{2,2,2}$.

Claim 6. In G, there are at most three vertices of degree 3.

Proof of Claim 6. Suppose to contrary that there are at least four vertices of degree 3 in G. By Claim 5, these 3-degree vertices cannot be adjacent to each other; otherwise, G contains J_4 as a subgraph. Since G cannot contain $Y_{2,2,2}$ as a subgraph; therefore, the tree G with at least four vertices of degree 3 will contain J_5 as a subgraph, a contradiction with Claim 5. □

Assuming that there are three vertices of degree 3 in G, by Claim 5 and by Lemma 14, G is the graph Z_{t_1,t_2} $(t_1 \geq 2, t_2 \geq 2)$.

Assume that there are two vertices of degree 3 in G, and then G is W_{t_1,t_2} $(t_1 \geq 2, t_2 \geq 1)$ by Lemma 14.

Finally, assume that there is only one vertex of degree 3 in G; then, by Claim 5 and Lemma 12, G is either $Y_{1,1,t_3}$ $(t_3 \geq 3)$ or Y_{1,t_2,t_3} $(t_2 \geq 2, t_3 \geq 2)$. The proof is completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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