Research Article

Characterization of Graphs Associated with the Ideal of Numerical Semigroups

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Received 29 June 2020; Accepted 30 July 2020; Published 7 September 2020

Academic Editor: Marco Fontana

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Let I be an ideal of a numerical semigroup Λ . We define an undirected graph $G_I(\Lambda)$ with vertex set $\{v_i: i \in (\Lambda \setminus I)^* = (\Lambda \setminus I) - \{0\}\}$ and edge set $\{v_i v_j \iff i + j \in I\}$. The aim of this article is to discuss the connectedness, girth, completeness, and some other related properties of the graph $G_I(\Lambda)$.

1. Introduction and Preliminaries

In mathematics, graph theory plays an important role in understanding the relationship between pair of objects through graphs. Today, graph theory has a number of applications in the field of engineering, computer science, and other sciences (see [1–3]). The idea of linking graph with algebraic structure gave rise to new research in the field of mathematics. This idea explores the interaction between structures of algebraic objects and graphs by using properties of graphs and algebraic structures. In the past, many researchers associated graphs with different algebraic structures (see [4–11]).

A graph $G(V(G), E(G))$ is an ordered pair with the vertex set $V(G)$ and the edge set $E(G)$. The order and the size of the graph G are the cardinality of its vertex set and edge, respectively. A graph G is connected if for any two vertices $u, v \in V(G)$, there is a path connecting u and v . The distance between two vertices $u, v \in V(G)$ is the length of the shortest path between them and is denoted by $d(u, v)$. The length of the longest path in G is the diameter of G , denoted by $\text{diam}(G)$. Length of a shortest cycle in the graph is referred

to as the girth of the graph. A graph is complete if for any two vertices, there is an edge between them. A subgraph H of a graph G is a clique if H is a complete graph, and the order of maximal clique is called the clique number of G , denoted by $cl(G)$. A graph is regular if all of its vertices have the same degree. For more undefined terminologies related to graph theory, see [12–14].

Numerical semigroup theory is the study of set of positive integer solutions of diophantine equations (see [15–18]), and this is very useful in the study of algebraic geometry codes (see [19–21]). Let us recall some definitions from the numerical semigroup theory, which we used in the later sections.

A subset $\Lambda \subseteq \mathbb{N}$ of nonnegative integers is called submonoid if the following holds:

- (1) $0 \in \Lambda$
- (2) If $x, y \in \Lambda$, then $x + y \in \Lambda$

A submonoid Λ is called numerical semigroup if $\mathbb{N} \setminus \Lambda$ is finite. The least positive integer in Λ , denoted by m_Λ , is known as the multiplicity of the numerical semigroup. The

elements of $\mathbb{N} \setminus \Lambda$ are called the gaps of Λ , and the largest of these gaps is known as the Frobenius number, denoted by F_Λ . It is well known that every numerical semigroup is finitely generated; that is, there exist $a_1, \dots, a_t$ such that $\Lambda = \langle a_1, \dots, a_t \rangle = \{n_1 a_1 + \dots + n_t a_t : n_1, \dots, n_t \in \mathbb{N}\}$. Moreover, every numerical semigroup has a unique minimal system of generators. The cardinality of the minimal system of generators is called the embedding dimension of numerical semigroup Λ , denoted by e_Λ . A subset I of numerical semigroup Λ is ideal (integral ideal) of Λ if $I + \Lambda \subseteq I$ (that is, $\forall x \in I$ and $s \in \Lambda$, the element $x + s$ is in I). An ideal I is an irreducible ideal if it cannot be written as intersection of two ideals which properly contain it.

A connection of graphs with numerical semigroups can be found in [22]. In this paper, we investigated the interplay between the structure of numerical semigroups and graph theoretic properties of specific graphs gained from numerical semigroups. The sketch of this paper is as follows.

In Section 2, we briefly describe the concept of connectedness, diameter, and girth of $G_I(\Lambda)$. In Section 3, we present a necessary and sufficient condition for a graph associated with an ideal of a numerical semigroup to be complete. In Section 4, we discuss the concept of clique number and cut point of $G_I(\Lambda)$.

2. Connectedness, Diameter, and Girth of $G_I(\Lambda)$

We start this section by showing that $G_I(\Lambda)$ is always a connected graph. Also, we show that the diameter of $G_I(\Lambda)$ is equal or less than 2 and girth of $G_I(\Lambda)$ is 3.

Proposition 1. *Let Λ be a numerical semigroup and I be an ideal of Λ . Then, $G_I(\Lambda)$ is a connected graph of finite order with at least one vertex of degree $|(\Lambda \setminus I)^*| - 1$.*

Proof. Choose $F_\Lambda \in \Lambda$, the Frobenius number, and $l \in I$, the least element. Then, note that $(F_\Lambda + l) + m \in I$ for all $m \geq 1$. This gives that $\Lambda \setminus I$ is always finite, and therefore, $(\Lambda \setminus I)^*$ is finite. We may assume that n is the largest element of Λ not in I as $(\Lambda \setminus I)^*$ is finite. This gives that $i + n \in I, \forall i \in (\Lambda \setminus I)^*$. Therefore, $G_I(\Lambda)$ is a connected graph of finite order with degree of v_n and is equal to $|(\Lambda \setminus I)^*| - 1$.

Example 1. Let $\Lambda = \{0, 3, 6, 7, 9, 10, 11, 12, 13, 14, \dots\}$ be a numerical semigroup and $I = \{11, 12, 13, \dots\}$ be an ideal of Λ . Then, $(\Lambda \setminus I)^* = \{3, 6, 7, 9, 10\}$ and the graph $G_I(\Lambda)$ is shown in Figure 1.

Proposition 2. *Let $G_I(\Lambda)$ be a graph associated with an ideal I of a numerical semigroup Λ . Then, $\text{diam}(G_I(\Lambda)) \leq 2$. Furthermore, if $G_I(\Lambda)$ contains a cycle, then $\text{gr}(G_I(\Lambda)) \leq 5$.*

Proof. Let $n \in (\Lambda \setminus I)^*$ be the largest element of Λ not in I ; then, v_n has an edge with every vertex v_k , where $k \in (\Lambda \setminus I)^*$ and $k \neq n$. Therefore, $d(v_n, v_k) = 1$. Now, for any two vertices v_i and v_j , $i \neq j$, we have

$$d(v_i, v_j) \leq d(v_i, v_n) + d(v_n, v_j) \leq 2. \quad (1)$$

FIGURE 1: Graph $G_I(\Lambda)$, where $\Lambda = \langle 3, 5, 7 \rangle$ and $I = \{11, 12, 13, \dots\}$.

This implies that

$$\text{diam}(G_I(\Lambda)) \leq 2. \quad (2)$$

Moreover, if any undirected graph G has a cycle, then $\text{gr}(G) \leq 2 \text{diam}(G) + 1$ (see [13], Proposition 1.3.2). Therefore, $\text{gr}(G_I(\Lambda)) \leq 5$.

Lemma 1. *Let $G_I(\Lambda)$ be a graph associated with an ideal I of a numerical semigroup Λ . If order of $G_I(\Lambda)$ is 4, then $G_I(\Lambda)$ must contain a cycle of length 3.*

Proof. We may assume that $(\Lambda \setminus I)^* = \{x_1, x_2, x_3, x_4\}$. If $G_I(\Lambda)$ is complete, then trivially it contains a cycle of length 3; otherwise, there is at least one pair of nonadjacent vertices. We have the following cases:

Case_1: if v_{x_1} and v_{x_2} are nonadjacent, then either $x_1 + x_2 = x_3$ or $x_1 + x_2 = x_4$. If $x_1 + x_2 = x_3$, then there are two more possibilities, either $x_1 + x_3 = x_4$ or $x_2 + x_3 = x_4$. So, $x_1 + x_2 = x_3$ and $x_1 + x_3 = x_4$ give $x_2 + x_3, x_2 + x_4$, and $x_3 + x_4 \in I$. This implies that there is a cycle $v_{x_2} - v_{x_3} - v_{x_4} - v_{x_2}$ of length 3. Also, $x_1 + x_2 = x_3$ and $x_2 + x_3 = x_4$ give $x_1 + x_3, x_1 + x_4$, and $x_3 + x_4 \in I$. This implies there is a cycle $v_{x_1} - v_{x_3} - v_{x_4} - v_{x_1}$ of length 3. Now, if $x_1 + x_2 = x_4$, then clearly $x_1 + x_3, x_1 + x_4$, and $x_3 + x_4 \in I$, and therefore, there is a cycle $v_{x_1} - v_{x_3} - v_{x_4} - v_{x_1}$ of length 3.

Case_2: if x_1 and x_3 are nonadjacent, then $x_2 + x_3, x_2 + x_4$, and $x_3 + x_4 \in I$, and therefore, there is a cycle $v_{x_2} - v_{x_3} - v_{x_4} - v_{x_2}$ of length 3.

Case_3: if x_2 and x_3 are nonadjacent, then $x_1 + x_3, x_1 + x_4$, and $x_3 + x_4 \in I$, and therefore, there is a cycle $v_{x_1} - v_{x_3} - v_{x_4} - v_{x_1}$ of length 3.

Proposition 3. *Let $G_I(\Lambda)$ be a graph associated with an ideal I of a numerical semigroup Λ . If order of $G_I(\Lambda) \geq 4$, then $G_I(\Lambda)$ must contain a cycle of length 3.*

Proof. This follows from Lemma 1.

Corollary 1. *Let $G_I(\Lambda)$ be a graph associated with an ideal I of a numerical semigroup Λ . If order of $G_I(\Lambda) \geq 4$, then $G_I(\Lambda)$ is not a bipartite graph.*

Corollary 2. Let $G_I(\Lambda)$ be a graph associated with an ideal I of a numerical semigroup Λ . If order of $G_I(\Lambda) \geq 4$, then $gr(G_I(\Lambda)) = 3$.

3. Completeness of $G_I(\Lambda)$

In this section, we provide a sufficient and necessary condition for $G_I(\Lambda)$ to be complete. Moreover, for a given numerical semigroup Λ , we computed the least number of ideals I for which graph $G_I(\Lambda)$ is complete.

Lemma 2. Any ideal of a numerical semigroup Λ is an intersection of irreducible ideals. Moreover, every irreducible ideal of numerical semigroup Λ is of the form $\Lambda \setminus B(x)$, where $B(x) = \{y \in \Lambda: x - y \in \Lambda\}$, for some $x \in \Lambda$.

Proof. For a proof, see [23].

Proposition 4. Let I be an irreducible ideal of a numerical semigroup Λ . If order of $G_I(\Lambda) \geq 3$, then $G_I(\Lambda)$ cannot be a complete graph.

Proof. Since I is an irreducible ideal, order of $G_I(\Lambda) = |B(x)^*|$ for some $x \in \Lambda$. We may assume that $B(x)^* = \{x_1, \dots, x_n, x\}$, where $n \geq 2$. Then, there must exist $x_i, x_j \in B(x)^*$ for $i \neq j$ such that $x - x_i = x_j$. This gives $x_i + x_j = x$, so there is no edge between v_{x_i} and v_{x_j} , and therefore, $G_I(\Lambda)$ is not a complete graph.

Example 2. Let $\Lambda = \langle 3, 5, 7 \rangle$ be a numerical semigroup. Choose $10 \in \Lambda$; then, $B^*(10) = \{3, 5, 7, 10\}$, and therefore, $I = \Lambda \setminus B^*(10) = \{6, 8, 9, 11, \dots\}$ is an irreducible ideal. As $|B^*(10)| = 4$, $G_I(\Lambda)$ is not a complete graph (see Figure 2).

Corollary 3. Let $G_I(\Lambda)$ is a graph of order ≥ 3 , associated with an irreducible ideal I of a numerical semigroup Λ . Then, $G_I(\Lambda)$ cannot be a regular graph.

Proof. This follows from Proposition 1 and Proposition 4.

Lemma 3. Let $\Lambda = \langle A \rangle$, where $A = \{a_1, \dots, a_r\}$ is the minimal system of generators. Then, for some $x \in \Lambda$, $B(x) = \{0, a_i, 2a_i, \dots, x\}$ if and only if x is the multiple of exactly one element $a_i \in A$.

Proof. Suppose that $B(x) = \{0, a_i, 2a_i, \dots, x\}$ and x is not a multiple of exactly one element of A . Then,

Case_1: if x is a multiple of an element of $a_j \in A$ other than a_i , then we can assume $x = na_j$. This gives $x - a_j = (n-1)a_j \in \Lambda$ and therefore $a_j \in B(x)$, which is not possible.

Case_2: if x is a linear combination of at least two elements $a_i, a_j \in A$, then $x = sa_i + ra_j$ for some positive integers s and r . Note that $x - a_j = sa_i + (r-1)a_j \in \Lambda$. This gives $a_j \in B(x)$, which is not possible.

The converse of this statement is straightforward.

FIGURE 2: Graph $G_I(\Lambda)$, where $\Lambda = \langle 3, 5, 7 \rangle$ and I is an irreducible ideal with $x = 10$.

Theorem 1. Let I be an ideal of a numerical semigroup Λ such that $I = \cap_{1 \leq j \leq n} I_j$, where I_j is an irreducible ideal and $n \geq 2$. Then, $G_I(\Lambda)$ is complete if and only if order of $G_{I_j}(\Lambda) \leq 2$ for each $1 \leq j \leq n$.

Proof. Let $\Lambda = \langle A \rangle$, where $A = \{a_1, \dots, a_r\}$ is the minimal system of generators. As for each $1 \leq j \leq n$, I_j is an irreducible ideal; therefore, we can assume that $I_j = \Lambda \setminus B(x_j)$ for some $x_j \in \Lambda$. This gives that the set of vertices of $G_I(\Lambda)$ is $\{v_i: i \in B(x_1)^* \cup \dots \cup B(x_n)^*\}$.

To prove the implication " $\Rightarrow$ ", suppose order of $G_{I_m}(\Lambda) \geq 3$ for some $1 \leq m \leq n$. Then, from Proposition 4, it follows that $G_{I_m}(\Lambda)$ is not a complete graph, so there must exist $x_i, x_j \in B(x_m)^*$ such that $x_i + x_j \in B(x_m)^*$; i.e., there is no edge between v_{x_i} and v_{x_j} . This gives $x_i, x_j \in B(x_1)^* \cup \dots \cup B(x_n)^*$ such that $x_i + x_j \in B(x_1)^* \cup \dots \cup B(x_n)^*$, and therefore, $G_I(\Lambda)$ is not complete.

Now, to prove the implication " $\Leftarrow$ ", we suppose on contrary that $G_I(\Lambda)$ is not a complete graph. Therefore, there exist $x, y, z \in B(x_1)^* \cup \dots \cup B(x_n)^*$ such that $z = x + y$. However, the order of $G_{I_j}(\Lambda) \leq 2$ for each $1 \leq j \leq n$; therefore, either $|B(x_j)^*| = 1$ or $|B(x_j)^*| = 2$. This gives that every element of $B(x_1)^* \cup \dots \cup B(x_n)^*$ is either an element of the minimal system of generators A or twice of some element of A (see Lemma 3). Note that z cannot be an element of A because $z = x + y$, so the only possibility is that z is twice of some element of A , say $z = 2a_s$, for some $1 \leq s \leq r$. This gives $x, y, z \in B(z)^*$, i.e., $|B(z)^*| \geq 3$, which is not possible. Hence, $G_I(\Lambda)$ is a complete graph.

Example 3. Let $\Lambda = \langle 3, 5 \rangle = \{0, 3, 5, 6, 8, 9, \dots\}$ be a numerical semigroup. Consider the irreducible ideals $I_1 = \Lambda \setminus B(3)^*$, $I_2 = \Lambda \setminus B(5)^*$, $I_3 = \Lambda \setminus B(6)^*$, $I_4 = \Lambda \setminus B(10)^*$, and $I_5 = \Lambda \setminus B(12)^*$, where $B(3)^* = \{3\}$, $B(5)^* = \{5\}$, $B(6)^* = \{3, 6\}$, $B(10)^* = \{5, 10\}$, and $B(12)^* = \{3, 6, 9, 12\}$. Let $I = I_1 \cap I_2 \cap I_3 \cap I_4 = \{8, 9, 11, \dots\}$ and $J = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5 = \{11, 12, \dots\}$ be two ideals of Λ . Then, by Theorem 1, it follows that $G_I(\Lambda)$ must be a complete graph (see Figure 3) while $G_J(\Lambda)$ cannot be a complete graph (see Figure 4).

Proposition 5. Let $\Lambda = \langle a_1, \dots, a_n \rangle$, where $n \geq 3$ be a numerical semigroup. Then, there are at least $2^n + 2^{n-1}$ ideals I such that $G_I(\Lambda)$ is complete.

FIGURE 3: Graph $G_I(\Lambda)$, where $\Lambda = \langle 3, 5, \rangle$ and $I = \{8, 9, 11, \dots\}$.

FIGURE 4: Graph $G_I(\Lambda)$, where $\Lambda = \langle 3, 5, \rangle$ and $I = \{11, 12, \dots\}$.

Proof. We prove it by induction on n . If $n = 3$, then $\Lambda = \langle a_1, a_2, a_3 \rangle$. We have the following cases:

Case_1: if I is irreducible, then the order of $G_I(\Lambda)$ is $|B^*(x)|$. From Theorem 1, it follows that $G_I(\Lambda)$ is complete if and only if $|G_I(\Lambda)| \leq 2$. This gives $|B^*(x)| = 1$ or 2. If $|B^*(x)| = 1$, then $B^*(x) = \{a_1\}$, $\{a_2\}$, or $\{a_3\}$, and if $|B^*(x)| = 2$, then $B^*(x) = \{a_1, 2a_1\}$, $\{a_2, 2a_2\}$, or $\{a_3, 2a_3\}$. But, note that $B^*(x) = \{a_2, 2a_2\}$ and $\{a_3, 2a_3\}$ may or may not exist because there is a possibility that $2a_2$ and $2a_3$ may be the multiple of more than one element of Λ or these are linear combination of two or more than two elements of Λ . So, in this case, there are at least 4 irreducible ideals I for which $G_I(\Lambda)$ is complete.

Case_2: now, if I is not irreducible, then I can be written as an intersection of irreducible ideals. We assume $I = \bigcap_{1 \leq j \leq n} I_j$, where I_j is an irreducible ideal and $n \geq 2$. This gives order of $G_I(\Lambda)$ as $|B^*(x)| = \bigcup B^*(x_j)$. So, if $G_I(\Lambda)$ is complete, then $|B^*(x_j)| \leq 2$, for all j , and therefore, $B^*(x_j)$ is one of $\{a_1\}$, $\{a_2\}$, $\{a_3\}$, $\{a_1, 2a_1\}$, $\{a_2, 2a_2\}$, and $\{a_3, 2a_3\}$. As $\{a_1\}$, $\{a_2\}$, $\{a_3\}$, and $\{a_1, 2a_1\}$ must exist in irreducible case, the least possibilities for $B^*(x)$ are $\{a_1, a_2\}$, $\{a_1, a_3\}$, $\{a_2, a_3\}$, $\{a_1, a_2, 2a_1\}$, $\{a_1, a_3, 2a_1\}$, $\{a_1, a_2, a_3\}$, and $\{a_1, a_2, a_3, 2a_1\}$. This gives that there are at least 7 reducible ideals for which $G_I(\Lambda)$ is complete. Moreover, if $\Lambda = I$, then $G_I(\Lambda)$ is an empty graph, which is trivially a complete graph. Hence, there are at least $12 = 2^3 + 2^2$ ideals for which $G_I(\Lambda)$ is complete.

Now, suppose that, as the induction hypothesis, $n \geq 4$ and, whenever, a numerical semigroup, say Λ_1 , has

$\{a_1, a_2, \dots, a_{n-1}\}$ as a minimal generating set, then there are at least $2^{n-1} + 2^{n-2}$ ideals J such that $G_J(\Lambda_1)$ is complete.

Now, let $\Lambda = \langle a_1, a_2, \dots, a_n \rangle$. Note that $\{a_1, a_2, \dots, a_n\} = \{a_1, a_2, \dots, a_{n-1}\} \cup \{a_n\}$. Then, induction hypothesis gives that there are at least $2(2^{n-1} + 2^{n-2}) = 2^n + 2^{n-1}$ ideals I such that $G_I(\Lambda)$ is complete.

4. Clique Number and Cut Point

In this section, we present some results on clique number and cut point of the graph $G_I(\Lambda)$.

Proposition 6. Let $G_I(\Lambda)$ be a graph associated with an irreducible ideal I of a numerical semigroup Λ . If order of $G_I(\Lambda)$ is n , then

$$cl(G_I(\Lambda)) = \begin{cases} \frac{n}{2} + 1, & \text{if } n \text{ is even,} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases} \quad (3)$$

Proof. Since I is irreducible, the order of $G_I(\Lambda)$ is exactly $|B(x)^*|$, for some $x \in \Lambda$. Assume that $B(x)^* = \{x_1, x_2, \dots, x_{n-1}, x\}$. We have the following cases:

Case_1: if $|B(x)^*|$ is even, then the cardinality of the set $B(x)^* - \{x\}$ is odd, and by definition of $B(x)$, we know that $x - x_1 = x_{n-1}$, $x - x_2 = x_{n-2}$, $\dots$, $x - x_{(n/2)-1} = x_{(n/2)+1}$. This gives that $(v_{x_1}, v_{x_{n-1}})$, $(v_{x_2}, v_{x_{n-2}})$, $\dots$, $(v_{x_{(n/2)-1}}, v_{x_{(n/2)+1}})$ are pairs of nonadjacent vertices. As cardinality of $B(x)^* - \{x\}$ is odd, it must contain an element $x_{(n/2)} = x/2$. Note that every element of the set $\{x_{(n/2)}, x_{(n/2)+1}, \dots, x_{n-1}, x\}$ is equal or greater than $x/2$, and therefore, the sum of any two different elements is greater than x . This gives that the graph corresponding to the set $\{x_{(n/2)}, x_{(n/2)+1}, \dots, x_{n-1}, x\}$ is a complete graph of maximum possible order.

Case_2: if $|B(x)^*|$ is odd, then the cardinality of the set $B(x)^* - \{x\}$ is even. In this case, we have $x - x_1 = x_{n-1}$, $x - x_2 = x_{n-2}$, $\dots$, $x - x_{(n+1)/2} = x_{(n+1)/2}$. This gives that $(v_{x_1}, v_{x_{n-1}})$, $(v_{x_2}, v_{x_{n-2}})$, $\dots$, $(v_{x_{(n+1)/2}}, v_{x_{(n+1)/2}})$ are pairs of nonadjacent vertices. Note that every element of the set $\{x_{(n+1)/2}, x_{((n+1)/2)+1}, \dots, x_{n-1}, x\}$ is greater than $x/2$, and therefore, the sum of any two different elements is greater than x . This gives that the graph corresponding to the set $\{x_{((n+1)/2)}, x_{((n+1)/2)+1}, \dots, x_{n-1}, x\}$ is a complete graph of maximum possible order.

Example 4. Let $\Lambda = \langle 3, 5 \rangle$ be a numerical semigroup and $I = \Lambda \setminus B^*(16) = \{9, 12, 14, 15, 17, \dots\}$ be an irreducible ideal of Λ . Then, Proposition 6 gives that the subset $\{8, 10, 11, 13, 16\}$ of $B^*(16) = \{3, 5, 6, 8, 10, 11, 13, 16\}$ is the maximal clique, and therefore, the clique number of graph

FIGURE 5: Graph $G_I(\Lambda)$, where $\Lambda = \langle 3, 5, \rangle$ and I is an irreducible ideal with $x = 16$.

FIGURE 6: Clique of graph $G_I(\Lambda)$ given in Figure 5.

$G_I(\Lambda)$ is 5. The graph $G_I(\Lambda)$ and its corresponding maximal clique are given in Figures 5 and 6, respectively:

Proposition 7. Let $\Lambda = \langle A \rangle$ be a numerical semigroup, where $A = \{a_1, a_2, \dots, a_n\}$ is the minimal system of generators of Λ and $G_I(\Lambda)$ is a graph associated with an irreducible ideal $I = \Lambda \setminus B^*(x)$, for some $x \in \Lambda$ such that the order of $G_I(\Lambda) \geq 3$. Then, v_x is the only cut point of $G_I(\Lambda)$ if and only if $B^*(x) = \{a_i, 2a_i, \dots, x\}$ for some $a_i \in A$.

Proof. Assume that there exist some $y_1, \dots, y_k \in B^*(x)$ such that $y_j \neq na_i$, where $1 \leq n \leq m$ and $1 \leq j \leq k$. Then, we have the following cases:

Case_1: if $j = 1$, then clearly $y_1 + na_i \in I$ for all $1 \leq n \leq m$; otherwise, y_1 is a multiple of a_i . This gives that for every two elements $x_1, x_2 \in B^*(x)$ other than x and y_1 , we have two different paths $v_{x_1} - v_x - v_{x_2}$ and $v_{x_1} - v_{y_1} - v_{x_2}$. This implies that v_x is not a cut point of $G_I(\Lambda)$.

Case_2: if $j > 1$, then assume that $y_k \in B^*(x)$ is the largest element such that $y_k \neq na_i$, where $1 \leq n \leq m$. Note that $y_k + y_1 = x$, where y_1 is the smallest element of $B^*(x)$ such that $y_1 \neq na_i$ and $1 \leq n \leq m$. Then, $y + y_k \in I$, for all $y \in B^*(x) - \{y_1\}$ and $y_1 + (m-1)a_i \in I$. This gives that for every two elements $x_1, x_2 \in B^*(x)$ other than x and y_k , we have two different paths, either $v_{x_1} - v_x - v_{x_2}$ and $v_{x_1} - v_{y_k} - v_{x_2}$ or $v_{x_1} - v_x - v_{x_2}$ and $v_{x_1} - v_{(m-1)a_i} - v_{y_k} - v_{x_2}$. This implies that v_x is not a cut point of $G_I(\Lambda)$.

Now, if $B^*(x) = \{a_i, 2a_i, \dots, x\}$, then clearly $a_i + ka_i \in B^*(x) \forall i \leq k \leq m-1$ and $a_i + x \in I$. This gives that $G_I(\Lambda) - \{v_x\}$ has at least two disconnected components and $y + x \in I$ for all $y \in B^*(x) - \{x\}$. Therefore, x is the only cut point.

FIGURE 7: Graph $G_I(\Lambda)$, where $\Lambda = \langle 4, 5, \rangle$ and I is an irreducible ideal with $x = 16$.

FIGURE 8: Graph $G_I(\Lambda) - \{v_{16}\}$.

Example 5. Let $\Lambda = \langle 4, 5, \rangle$ be a numerical semigroup and $I = \Lambda \setminus B^*(16)$ be an irreducible ideal, where $B^*(16) = \{4, 8, 12, 16\}$. Then, v_{16} is a cut point of graph $G_I(\Lambda)$ (see Figures 7 and 8).

5. Conclusion

In this article, graphs associated with ideals of a numerical semigroup have been studied and it has been proved that these graphs are connected. We investigated some properties such as girth, regularity, clique number, and cut point of these graphs. Also, a necessary and sufficient condition has been given for a graph associated with an ideal of a numerical semigroup to be complete.

Data Availability

No data were used to support this study.

Disclosure

This research was carried out as a part of the employment of the authors.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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