Research Article

$(\tau_1, \tau_2)\alpha$ -Continuity for Multifunctions

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The main goal of this article is to introduce the concepts of upper and lower $(\tau_1, \tau_2)\alpha$ -continuous multifunctions. Characterizations of upper and lower $(\tau_1, \tau_2)\alpha$ -continuous multifunctions are investigated. The relationships between upper and lower $(\tau_1, \tau_2)\alpha$ -continuous multifunctions and the other types of continuity are discussed.

1. Introduction

General topology is important in many fields of applied sciences as well as branches of mathematics. Continuity is a basic concept for the study in topological spaces. Generalization of this concept by using weaker forms of open sets such as α -sets [1], semi-open sets [2], preopen sets [3], and semi-preopen sets [4] is one of the main research topics of general topology. In 1965, Njåstad [1] introduced a weak form of open sets called α -sets. Mashhour et al. [5] defined a function to be α -continuous if the inverse image of each open set is an α -set. Noiri [6] called α -continuous functions strong semi-continuous and investigated α -continuous functions in [7] and investigated α -continuous functions. In [8, 9], the authors investigated a class of functions called almost α -continuous or almost feebly continuous. In 1987, Noiri [10] introduced a class of functions called weakly α -continuous functions. Some characterizations of weakly α -continuous functions are investigated in [11–13]. Neubrunn [14] extended these functions to multifunction and introduced the notions of upper and lower α -continuous multifunctions. Popa and Noiri [15] investigated several characterizations of upper and lower α -continuous multifunctions and some basic properties of such multifunctions. In [16], the present authors introduced upper and lower almost α -continuous multifunctions and obtained some characterizations of upper and lower almost α -continuous multifunctions and several properties of such

multifunctions. Recently, Popa and Noiri [17] investigated some characterizations and several properties concerning upper and lower weakly α -continuous multifunctions. In this paper, we introduce the notions of upper and lower $(\tau_1, \tau_2)\alpha$ -continuous multifunctions and investigate some characterizations of upper and lower $(\tau_1, \tau_2)\alpha$ -continuous multifunctions. Section 4 is devoted to introducing and studying upper and lower almost $(\tau_1, \tau_2)\alpha$ -continuous multifunctions. In Section 5, several interesting characterizations of upper and lower weakly $(\tau_1, \tau_2)\alpha$ -continuous multifunctions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2) . The closure of A and the interior of A with respect to τ_i are denoted by $\tau_i\text{-Cl}(A)$ and $\tau_i\text{-Int}(A)$, respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$ -semi-open (resp. $\tau_1\tau_2$ -regular open [18] and $\tau_1\tau_2$ -regular closed [19]) if $A \subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(A))$ (resp. $A = \tau_1\text{-Int}(\tau_2\text{-Cl}(A))$ and $A = \tau_1\text{-Cl}(\tau_2\text{-Int}(A))$). The complement of a $\tau_1\tau_2$ -semi-open set is said to be $\tau_1\tau_2$ -semi-closed. The $\tau_1\tau_2$ -semi-closure of A is defined by the intersection of $\tau_1\tau_2$ -semi-closed sets containing A and is denoted by $\tau_1\tau_2\text{-sCl}(A)$. The $\tau_1\tau_2$ -semi-interior of A is defined by the union of $\tau_1\tau_2$ -semi-

open sets contained in A and is denoted by $\tau_1\tau_2\text{-sInt}(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$ -closed [20] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$ -closed set is said to be $\tau_1\tau_2$ -open. The intersection of all $\tau_1\tau_2$ -closed sets containing A is called the $\tau_1\tau_2$ -closure of A and is denoted by $\tau_1\tau_2\text{-Cl}(A)$. The union of all $\tau_1\tau_2$ -open sets contained in A is called the $\tau_1\tau_2$ -interior of A and is denoted by $\tau_1\tau_2\text{-Int}(A)$.

By a multifunction $F: X \longrightarrow Y$, we mean a point-to-set correspondence from X into Y , and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F: X \longrightarrow Y$, following [21], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X | F(x) \subseteq B\}$ and $F^-(B) = \{x \in X | F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X | y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Then, F is said to be surjection if $F(X) = Y$, or equivalent, if for each $y \in Y$ there exists $x \in X$ such that $y \in F(x)$ and F is called injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

Definition 1 (see [22]). A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be:

- (1) upper (τ_1, τ_2) -continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \subseteq V$, there exists a $\tau_1\tau_2$ -open subset U of X containing x such that $F(U) \subseteq V$;
- (2) lower (τ_1, τ_2) -continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$ -open subset U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$;
- (3) upper (resp. lower) (τ_1, τ_2) -continuous if F has this property at each point of X .

Definition 2. A subset A of a bitopological space (X, τ_1, τ_2) is said to be:

- (i) $(\tau_1, \tau_2)\alpha$ -open if $A \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$;
- (ii) $(\tau_1, \tau_2)\alpha$ -closed if its complement is $(\tau_1, \tau_2)\alpha$ -open.

Definition 3. Let A be a subset of a bitopological (X, τ_1, τ_2) . Then,

- (i) The intersection of all $(\tau_1, \tau_2)\alpha$ -closed sets containing A is called the $(\tau_1, \tau_2)\alpha$ -closure of A and is denoted by $(\tau_1, \tau_2)\alpha\text{-Cl}(A)$.
- (ii) The union of all $(\tau_1, \tau_2)\alpha$ -open sets contained in A is called the $(\tau_1, \tau_2)\alpha$ -interior of A and is denoted by $(\tau_1, \tau_2)\alpha\text{-Int}(A)$.

Lemma 1 (see [20]). For a subset A of a bitopological space (X, τ_1, τ_2) , the following properties hold:

- (1) $\tau_1\tau_2\text{-sCl}(A) = \tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cup A$
- (2) If A is $\tau_1\tau_2$ -open, then $\tau_1\tau_2\text{-sCl}(A) = \tau_1\text{-Int}(\tau_2\text{-Cl}(A))$

Lemma 2. For a subset A of a bitopological space (X, τ_1, τ_2) , the following properties are equivalent:

- (1) A is $(\tau_1, \tau_2)\alpha$ -open
- (2) $U \subseteq A \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(U))$ for some $\tau_1\tau_2$ -open set U
- (3) $U \subseteq A \subseteq \tau_1\tau_2\text{-sCl}(U)$ for some $\tau_1\tau_2$ -open set U
- (4) $A \subseteq \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(A))$

Proof

(1) $\implies$ (2): Suppose that A is a $(\tau_1, \tau_2)\alpha$ -open set. Then, we have

$$A \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))). \quad (1)$$

Put $U = \tau_1\tau_2\text{-Int}(A)$; then, $U \subseteq A \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(U))$.

(2) $\implies$ (3): This follows from Lemma 1 (2).

(3) $\implies$ (4): Suppose that $U \subseteq A \subseteq \tau_1\tau_2\text{-sCl}(U)$ for some $\tau_1\tau_2$ -open set U . Then, we have $U \subseteq \tau_1\tau_2\text{-Int}(A)$, and hence $A \subseteq \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(A))$.

(4) $\implies$ (1): Suppose that $A \subseteq \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(A))$. Since $\tau_1\tau_2\text{-Int}(A)$ is $\tau_1\tau_2$ -open and by Lemma 1 (2), $A \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. Thus, A is $(\tau_1, \tau_2)\alpha$ -open. $\square$

Lemma 3. For a subset A of a bitopological space (X, τ_1, τ_2) , the following properties hold:

- (1) A is $(\tau_2, \tau_2)\alpha$ -closed if and only if $\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A)) \subseteq A$
- (2) $\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A)) = \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$
- (3) $\alpha(\tau_1, \tau_2)\text{-Cl}(A) = A \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$

Proof

- (1) Let A be a $(\tau_1, \tau_2)\alpha$ -closed set. Then, $X - A$ is $(\tau_1, \tau_2)\alpha$ -open and by Lemma 2, $X - A \subseteq \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(X - A)) = X - \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A))$. Thus, $\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A)) \subseteq A$.

Conversely, suppose that $\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A)) \subseteq A$. Then,

$$\begin{aligned} X - A &\subseteq X - \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A)) \\ &= \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(X - A)). \end{aligned} \quad (2)$$

By Lemma 2, we have $X - A$ is $(\tau_1, \tau_2)\alpha$ -open, and hence A is $(\tau_1, \tau_2)\alpha$ -closed.

- (2) Since $X - \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A)) = \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(X - A))$, by Lemma 1 (2),

$$\begin{aligned} X - \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A)) &= \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(X - A))) \\ &= X - \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))), \end{aligned} \quad (3)$$

and hence $\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A)) = \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$.

(3) We observe that

$$\begin{aligned} &\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))))) \\ &\subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A \cup (\tau_1\tau_2\text{-Cl}(A))))) \\ &\subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))) \\ &\subseteq A \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))). \end{aligned} \quad (4)$$

Thus, $A \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$ is $(\tau_1, \tau_2)\alpha$ -closed, and hence $(\tau_1, \tau_2)\alpha\text{-Cl}(A) \subseteq A \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$. On the other hand, since $(\tau_1, \tau_2)\alpha\text{-Cl}(A)$ is $(\tau_1, \tau_2)\alpha$ -closed, we have

$$\begin{aligned} &\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))) \\ &\subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}((\tau_1, \tau_2)\alpha\text{-Cl}(A)))) \\ &\subseteq (\tau_1, \tau_2)\alpha\text{-Cl}(A). \end{aligned} \quad (5)$$

Therefore, $A \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))) \subseteq (\tau_1, \tau_2)\alpha\text{-Cl}(A)$. Consequently, we obtain $(\tau_1, \tau_2)\alpha\text{-Cl}(A) = A \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$. $\square$

3. Characterizations of Upper and Lower $(\tau_1, \tau_2)\alpha$ -Continuous Multifunctions

In this section, we introduce the notions of upper and lower $(\tau_1, \tau_2)\alpha$ -continuous multifunctions. Moreover, several characterizations of upper and lower $(\tau_1, \tau_2)\alpha$ -continuous multifunctions are discussed.

Definition 4. A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be:

- (1) upper $(\tau_1, \tau_2)\alpha$ -continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \subseteq V$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq V$;
- (2) lower $(\tau_1, \tau_2)\alpha$ -continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$;
- (3) upper (resp. lower) $(\tau_1, \tau_2)\alpha$ -continuous if F has this property at each point of X .

Remark 1. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following implication holds:

upper (τ_1, τ_2) -continuity $\implies$ upper $(\tau_1, \tau_2)\alpha$ -continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. Let $X = \{a, b, c, d\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{b\}, X\}$. Let $Y = \{-2, -1, 0, 1, 2\}$ with topologies

$$\sigma_1 = \{\emptyset, \{-1, -2, 2\}, \{0, 1\}, Y\}, \quad (6)$$

and $\sigma_2 = \{\emptyset, \{-1, -2, 2\}, Y\}$. A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is defined as follows:

$$F(x) = \begin{cases} \{-1, -2\}, & \text{if } x = a, \\ \{-1\}, & \text{if } x = b, \\ \{0, 1\}, & \text{if } x = c, \\ \{-1, 0\}, & \text{if } x = d. \end{cases} \quad (7)$$

Then, F is upper $(\tau_1, \tau_2)\alpha$ -continuous but F is not upper (τ_1, τ_2) -continuous.

Theorem 1. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is upper $(\tau_1, \tau_2)\alpha$ -continuous at a point x of X
- (2) $x \in \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^+(V)))$ for each $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$
- (3) For each $\tau_1\tau_2$ -semiopen subset U of X containing x and each $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$, there exists a nonempty $\tau_1\tau_2$ -open subset G of X such that $G \subseteq U$ and $F(G) \subseteq V$

Proof

(1) $\implies$ (2): Let V be any $\sigma_1\sigma_2$ -open subset of Y such that $F(x) \subseteq V$. Then, there exists a $(\tau_1, \tau_2)\alpha$ -open set U containing x such that $F(U) \subseteq V$; hence, $x \in U \subseteq F^+(V)$. Since U is $(\tau_1, \tau_2)\alpha$ -open, by Lemma 2, we have $x \in U \subseteq \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(U)) \subseteq \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^+(V)))$.

(2) $\implies$ (3): Let V be any $\sigma_1\sigma_2$ -open subset of Y such that $F(x) \subseteq V$. Then, we have $x \in \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^+(V)))$. Let U be any $\tau_1\tau_2$ -semi-open set containing x . Thus, $U \cap [\tau_1\tau_2\text{-Int}(F^+(V))] \neq \emptyset$ and $U \cap [\tau_1\tau_2\text{-Int}(F^+(V))]$ is

$\tau_1\tau_2$ -semi-open. Put $G = \tau_1\tau_2\text{-Int}(U \cap [\tau_1\tau_2\text{-Int}(F^+(V))])$; then, G is a nonempty $\tau_1\tau_2$ -open subset of X such that $G \subseteq U$ and $F(G) \subseteq V$.

(3) $\implies$ (1): Let $\tau_1\tau_2\text{-SO}(X, x)$ be the family of all $\tau_1\tau_2$ -semi-open subsets of X containing x . Let V be any $\sigma_1\sigma_2$ -open subset of Y containing $F(x)$. For each $U \in \tau_1\tau_2\text{-SO}(X, x)$, there exists a nonempty $\tau_1\tau_2$ -open set H_U such that $H_U \subseteq U$ and $F(H_U) \subseteq V$. Let $W = \cup\{H_U | U \in \tau_1\tau_2\text{-SO}(X, x)\}$. Then, we have W which is $\tau_1\tau_2$ -open in X , $x \in \tau_1\tau_2\text{-sCl}(W)$, and $F(W) \subseteq V$. Put $G = W \cup \{x\}$. By Lemma 1, we have

$$\begin{aligned} W &\subseteq G \subseteq \tau_1\tau_2\text{-sCl}(G) = \tau_1\text{-Int}(\tau_2\text{-Cl}(G)) \cup G \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W \cup \{x\})) \cup (W \cup \{x\}) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W) \cup \tau_2\text{-Cl}(\{x\})) \cup (W \cup \{x\}) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W)) \cup (W \cup \{x\}) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W)) \\ &= \tau_1\tau_2\text{-sCl}(W), \end{aligned} \quad (8)$$

and by Lemma 2, G is a $(\tau_1, \tau_2)\alpha$ -open subset of X containing x such that $F(G) \subseteq V$. This shows that F is upper $(\tau_1, \tau_2)\alpha$ -continuous at x . $\square$

Theorem 2. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is lower $(\tau_1, \tau_2)\alpha$ -continuous at a point x of X
- (2) $x \in \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F(V)))$ for each $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$
- (3) For each $\tau_1\tau_2$ -semiopen subset U of X containing x and each $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$, there exists a nonempty $\tau_1\tau_2$ -open subset G of X such that $G \subseteq U$ and $F(z) \cap V \neq \emptyset$ for every $z \in G$

Proof. The proof is similar to that of Theorem 1. $\square$

Definition 5. A subset A of a bitopological space (X, τ_1, τ_2) is said to be a $\tau_1\tau_2$ -neighbourhood (resp. $(\tau_1, \tau_2)\alpha$ -neighbourhood) of $x \in X$ if there exists a $\tau_1\tau_2$ -open (resp. $(\tau_1, \tau_2)\alpha$ -open) set U such that $x \in U \subseteq A$.

Theorem 3. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is upper $(\tau_1, \tau_2)\alpha$ -continuous

- (2) $F^+(V)$ is $(\tau_1, \tau_2)\alpha$ -open in X for each $\sigma_1\sigma_2$ -open subset V of Y
- (3) $F(H)$ is $(\tau_1, \tau_2)\alpha$ -closed in X for each $\sigma_1\sigma_2$ -closed subset H of Y
- (4) $\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(F(B))) \subseteq F(\sigma_1\sigma_2\text{-Cl}(B))$ for each subset B of Y
- (5) $F^+(V) \subseteq \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^+(V)))$ for each $\sigma_1\sigma_2$ -open subset V of Y
- (6) $(\tau_1, \tau_2)\alpha\text{-Cl}(F(B)) \subseteq F(\sigma_1\sigma_2\text{-Cl}(B))$ for each subset B of Y
- (7) For each $x \in X$ and each $\sigma_1\sigma_2$ -neighbourhood V of $F(x)$, $F^+(V)$ is a $(\tau_1, \tau_2)\alpha$ -neighbourhood of x
- (8) For each $x \in X$ and each $\sigma_1\sigma_2$ -neighbourhood V of $F(x)$, there exists a $(\tau_1, \tau_2)\alpha$ -neighbourhood U of x such that $F(U) \subseteq V$

Proof

(1) $\implies$ (2): Let V be any $\sigma_1\sigma_2$ -open subset of Y and $x \in F^+(V)$. By Theorem 1, $x \in \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^+(V)))$, and hence

$$F^+(V) \subseteq \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^+(V))). \quad (9)$$

It follows from Lemma 2 that $F^+(V)$ is $(\tau_1, \tau_2)\alpha$ -open in X .

(2) $\implies$ (3): This follows from the fact that $F^+(Y - B) = X - F(B)$ for any subset B of Y .

(3) $\implies$ (4): Let B be any subset of Y . Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$ -closed in Y and by (3), $F(\sigma_1\sigma_2\text{-Cl}(B))$ is $(\tau_1, \tau_2)\alpha$ -closed in X . By Lemma 3 (1),

$$\begin{aligned} &\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(F^-(B))) \\ &\subseteq \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(B)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B)). \end{aligned} \quad (10)$$

(4) $\implies$ (5): Let V be any $\sigma_1\sigma_2$ -open subset of Y . Then, we have $Y - V$ is $\sigma_1\sigma_2$ -closed in Y . By (4),

$$\begin{aligned} X - \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^+(V))) &= \tau_1\tau_2\text{-sInt}(X - \tau_1\tau_2\text{-Int}(F^+(V))) \\ &= \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(X - F^+(V))) \\ &= \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(F^-(Y - V))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - V)) \\ &= X - F^+(V), \end{aligned} \quad (11)$$

and hence $F^+(V) \subseteq \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^+(V)))$.

(5) $\implies$ (6): Let B be any subset of Y . Then, $Y - \sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$ -open in Y , and by (5),

$$\begin{aligned} X - F^-(\sigma_1\sigma_2\text{-Cl}(B)) &= F^+(Y - \sigma_1\sigma_2\text{-Cl}(B)) \\ &\subseteq \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^+(Y - \sigma_1\sigma_2\text{-Cl}(B)))) \\ &= \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(X - F^-(\sigma_1\sigma_2\text{-Cl}(B)))) \\ &= \tau_1\tau_2\text{-sCl}(X - \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(B)))) \\ &= X - \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(B)))). \end{aligned} \quad (12)$$

Thus, $\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(F(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F(\sigma_1\sigma_2\text{-Cl}(B))$. By Lemma 3,

$$\begin{aligned} \alpha(\tau_1, \tau_2)\text{-Cl}(F^-(B)) &\subseteq \alpha(\tau_1, \tau_2)\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(B))) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(B)) \cup \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(B))))) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(B)) \cup \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(B)))) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(B)). \end{aligned} \quad (13)$$

(6) $\implies$ (3): Let H be any $\sigma_1\sigma_2$ -closed subset of Y . By (6), we have

$$\alpha(\tau_1, \tau_2)\text{-Cl}(F^-(H)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(H)) = F^-(H), \quad (14)$$

and hence $F(H)$ is $(\tau_1, \tau_2)\alpha$ -closed in X .

(2) $\implies$ (7): Let $x \in X$ and V be a $\sigma_1\sigma_2$ -neighbourhood of $F(x)$. Then, there exists a $\sigma_1\sigma_2$ -open subset U of Y such that $F(x) \subseteq U \subseteq V$. Thus, $x \in F^+(U) \subseteq F^+(V)$. By (2), we have $F^+(U)$ is $(\tau_1, \tau_2)\alpha$ -open in X and hence $F^+(V)$ is a $(\tau_1, \tau_2)\alpha$ -neighbourhood of x .

(7) $\implies$ (8): Let $x \in X$ and V be a $\sigma_1\sigma_2$ -neighbourhood of $F(x)$. By (7), $F^+(V)$ is a $(\tau_1, \tau_2)\alpha$ -neighbourhood of x . Put $U = F^+(V)$; then, U is a $(\tau_1, \tau_2)\alpha$ -neighbourhood of x and $F(U) \subseteq V$.

(8) $\implies$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$ -open subset of Y such that $F(x) \subseteq V$. Then, V is a $\sigma_1\sigma_2$ -neighbourhood of $F(x)$ and by (8), there exists a $(\tau_1, \tau_2)\alpha$ -neighbourhood U of x such that $F(U) \subseteq V$. Since U is $(\tau_1, \tau_2)\alpha$ -neighbourhood of x , there exists a $(\tau_1, \tau_2)\alpha$ -open set W such that $x \in W \subseteq U$; hence, $F(W) \subseteq V$. This shows that F is upper $(\tau_1, \tau_2)\alpha$ -continuous at x . Consequently, we obtain that F is upper $(\tau_1, \tau_2)\alpha$ -continuous. $\square$

Theorem 4. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is lower $(\tau_1, \tau_2)\alpha$ -continuous
- (2) $F(V)$ is $(\tau_1, \tau_2)\alpha$ -open in X for every $\sigma_1\sigma_2$ -open subset V of Y
- (3) $F^+(H)$ is $(\tau_1, \tau_2)\alpha$ -closed in X for every $\sigma_1\sigma_2$ -closed subset H of Y
- (4) $\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(F^+(B))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y
- (5) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y
- (6) $F((\tau_1, \tau_2)\alpha\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$ for every subset A of X
- (7) $F(\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A))) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$ for every subset A of X
- (8) $F(\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$ for every subset A of X

Proof

(1) $\implies$ (2): Let V be any $\sigma_1\sigma_2$ -open subset of Y and $x \in F(V)$. By Theorem 2, $x \in \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F(V)))$, and hence

$$F^-(V) \subseteq \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^-(V))). \quad (15)$$

It follows from Lemma 2 that $F(V)$ is $(\tau_1, \tau_2)\alpha$ -open in X .

(2) $\implies$ (3): This follows from the fact that $F^+(Y - B) = X - F(B)$ for any subset B of Y .

(3) $\implies$ (4): Let B be any subset of Y . Then, we have $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$ -closed in Y and by (3), $F^+(\sigma_1\sigma_2\text{-Cl}(B))$ is $\alpha(\tau_1, \tau_2)$ -closed in X . By Lemma 3 (1),

$$\begin{aligned} \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(F^+(B))) &\subseteq \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Cl}(B)))) \\ &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B)). \end{aligned} \quad (16)$$

(4) $\implies$ (5): Let B be any subset of Y . By (4) and Lemma 3, we have

$$\begin{aligned} (\tau_1, \tau_2)\alpha\text{-Cl}(F^+(B)) &\subseteq (\tau_1, \tau_2)\alpha\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Cl}(B))) \\ &= F^+(\sigma_1\sigma_2\text{-Cl}(B)) \cup \tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Cl}(B)))) \\ &= F^+(\sigma_1\sigma_2\text{-Cl}(B)). \end{aligned} \quad (17)$$

(5) $\implies$ (6): Let A be any subset of X . Since $A \subseteq F^+(F(A))$, we have

$$\begin{aligned} (\tau_1, \tau_2)\alpha\text{-Cl}(A) &\subseteq (\tau_1, \tau_2)\alpha\text{-Cl}(F^+(F(A))) \\ &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(F(A))), \end{aligned} \quad (18)$$

and hence $F((\tau_1, \tau_2)\alpha\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$.

(6) $\implies$ (7): This follows immediately from Lemma 3.

(7) $\implies$ (8): Let A be any subset of X . By (7) and Lemma 3 (2), we have

$$\begin{aligned} &F(\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))) \\ &= F(\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A))) \\ &\subseteq \sigma_1\sigma_2\text{-Cl}(F(A)). \end{aligned} \quad (19)$$

(8) $\implies$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$ -open subset of Y such that $F(x) \cap V \neq \emptyset$. Then, we have $x \in F(V)$. We shall show that $F(V)$ is $(\tau_1, \tau_2)\alpha$ -open in X . By the hypothesis, $F(\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^+(Y - V))))) \subseteq \sigma_1\sigma_2\text{-Cl}(F(F^+(Y - V))) \subseteq Y - V$. Thus, $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^+(Y - V)))) \subseteq F^+(Y - V) = X - F(V)$, and hence $F(V) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F(V))))$. This shows that $F(V)$ is $\alpha(\tau_1, \tau_2)$ -open in X . Put $U = F(V)$; then, U is a $(\tau_1, \tau_2)\alpha$ -open set containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. Consequently, we obtain that F is lower $(\tau_1, \tau_2)\alpha$ -continuous. $\square$

Definition 6. A function $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be $(\tau_1, \tau_2)\alpha$ -continuous if for every $\sigma_1\sigma_2$ -open subset V of Y , $f^{-1}(V)$ is $(\tau_1, \tau_2)\alpha$ -open in X .

Corollary 1. For a function $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) f is $(\tau_1, \tau_2)\alpha$ -continuous
- (2) $f^{-1}(F)$ is $(\tau_1, \tau_2)\alpha$ -closed in X for each $\sigma_1\sigma_2$ -closed subset F of Y
- (3) $\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(f^{-1}(B))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$ for each subset B of Y
- (4) $(\tau_1, \tau_2)\alpha\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$ for each subset B of Y
- (5) For each $x \in X$ and each $\sigma_1\sigma_2$ -neighbourhood V of $f(x)$, $f^{-1}(V)$ is a $(\tau_1, \tau_2)\alpha$ -neighbourhood of x
- (6) For each $x \in X$ and each $\sigma_1\sigma_2$ -neighbourhood V of $f(x)$, there exists a $(\tau_1, \tau_2)\alpha$ -neighbourhood U of x such that $f(U) \subseteq V$
- (7) $f((\tau_1, \tau_2)\alpha\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(f(A))$ for each subset A of X
- (8) $f(\tau_1\tau_2\text{-sInt}(\tau_1\tau_2\text{-Cl}(A))) \subseteq \sigma_1\sigma_2\text{-Cl}(f(A))$ for each subset A of X
- (9) $f(\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))) \subseteq \sigma_1\sigma_2\text{-Cl}(f(A))$ for each subset A of X

Lemma 4 (see [20]). For subsets A, B of a bitopological space (X, τ_1, τ_2) , the following properties hold:

- (1) $A \subseteq \tau_1\tau_2\text{-ker}(A)$
- (2) If $A \subseteq B$, then $\tau_1\tau_2\text{-ker}(A) \subseteq \tau_1\tau_2\text{-ker}(B)$
- (3) If A is $\tau_1\tau_2$ -open, then $\tau_1\tau_2\text{-ker}(A) = A$
- (4) $x \in \tau_1\tau_2\text{-ker}(A)$ if and only if $A \cap H \neq \emptyset$ for any $\tau_1\tau_2$ -closed set H containing x

Theorem 5. Let $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a multifunction. If

$$F^+(\sigma_1\sigma_2\text{-ker}(A)) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^+(A)), \quad (20)$$

for every subset A of X , then F is upper $(\tau_1, \tau_2)\alpha$ -continuous.

Proof. Let V be any $\sigma_1\sigma_2$ -open subset of Y . By Lemma 4 (3), we have $F^+(V) = F^+(\sigma_1\sigma_2\text{-ker}(V)) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^+(V))$, and hence $(\tau_1, \tau_2)\alpha\text{-Int}(F^+(V)) = F^+(V)$. Thus, $F^+(V)$ is $(\tau_1, \tau_2)\alpha$ -open, by Theorem 3, F is upper $(\tau_1, \tau_2)\alpha$ -continuous. $\square$

Theorem 6. Let $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a multifunction. If

$$F^-(\sigma_1\sigma_2\text{-ker}(A)) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^-(A)), \quad (21)$$

for every subset A of X , then F is lower $(\tau_1, \tau_2)\alpha$ -continuous.

Proof. The proof is similar to that of Theorem 5. $\square$

Definition 7 (see [20]). A collection $\mathcal{U}$ of subsets of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$ -locally finite if for each $x \in X$ has a $\tau_1\tau_2$ -neighbourhood which intersects only finitely many elements of $\mathcal{U}$.

Definition 8 (see [20]). A subset A of a bitopological space (X, τ_1, τ_2) is said to be

- (1) $\tau_1\tau_2$ -paracompact if every cover of A by $\tau_1\tau_2$ -open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$ -open sets of X and is $\tau_1\tau_2$ -locally finite in X
- (2) $\tau_1\tau_2$ -regular if for each $x \in A$ and each $\tau_1\tau_2$ -open set U of X containing x , there exists a $\tau_1\tau_2$ -open set V of X such that $x \in V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq U$

Lemma 5 (see [20]). If A is a $\tau_1\tau_2$ -regular $\tau_1\tau_2$ -paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$ -open neighbourhood of A , then there exists a $\tau_1\tau_2$ -open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq U$.

Definition 9. A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is called punctually (τ_1, τ_2) -paracompact (resp. punctually (τ_1, τ_2) -regular) if for each $x \in X$, $F(x)$ is $\tau_1\tau_2$ -paracompact (resp. $\tau_1\tau_2$ -regular).

For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, by

$$\alpha\text{Cl}F_{\otimes}: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2), \quad (22)$$

we shall denote a multifunction defined as follows: $\alpha\text{Cl}F_{\otimes}(x) = (\sigma_1, \sigma_2)\alpha\text{-Cl}(F(x))$ for each $x \in X$.

Lemma 6. Let (X, τ_1, τ_2) be a bitopological space. Then, $(\tau_1, \tau_2)\alpha\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(A)$ for every subset A of X .

Lemma 7. If $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is punctually (τ_1, τ_2) -paracompact and punctually (τ_1, τ_2) -regular, then $\alpha\text{Cl}F_{\otimes}^+(V) = F^+(V)$ for every $\sigma_1\sigma_2$ -open subset V of Y .

Proof. Let V be any $\sigma_1\sigma_2$ -open subset V of Y and $x \in \alpha\text{Cl}F_{\otimes}^+(V)$. Then, we have $(\sigma_1, \sigma_2)\alpha\text{-Cl}(F(x)) \subseteq V$ and $F(x) \subseteq V$. Thus, $x \in F^+(V)$, and hence $\alpha\text{Cl}F_{\otimes}^+(V) \subseteq F^+(V)$. On the other hand, let $x \in F^+(V)$. Then, we have $F(x) \subseteq V$ and by Lemma 5, there exists a $\sigma_1\sigma_2$ -open subset U of Y such that $F(x) \subseteq \sigma_1\sigma_2\text{-Cl}(U) \subseteq U \subseteq V$. By Lemma 6, $(\sigma_1, \sigma_2)\alpha\text{-Cl}(F(x)) \subseteq \sigma_1\sigma_2\text{-Cl}(U) \subseteq V$, and hence $x \in \alpha\text{Cl}F_{\otimes}^+(V)$. Thus,

$F^+(V) \subseteq \alpha\text{Cl}F_{\otimes}^+(V)$. Consequently, we obtain $\alpha\text{Cl}F_{\otimes}^+(V) = F^+(V)$. $\square$

Theorem 7. Let $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be punctually (τ_1, τ_2) -paracompact and punctually (τ_1, τ_2) -regular. Then, F is upper $(\tau_1, \tau_2)\alpha$ -continuous if and only if $\alpha\text{Cl}F_{\otimes}: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is upper $(\tau_1, \tau_2)\alpha$ -continuous.

Proof. Suppose that F is upper $(\tau_1, \tau_2)\alpha$ -continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$ -open subset of Y such that $\alpha\text{Cl}F_{\otimes}(x) \subseteq V$. By Lemma 7, we have $x \in \alpha\text{Cl}F_{\otimes}^+(V) = F^+(V)$. Since F is upper $(\tau_1, \tau_2)\alpha$ -continuous, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq V$. Since $F(z)$ is $\sigma_1\sigma_2$ -paracompact and $\sigma_1\sigma_2$ -regular for each $z \in U$, by Lemma 5, there exists a $\sigma_1\sigma_2$ -open set H such that $F(z) \subseteq H \subseteq \sigma_1\sigma_2\text{-Cl}(H) \subseteq V$. By Lemma 6, we have $(\sigma_1, \sigma_2)\alpha\text{-Cl}(F(z)) \subseteq \sigma_1\sigma_2\text{-Cl}(H) \subseteq V$ for each $z \in U$, and hence $\alpha\text{Cl}F_{\otimes}(U) \subseteq V$. This shows that $\alpha\text{Cl}F_{\otimes}$ is upper $(\tau_1, \tau_2)\alpha$ -continuous.

Conversely, suppose that $\alpha\text{Cl}F_{\otimes}: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is upper $(\tau_1, \tau_2)\alpha$ -continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$ -open subset of Y such that $F(x) \subseteq V$. By Lemma 7, we have $x \in F^+(V) = \alpha\text{Cl}F_{\otimes}^+(V)$, and hence $\alpha\text{Cl}F_{\otimes}(x) \subseteq V$. Since $\alpha\text{Cl}F_{\otimes}$ is upper $(\tau_1, \tau_2)\alpha$ -continuous, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $\alpha\text{Cl}F_{\otimes}(U) \subseteq V$; hence, $F(U) \subseteq V$. This shows that F is upper $(\tau_1, \tau_2)\alpha$ -continuous. $\square$

Lemma 8. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, it follows that for each $(\sigma_1, \sigma_2)\alpha$ -open subset V of Y $\alpha\text{Cl}F_{\otimes}^-(V) = F^-(V)$.

Proof. Suppose that V is any $(\sigma_1, \sigma_2)\alpha$ -open subset of Y . Let $x \in \alpha\text{Cl}F_{\otimes}^-(V)$. Then, we have $(\sigma_1, \sigma_2)\alpha\text{-Cl}(F(x)) \cap V \neq \emptyset$, and hence $F(x) \cap V \neq \emptyset$. Thus, $x \in F^-(V)$. This shows that $\alpha\text{Cl}F_{\otimes}^-(V) \subseteq F^-(V)$. On the other hand, let $x \in F^-(V)$. Then, we have $\emptyset \neq F(x) \cap V \subseteq (\sigma_1, \sigma_2)\alpha\text{-Cl}(F(x)) \cap V$, and hence $x \in \alpha\text{Cl}F_{\otimes}^-(V)$. Therefore, $F^-(V) \subseteq \alpha\text{Cl}F_{\otimes}^-(V)$. Consequently, we obtain $\alpha\text{Cl}F_{\otimes}^-(V) = F^-(V)$. $\square$

Theorem 8. A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is lower $(\tau_1, \tau_2)\alpha$ -continuous if and only if $\alpha\text{Cl}F_{\otimes}: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is lower $(\tau_1, \tau_2)\alpha$ -continuous.

Proof. By utilizing Lemma 8, this can be proved similarly to that of Theorem 7. $\square$

Definition 10. A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$ -compact [20] (resp. $(\tau_1, \tau_2)\alpha$ -compact) if every cover of X by $\tau_1\tau_2$ -open (resp. $(\tau_1, \tau_2)\alpha$ -open) sets of X has a finite subcover.

Theorem 9. Let $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be an upper $(\tau_1, \tau_2)\alpha$ -continuous surjective multifunction such that $F(x)$ is $\sigma_1\sigma_2$ -compact for each $x \in X$. If (X, τ_1, τ_2) is $(\tau_1, \tau_2)\alpha$ -compact, then (Y, σ_1, σ_2) is $\sigma_1\sigma_2$ -compact.

Proof. Suppose that (X, τ_1, τ_2) is $(\tau_1, \tau_2)\alpha$ -compact. Let $\{V_\gamma\}_{\gamma \in \Gamma}$ be a $\sigma_1\sigma_2$ -open cover of Y . For each $x \in X$, $F(x)$ is $\sigma_1\sigma_2$ -compact, there exists a finite subset $\Gamma(x)$ of Γ such that

$F(x) \subseteq \cup\{V_\gamma | \gamma \in \Gamma(x)\}$. Put $V(x) = \cup\{V_\gamma | \gamma \in \Gamma(x)\}$. Since F is upper $(\tau_1, \tau_2)\alpha$ -continuous, there exists a $(\tau_1, \tau_2)\alpha$ -open set $U(x)$ containing x such that $F(U(x)) \subseteq V(x)$. The family $\{U(x) | x \in X\}$ is a $(\tau_1, \tau_2)\alpha$ -open cover of X and there exists a finite number of points, say, $x_1, x_2, \dots, x_n$ in X such that $X = \cup_{i=1}^n U(x_i)$. Thus, $Y = F(X) = F(\cup_{i=1}^n U(x_i)) = \cup_{i=1}^n F(U(x_i)) \subseteq \cup_{i=1}^n V(x_i) = \cup_{i=1}^n \cup_{\gamma \in \Gamma(x_i)} V_\gamma$. This shows that (Y, σ_1, σ_2) is $\sigma_1\sigma_2$ -compact. $\square$

4. Characterizations of Upper and Lower Almost $(\tau_1, \tau_2)\alpha$ -Continuous Multifunctions

In this section, we introduce the notions of upper and lower almost $(\tau_1, \tau_2)\alpha$ -continuous multifunctions and investigate some characterizations of these multifunctions.

Definition 11. A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be:

- (1) upper almost $(\tau_1, \tau_2)\alpha$ -continuous at a point $x \in X$ if for each $\tau_1\tau_2$ -semi-open subset U of X containing x and each $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$, there exists a nonempty $(\tau_1, \tau_2)\alpha$ -open subset G of X such that $G \subseteq U$ and $F(G) \subseteq \sigma_1\sigma_2\text{-sCl}(V)$;
- (2) lower almost $(\tau_1, \tau_2)\alpha$ -continuous at a point $x \in X$ if for each $\tau_1\tau_2$ -semi-open subset U of X containing x and each $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$, there exists a nonempty $(\tau_1, \tau_2)\alpha$ -open subset G of X such that $G \subseteq U$ and $\sigma_1\sigma_2\text{-sCl}(V) \cap F(z) \neq \emptyset$ for every $z \in G$;
- (3) upper almost (resp. lower almost) $(\tau_1, \tau_2)\alpha$ -continuous if F has this property at each point of X .

Remark 2. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following implication holds:

upper $(\tau_1, \tau_2)\alpha$ -continuity $\implies$ upper almost $(\tau_1, \tau_2)\alpha$ -continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 2. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{c\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{c\}, X\}$. Let $Y = \{-2, -1, 0, 1, 2\}$ with topologies $\sigma_1 = \{\emptyset, \{-1, 0, 1\}, \{-2, 2\}, Y\}$ and $\sigma_2 = \{\emptyset, \{-1, 0, 1\}, Y\}$. Define a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ as follows:

$$F(x) = \begin{cases} \{0\}, & \text{if } x = a, \\ \{-1, 1\}, & \text{if } x = b, \\ \{-2, 2\}, & \text{if } x = c. \end{cases} \quad (23)$$

Then, F is upper almost $(\tau_1, \tau_2)\alpha$ -continuous but F is not upper $(\tau_1, \tau_2)\alpha$ -continuous.

Theorem 10. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is upper almost $(\tau_1, \tau_2)\alpha$ -continuous at $x \in X$

- (2) For every $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-sCl}(V)$
- (3) $x \in (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$
- (4) $x \in \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))))$ for every $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$

Proof

(1) $\implies$ (2): Let $\tau_1\tau_2\text{-SO}(X, x)$ be the family of all $\tau_1\tau_2$ -semi-open subsets of X containing x . Let V be any $\sigma_1\sigma_2$ -open subset of Y containing $F(x)$. For each $U \in \tau_1\tau_2\text{-SO}(X, x)$, there exists a nonempty $\tau_1\tau_2$ -open subset G_U such that $G_U \subseteq U$ and $F(G_U) \subseteq \sigma_1\sigma_2\text{-sCl}(V)$. Let $W = \cup\{G_U | U \in \tau_1\tau_2\text{-SO}(X, x)\}$. Then, W is $\tau_1\tau_2$ -open, $x \in \tau_1\tau_2\text{-sCl}(W)$ and $F(W) \subseteq \sigma_1\sigma_2\text{-sCl}(V)$. Put $H = W \cup \{x\}$; by Lemma 1,

$$\begin{aligned} W &\subseteq H \subseteq \tau_1\tau_2\text{-sCl}(H) = \tau_1\text{-Int}(\tau_2\text{-Cl}(H)) \cup H \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W \cup \{x\})) \cup (W \cup \{x\}) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W) \cup \tau_2\text{-Cl}(\{x\})) \cup (W \cup \{x\}) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W)) \cup (W \cup \{x\}) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W)) = \tau_1\tau_2\text{-sCl}(W), \end{aligned} \quad (24)$$

and by Lemma 2, H is a $(\tau_1, \tau_2)\alpha$ -open subset of X containing x such that $F(H) \subseteq \sigma_1\sigma_2\text{-sCl}(V)$.

(2) $\implies$ (3): Let V be any $\sigma_1\sigma_2$ -open subset of Y containing $F(x)$. By (2), there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$, and hence $x \in (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$.

(3) $\implies$ (4): Let V be any $\sigma_1\sigma_2$ -open subset of Y containing $F(x)$. By (3), we have $x \in (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$. Put $U = (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$; then, U is a $(\tau_1, \tau_2)\alpha$ -open subset of X such that $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-sCl}(V))$. Consequently, we obtain $x \in \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))))$.

(4) $\implies$ (1): Let U be any $\tau_1\tau_2$ -semi-open subset of X containing x and V be any $\sigma_1\sigma_2$ -open subset of Y containing $F(x)$. By (4) and Lemma 1 (2),

$$\begin{aligned} x &\in \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V))))) \\ &= \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))). \end{aligned} \quad (25)$$

Thus, $\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V))) \cap U \neq \emptyset$ and $\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V))) \cap U$ is $\tau_1\tau_2$ -semi-open. Put $G = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V))) \cap U)$; then, G is a nonempty $\tau_1\tau_2$ -open subset of X such that $G \subseteq U$ and $F(G) \subseteq \sigma_1\sigma_2\text{-sCl}(V)$. This shows that F is upper almost $(\tau_1, \tau_2)\alpha$ -continuous at x . $\square$

Theorem 11. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is lower almost $(\tau_1, \tau_2)\alpha$ -continuous at $x \in X$
- (2) For every $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $\sigma_1\sigma_2\text{-sCl}(V) \cap F(z) \neq \emptyset$ for every $z \in U$
- (3) $x \in (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$

- (4) $x \in \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))))$ for every $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$

Proof. The proof is similar to that of Theorem 10. $\square$

Lemma 9. Let (X, τ_1, τ_2) be a bitopological space. If V is $\tau_1\tau_2$ -open in X , then $\tau_1\tau_2\text{-sCl}(V)$ is $\tau_1\tau_2$ -regular open.

Proof. Let V be a $\tau_1\tau_2$ -open set. By Lemma 1 (2), $\tau_1\tau_2\text{-sCl}(V) = \tau_1\text{-Int}(\tau_2\text{-Cl}(V))$, and hence

$$\begin{aligned}\tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-sCl}(V))) &= \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(\tau_2\text{-Cl}(V)))) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(V)) = \tau_1\tau_2\text{-sCl}(V).\end{aligned}\tag{26}$$

Thus, $\tau_1\tau_2\text{-sCl}(V)$ is $\tau_1\tau_2$ -regular open. $\square$

Theorem 12. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is upper almost $(\tau_1, \tau_2)\alpha$ -continuous
- (2) For each $x \in X$ and each $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-sCl}(V)$
- (3) For each $x \in X$ and each $\sigma_1\sigma_2$ -regular open subset V of Y containing $F(x)$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq V$
- (4) $F^+(V)$ is $(\tau_1, \tau_2)\alpha$ -open in X for every $\sigma_1\sigma_2$ -regular open subset V of Y
- (5) $F^-(K)$ is $(\tau_1, \tau_2)\alpha$ -closed in X for every $\sigma_1\sigma_2$ -regular closed subset K of Y
- (6) $F^+(V) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (7) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2\text{-sInt}(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y
- (8) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(K)))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y
- (9) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y
- (10) $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(K))))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y
- (11) $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-sInt}(K))))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y
- (12) $F^+(V) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(K)))))$ for every $\sigma_1\sigma_2$ -open subset V of Y

exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq V$. Thus, $x \in \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(U))) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(V))))$, and hence $F^+(V) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(V))))$. This shows that $F^+(V)$ is $(\tau_1, \tau_2)\alpha$ -open in X .

(4) $\implies$ (5): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y .

(5) $\implies$ (6): Let V be any $\sigma_1\sigma_2$ -open subset of Y and $x \in F^+(V)$. Then, we have $F(x) \subseteq V \subseteq \sigma_1\sigma_2\text{-sCl}(V)$, and hence

$$x \in F^+(\sigma_1\sigma_2\text{-sCl}(V)) = X - F^-(Y - \sigma_1\sigma_2\text{-sCl}(V)).\tag{27}$$

By Lemma 9, $Y - \sigma_1\sigma_2\text{-sCl}(V)$ is $\sigma_1\sigma_2$ -regular closed in Y , and by (5), $F^-(Y - \sigma_1\sigma_2\text{-sCl}(V))$ is $(\tau_1, \tau_2)\alpha$ -closed in X . This implies that $F^+(\sigma_1\sigma_2\text{-sCl}(V))$ is $(\tau_1, \tau_2)\alpha$ -open in X , and hence $x \in (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$. Consequently, we obtain $F^+(V) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$.

(6) $\implies$ (7): Let K be any $\sigma_1\sigma_2$ -closed subset of Y . Then, we have $Y - K$ is $\sigma_1\sigma_2$ -open in Y and by (6),

$$\begin{aligned}X - F^-(K) &= F^+(Y - K) \\ &\subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(Y - K))) \\ &= (\tau_1, \tau_2)\alpha\text{-Int}(F^+(Y - \sigma_1\sigma_2\text{-sInt}(K))) \\ &= (\tau_1, \tau_2)\alpha\text{-Int}(X - F^-(\sigma_1\sigma_2\text{-sInt}(K))) \\ &= X - (\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2\text{-sInt}(K))).\end{aligned}\tag{28}$$

Therefore, $(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2\text{-sInt}(K))) \subseteq F^-(K)$.

(7) $\implies$ (8): The proof is obvious since $\sigma_1\sigma_2\text{-sInt}(K) = \sigma_1\text{-Cl}(\sigma_2\text{-Int}(K))$ for every $\sigma_1\sigma_2$ -closed subset K of Y .

(8) $\implies$ (9): The proof is obvious.

(9) $\implies$ (10): Let K be any $\sigma_1\sigma_2$ -closed subset of Y . By (9) and Lemma 3 (3), we obtain

Proof

- (1) $\implies$ (2): The proof follows from Theorem 10.
- (2) $\implies$ (3): This is obvious.
- (3) $\implies$ (4): Let V be any $\sigma_1\sigma_2$ -regular open subset of Y and $x \in F^+(V)$. Then, we have $F(x) \subseteq V$. By (3), there

$$\begin{aligned}
& \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(K)))))) \\
& \subseteq (\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(K)))) \\
& = (\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(K)))))) \\
& \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(K)) = F^-(K).
\end{aligned} \tag{29}$$

(10) $\implies$ (11): The proof is obvious since $\sigma_1\sigma_2\text{-sInt}(K) = \sigma_1\text{-Cl}(\sigma_2\text{-Int}(K))$ for every $\sigma_1\sigma_2$ -closed subset K of Y .

(11) $\implies$ (12): Let V be any $\sigma_1\sigma_2$ -open subset of Y . Then, we have $Y - V$ is $\sigma_1\sigma_2$ -closed in Y and by (11),

$$\begin{aligned}
& \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-sInt}(Y - V)))))) \\
& \subseteq F^-(Y - V) = X - F^+(V).
\end{aligned} \tag{30}$$

Moreover, we have

$$\begin{aligned}
& \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-sInt}(Y - V)))))) \\
& = \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-sCl}(V)))))) \\
& = \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(X - F^+(\sigma_1\sigma_2\text{-sCl}(V)))))) \\
& = X - \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V))))),
\end{aligned} \tag{31}$$

and hence $F^+(V) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V))))))$.

(12) $\implies$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$ -open subset of Y containing $F(x)$. By (12), we have $x \in F^+(V) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V))))))$, and hence F is upper almost $(\tau_1, \tau_2)\alpha$ -continuous at x by Theorem 10. Consequently, we obtain F is upper almost $(\tau_1, \tau_2)\alpha$ -continuous. $\square$

Theorem 13. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is lower almost $(\tau_1, \tau_2)\alpha$ -continuous
- (2) For each $x \in X$ and each $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $U \subseteq F(\sigma_1\sigma_2\text{-sCl}(V))$
- (3) For each $x \in X$ and each $\sigma_1\sigma_2$ -regular open subset V of Y such that

$$F(x) \cap V \neq \emptyset, \tag{32}$$

there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $U \subseteq F(V)$

- (4) $F(V)$ is $(\tau_1, \tau_2)\alpha$ -open in X for every $\sigma_1\sigma_2$ -regular open subset V of Y
- (5) $F^+(K)$ is $(\tau_1, \tau_2)\alpha$ -closed in X for every $\sigma_1\sigma_2$ -regular closed subset K of Y
- (6) $F(V) \subseteq \alpha(\tau_1, \tau_2)\text{-Int}(F(\sigma_1\sigma_2\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (7) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(\sigma_1\sigma_2\text{-sInt}(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y

(8) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(K)))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y

(9) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y

(10) $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^+(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(K)))))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y

(11) $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-sInt}(K)))))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y

(12) $F(V) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F(\sigma_1\sigma_2\text{-sCl}(K))))))$ for every $\sigma_1\sigma_2$ -open subset V of Y

Proof. The proof is similar to that of Theorem 12. $\square$

Definition 12. A function $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be almost $(\tau_1, \tau_2)\alpha$ -continuous if $f^{-1}(V)$ is $(\tau_1, \tau_2)\alpha$ -open in X for every $\sigma_1\sigma_2$ -regular open subset V of Y .

Corollary 2. For a function $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) f is almost $(\tau_1, \tau_2)\alpha$ -continuous
- (2) For each $x \in X$ and each $\sigma_1\sigma_2$ -open subset V of Y containing $f(x)$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-sCl}(V)$
- (3) For each $x \in X$ and each $\sigma_1\sigma_2$ -regular open subset V of Y containing $f(x)$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $f(U) \subseteq V$
- (4) For each $x \in X$ and each $\sigma_1\sigma_2$ -open subset V of Y containing $f(x)$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $f(U) \subseteq \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))$
- (5) $f^{-1}(F)$ is $(\tau_1, \tau_2)\alpha$ -closed in X for every $\sigma_1\sigma_2$ -regular closed subset F of Y

- (6) $f^{-1}(V) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (7) $(\tau_1, \tau_2)\alpha\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-sInt}(F))) \subseteq f^{-1}(F)$ for every $\sigma_1\sigma_2$ -closed subset F of Y
- (8) $(\tau_1, \tau_2)\alpha\text{-Cl}(f^{-1}(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(F)))) \subseteq f^{-1}(F)$ for every $\sigma_1\sigma_2$ -closed subset F of Y
- (9) $(\tau_1, \tau_2)\alpha\text{-Cl}(f^{-1}(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y
- (10) $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(F))))) \subseteq f^{-1}(F)$ for every $\sigma_1\sigma_2$ -closed subset F of Y
- (11) $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-sInt}(F))))) \subseteq f^{-1}(F)$ for every $\sigma_1\sigma_2$ -closed subset F of Y
- (12) $f^{-1}(V) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-sCl}(V)))))$ for every $\sigma_1\sigma_2$ -open subset V of Y

Let $\{(X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \mid \gamma \in \Gamma\}$ be a family of bitopological spaces. Let $(X^*, \tau_1^*, \tau_2^*)$ be the product space, where $X^* = \prod_{\gamma \in \Gamma} X_\gamma$ and τ_i^* denotes the product topology of $\{\tau_i(\gamma) \mid \gamma \in \Gamma\}$ for $i = 1, 2$.

Lemma 10. Let $A_{\gamma k}$ be a nonempty subset of $X_{\gamma k}$ for $k = 1, 2, \dots, n$. Then, $\prod_{k=1}^n A_{\gamma k} \times \prod_{\gamma \neq \gamma_0} X_\gamma$ is $(\tau_1^*, \tau_2^*)\alpha$ -open in X^* if and only if $A_{\gamma k}$ is $(\tau_1(\gamma_k), \tau_2(\gamma_k))\alpha$ -open in $X_{\gamma k}$ for each $k = 1, 2, \dots, n$.

Proof. The proof is similar to that of [23] (Theorem 4.6) $\square$

Let $\{(X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \mid \gamma \in \Gamma\}$ and $\{(Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma)) \mid \gamma \in \Gamma\}$ be two arbitrary families of bitopological spaces with the same set of indices. Let $F_\gamma: (X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \longrightarrow (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$ be a multifunction for each $\gamma \in \Gamma$. Let $F^*: (X^*, \tau_1^*, \tau_2^*) \longrightarrow (Y^*, \sigma_1^*, \sigma_2^*)$ be the product multifunction defined by $F^*(x^*) = \prod_{\gamma \in \Gamma} \{F_\gamma(x_\gamma)\}$ for each $x^* = \{x_\gamma\} \in \prod_{\gamma \in \Gamma} X_\gamma$, where τ_i^* and σ_i^* denote the product topologies for $i = 1, 2$.

Theorem 14. If $F^*: (X^*, \tau_1^*, \tau_2^*) \longrightarrow (Y^*, \sigma_1^*, \sigma_2^*)$ is upper almost $(\tau_1^*, \tau_2^*)\alpha$ -continuous, then

$$F_\gamma: (X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \longrightarrow (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma)), \quad (33)$$

is upper almost $(\tau_1(\gamma), \tau_2(\gamma))\alpha$ -continuous for each $\gamma \in \Gamma$.

Proof. Let V_γ be a $\sigma_1(\gamma)\sigma_2(\gamma)$ -regular open subset of Y_γ . Since F^* is upper almost $(\tau_1^*, \tau_2^*)\alpha$ -continuous,

$$F^{*+} \left(V_\gamma \times \prod_{\gamma \neq \gamma_0} Y_{\gamma_0} \right) = F_\gamma^+(V_\gamma) \times \prod_{\gamma \neq \gamma_0} Y_{\gamma_0}, \quad (34)$$

is $(\tau_1^*, \tau_2^*)\alpha$ -open in X^* , and by Lemma 10, $F_\gamma^+(V_\gamma)$ is $(\tau_1(\gamma), \tau_2(\gamma))\alpha$ -open in X_γ . Thus, F_γ is upper almost $(\tau_1(\gamma), \tau_2(\gamma))\alpha$ -continuous. $\square$

Theorem 15. If $F^*: (X^*, \tau_1^*, \tau_2^*) \longrightarrow (Y^*, \sigma_1^*, \sigma_2^*)$ is lower almost $(\tau_1^*, \tau_2^*)\alpha$ -continuous, then

$$F_\gamma: (X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \longrightarrow (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma)), \quad (35)$$

is lower almost $(\tau_1(\gamma), \tau_2(\gamma))\alpha$ -continuous for each $\gamma \in \Gamma$.

Proof. The proof is similar to that of Theorem 14. $\square$

Theorem 16. Let (X, τ_1, τ_2) and $(Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$ be bitopological spaces for each $\gamma \in \Gamma$. Let $F_\gamma: (X, \tau_1, \tau_2) \longrightarrow (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$ be a multifunction for each $\gamma \in \Gamma$ and $F: (X, \tau_1, \tau_2) \longrightarrow (Y^*, \sigma_1^*, \sigma_2^*)$ a multifunction defined by $F(x) = \prod_{\gamma \in \Gamma} \{F_\gamma(x)\}$ for each $x \in X$. If F is upper almost $(\tau_1, \tau_2)\alpha$ -continuous, then F_γ is upper almost $(\tau_1, \tau_2)\alpha$ -continuous for each $\gamma \in \Gamma$.

Proof. Let $x \in X$, $\gamma \in \Gamma$, and V_γ be any $\sigma_1(\gamma)\sigma_2(\gamma)$ -regular open subset of Y_γ containing $F_\gamma(x)$. Then, $\pi_\gamma^{-1}(V_\gamma) = V_\gamma \times \prod_{\gamma \neq \gamma_0} Y_{\gamma_0}$ is $\sigma_1^*\sigma_2^*$ -regular open in $\prod_{\gamma \in \Gamma} Y_\gamma$ containing $F(x)$, where $\pi_\gamma: (Y^*, \sigma_1^*, \sigma_2^*) \longrightarrow (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$ is the projection for each $\gamma \in \Gamma$. Since F is upper almost $(\tau_1, \tau_2)\alpha$ -continuous, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq \pi_\gamma^{-1}(V_\gamma)$. Therefore, $F_\gamma(U) \subseteq \pi_\gamma(F(U)) \subseteq \pi_\gamma(\pi_\gamma^{-1}(V_\gamma)) = V_\gamma$. This shows that F_γ is upper almost $(\tau_1, \tau_2)\alpha$ -continuous for each $\gamma \in \Gamma$. $\square$

Theorem 17. Let (X, τ_1, τ_2) and $(Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$ be bitopological spaces for each $\gamma \in \Gamma$. Let $F_\gamma: (X, \tau_1, \tau_2) \longrightarrow (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$ be a multifunction for each $\gamma \in \Gamma$ and $F: (X, \tau_1, \tau_2) \longrightarrow (Y^*, \sigma_1^*, \sigma_2^*)$ a multifunction defined by $F(x) = \prod_{\gamma \in \Gamma} \{F_\gamma(x)\}$ for each $x \in X$. If F is lower almost $(\tau_1, \tau_2)\alpha$ -continuous, then F_γ is lower almost $(\tau_1, \tau_2)\alpha$ -continuous for each $\gamma \in \Gamma$.

Proof. The proof is similar to that of Theorem 16. $\square$

5. Characterizations of Upper and Lower Weakly $(\tau_1, \tau_2)\alpha$ -Continuous Multifunctions

In this section, we introduce the concepts of upper and lower weakly $(\tau_1, \tau_2)\alpha$ -continuous multifunctions. Furthermore, some characterizations of upper and lower weakly $(\tau_1, \tau_2)\alpha$ -continuous multifunctions are investigated.

Definition 13. A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be:

- (1) upper weakly $(\tau_1, \tau_2)\alpha$ -continuous at a point $x \in X$ if for each $\tau_1\tau_2$ -semi-open subset U of X containing x and each $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$, there exists a nonempty $\tau_1\tau_2$ -open subset G of X such that $G \subseteq U$ and $F(G) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$;
- (2) lower weakly $(\tau_1, \tau_2)\alpha$ -continuous at a point $x \in X$ if for each $\tau_1\tau_2$ -semi-open subset U of X containing x and each $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$, there exists a nonempty $\tau_1\tau_2$ -open subset G of X such that $G \subseteq U$ and $F(z) \cap \sigma_1\sigma_2\text{-Cl}(V) \neq \emptyset$ for every $z \in G$;
- (3) upper weakly (resp. lower weakly) $(\tau_1, \tau_2)\alpha$ -continuous if F has this property at each point of X .

Remark 3. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following implication holds:

upper almost $(\tau_1, \tau_2)\alpha$ -continuity $\implies$ upper weakly $(\tau_1, \tau_2)\alpha$ -continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 3. Let $X = \{a, b, c, d\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, X\}$ and $\tau_2 = \{\emptyset, X\}$. Let $Y = \{-1, 0, 1, 2\}$ with topologies $\sigma_1 = \{\emptyset, \{-1, 2\}, \{0, 1\}, Y\}$, and $\sigma_2 = \{\emptyset, \{-1, 2\}, Y\}$. A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is defined as follows:

$$F(x) = \begin{cases} \{-1\}, & \text{if } x = a, \\ \{1\}, & \text{if } x = b, \\ \{0, 1\}, & \text{if } x = c, \\ \{-1, 0\}, & \text{if } x = d. \end{cases} \quad (36)$$

Then, F is upper weakly $(\tau_1, \tau_2)\alpha$ -continuous but F is not upper almost $(\tau_1, \tau_2)\alpha$ -continuous.

Theorem 18. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is upper weakly $(\tau_1, \tau_2)\alpha$ -continuous at $x \in X$

- (2) For any $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$
- (3) $x \in (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_1\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$
- (4) $x \in \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_1\text{-Cl}(V)))))$ for every $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$

Proof

(1) $\implies$ (2): Let $\tau_1\tau_2\text{-SO}(X, x)$ be the family of all $\tau_1\tau_2$ -semi-open subsets of X containing x . Let V be any $\sigma_1\sigma_2$ -open subset of Y containing $F(x)$. For each $U \in \tau_1\tau_2\text{-SO}(X, x)$, there exists a nonempty $\tau_1\tau_2$ -open set H_U such that $H_U \subseteq U$ and $F(H_U) \subseteq V$. Let $W = \bigcup \{H_U \mid U \in \tau_1\tau_2\text{-SO}(X, x)\}$. Then, W is $\tau_1\tau_2$ -open in X , $x \in \tau_1\tau_2\text{-sCl}(W)$, and $F(W) \subseteq V$. Put $G = W \cup \{x\}$. By Lemma 1, we have

$$\begin{aligned} W \subseteq G \subseteq \tau_1\tau_2\text{-sCl}(G) &= \tau_1\text{-Int}(\tau_2\text{-Cl}(G)) \cup G \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W \cup \{x\})) \cup (W \cup \{x\}) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W) \cup \tau_2\text{-Cl}(\{x\})) \cup (W \cup \{x\}) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W)) \cup (W \cup \{x\}) \\ &= \tau_1\text{-Int}(\tau_2\text{-Cl}(W)) \\ &= \tau_1\tau_2\text{-sCl}(W), \end{aligned} \quad (37)$$

and by Lemma 2, G is a $(\tau_1, \tau_2)\alpha$ -open subset of X containing x such that $F(G) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$.

(2) $\implies$ (3): Let V be any $\sigma_1\sigma_2$ -open subset of Y containing $F(x)$. By (2), there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This implies that $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$, and hence

$$x \in (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_1\text{-Cl}(V))). \quad (38)$$

(3) $\implies$ (4): Let V be any $\sigma_1\sigma_2$ -open subset of Y . By (3), we have

$$x \in (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_1\text{-Cl}(V))). \quad (39)$$

Then, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X such that $x \in U \subseteq F^+(\sigma_1\sigma_1\text{-Cl}(V))$. Thus, $x \in \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_1\text{-Cl}(V)))))$.

(4) $\implies$ (1): Let U be any $\tau_1\tau_2$ -semi-open set containing x and V be any $\sigma_1\sigma_2$ -open set containing $F(x)$. By (4) and Lemma 1 (2), we have

$$\begin{aligned} x \in \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_1\text{-Cl}(V))))) \\ = \tau_1\tau_2\text{-sCl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_1\text{-Cl}(V)))). \end{aligned} \quad (40)$$

Thus, $\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_1\text{-Cl}(V))) \cap U \neq \emptyset$ and $\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_1\text{-Cl}(V))) \cap U$ is $\tau_1\tau_2$ -semi-open. Put $G = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_1\text{-Cl}(V))) \cap U)$; then G is a nonempty $\sigma_1\sigma_2$ -open subset of X such that $G \subseteq U$ and $F(G) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that F is upper weakly $(\tau_1, \tau_2)\alpha$ -continuous at x . $\square$

Theorem 19. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is lower weakly $(\tau_1, \tau_2)\alpha$ -continuous at $x \in X$
- (2) For any $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for every $z \in U$
- (3) $x \in (\tau_1, \tau_2)\alpha\text{-Int}(F^-(\sigma_1\sigma_1\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$
- (4) $x \in \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^-(\sigma_1\sigma_1\text{-Cl}(V)))))$ for every $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$

Proof. The proof is similar to that of Theorem 18. $\square$

Definition 14. Let A be a subset of a bitopological space (X, τ_1, τ_2) . A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$ -cluster point of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$ -open set U containing x .

The set of all $(\tau_1, \tau_2)\theta$ -cluster point of A is called the $(\tau_1, \tau_2)\theta$ -closure of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Cl}(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$ -closed if $A = (\tau_1, \tau_2)\theta\text{-Cl}(A)$. The complement of a $(\tau_1, \tau_2)\theta$ -closed set is said to be $(\tau_1, \tau_2)\theta$ -open. The union of all $(\tau_1, \tau_2)\theta$ -open sets contained in A is called the $(\tau_1, \tau_2)\theta$ -interior of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Int}(A)$.

Lemma 11. For a subset A of a bitopological space (X, τ_1, τ_2) , the following properties hold:

- (1) If A is $\tau_1\tau_2$ -open in X , then $\tau_1\tau_2\text{-Cl}(A) = (\tau_1, \tau_2)\theta\text{-Cl}(A)$
- (2) $(\tau_1, \tau_2)\theta\text{-Cl}(A)$ is $\tau_1\tau_2$ -closed in X

Proof

- (1) In general, $\tau_1\tau_2\text{-Cl}(A) \subseteq (\tau_1, \tau_2)\theta\text{-Cl}(A)$. Suppose that $x \notin \tau_1\tau_2\text{-Cl}(A)$. Then, there exists a $\tau_1\tau_2$ -open set U containing x such that $U \cap A = \emptyset$; hence, $\tau_1\tau_2\text{-Cl}(U) \cap A = \emptyset$. This shows that $x \notin (\tau_1, \tau_2)\theta\text{-Cl}(A)$. Thus, $(\tau_1, \tau_2)\theta\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(A)$. Consequently, we obtain $\tau_1\tau_2\text{-Cl}(A) = (\tau_1, \tau_2)\theta\text{-Cl}(A)$.
- (2) Let $x \in X - (\tau_1, \tau_2)\theta\text{-Cl}(A)$. Then, we have $x \notin (\tau_1, \tau_2)\theta\text{-Cl}(A)$. There exists a $\tau_1\tau_2$ -open set U_x containing x such that $\tau_1\tau_2\text{-Cl}(U_x) \cap A = \emptyset$. This implies that $(\tau_1, \tau_2)\theta\text{-Cl}(A) \cap U_x = \emptyset$, and hence $x \in U_x \subseteq X - (\tau_1, \tau_2)\theta\text{-Cl}(A)$. Thus, $X - (\tau_1, \tau_2)\theta\text{-Cl}(A) = \bigcup_{x \in X - (\tau_1, \tau_2)\theta\text{-Cl}(A)} U_x$. This shows that $(\tau_1, \tau_2)\theta\text{-Cl}(A)$ is $\tau_1\tau_2$ -closed. $\square$

Definition 15. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$ -closed (resp. $(\tau_1, \tau_2)r$ -open) if $A = \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$ (resp. $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$).

Theorem 20. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is upper weakly $(\tau_1, \tau_2)\alpha$ -continuous
- (2) For each $x \in X$ and each $\sigma_1\sigma_2$ -open subset V of Y containing $F(x)$, there exists a $(\tau_1, \tau_2)\alpha$ -open

subset U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$

- (3) $F^+(V) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (4) $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y
- (5) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y
- (6) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y
- (7) $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$ for every subset B of Y
- (8) $F^+(V) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (9) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$ for every $(\sigma_1, \sigma_2)r$ -closed subset K of Y
- (10) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (11) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^-(\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ for every subset B of Y
- (12) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$ -open subset V of Y

Proof

(1) $\implies$ (2): The proof follows immediately from Theorem 18.

(2) $\implies$ (3): Let V be any $\sigma_1\sigma_2$ -open subset of Y and $x \in F^+(V)$. Then, we have $F(x) \subseteq V$ and by (2), there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Therefore, we have $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$, and hence $x \in U \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))))$. Consequently, we obtain $F^+(V) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))))$.

(3) $\implies$ (4): Let K be any $\sigma_1\sigma_2$ -closed subset of Y . Then, we have $Y - K$ is $\sigma_1\sigma_2$ -open in Y and by (3),

$$\begin{aligned}
 X - F^-(K) &= F^+(Y - K) \\
 &\subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(Y - K))))) \\
 &= \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(Y - \sigma_1\sigma_2\text{-Int}(K))))) \\
 &= \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(X - F^-(\sigma_1\sigma_2\text{-Int}(K))))) \\
 &= \tau_1\text{-Int}(\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))))) \\
 &= \tau_1\text{-Int}(X - \tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))))) \\
 &= X - \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K)))))
 \end{aligned} \tag{41}$$

Thus, $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F(\sigma_1\sigma_2\text{-Int}(K)))))) \subseteq F(K)$.
 (4) $\implies$ (5): Let K be any $\sigma_1\sigma_2$ -closed subset of Y . By (4), we have

$$\begin{aligned} \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K)))))) \\ \subseteq F^-(K), \end{aligned} \quad (42)$$

and hence $(\tau_1, \tau_2)\alpha\text{-Cl}(F(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F(K)$ by Lemma 3 (3).

(5) $\implies$ (6): Let B be any subset of Y . Then, we have $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$ -closed in Y , and by (5), $(\tau_1, \tau_2)\alpha\text{-Cl}(F(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F(\sigma_1\sigma_2\text{-Cl}(B))$.

(6) $\implies$ (7): Let B be any subset of Y . By (6), we have

$$\begin{aligned} F^+(\sigma_1\sigma_2\text{-Int}(B)) &= X - F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - (\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(Y - B))) \\ &= X - (\tau_1, \tau_2)\alpha\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= X - (\tau_1, \tau_2)\alpha\text{-Cl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))). \end{aligned} \quad (43)$$

(7) $\implies$ (8): The proof is obvious.

(8) $\implies$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$ -open subset of Y containing $F(x)$. By (8), we have

$$\begin{aligned} x \in F^+(V) &\subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))))), \end{aligned} \quad (44)$$

and hence F is upper weakly $(\tau_1, \tau_2)\alpha$ -continuous at x by Theorem 18. Consequently, we obtain that F is upper weakly $(\tau_1, \tau_2)\alpha$ -continuous.

(5) $\implies$ (9): The proof is obvious.

(9) $\implies$ (10): Let V be any $\sigma_1\sigma_2$ -open subset of Y . Then, we have $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)r$ -closed in Y , and by (9),

$$\begin{aligned} (\tau_1, \tau_2)\alpha\text{-Cl}(F^-(V)) &\subseteq (\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2 \\ &\quad \text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)). \end{aligned} \quad (45)$$

(10) $\implies$ (8): Let V be any $\sigma_1\sigma_2$ -open subset of Y . By (10), we have

$$\begin{aligned} X - (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) &= (\tau_1, \tau_2)\alpha\text{-Cl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= (\tau_1, \tau_2)\alpha\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))), \end{aligned} \quad (46)$$

and hence $F^+(V) \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(10) $\implies$ (11): Let B be any subset of Y . Then, $\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ is $\sigma_1\sigma_2$ -open in Y . By (10) and Lemma 11 (1),

$$\begin{aligned} &(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B)). \end{aligned} \quad (47)$$

(11) $\implies$ (12): Let V be any $\sigma_1\sigma_2$ -open subset of Y . By (11) and Lemma 11 (1), we have $(\tau_1, \tau_2)\alpha\text{-Cl}(F(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F(\sigma_1\sigma_2\text{-Cl}(V))$.

(12) $\implies$ (10): Let V be any $\sigma_1\sigma_2$ -open subset of Y . By (12), we have

$$\begin{aligned} &(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)), \end{aligned} \quad (48)$$

and hence

$$\begin{aligned} &(\tau_1, \tau_2)\alpha\text{-Cl}(F^-(V)) \subseteq (\tau_1, \tau_2)\alpha \\ &\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)). \end{aligned} \quad (49)$$

□

Theorem 21. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is lower weakly $(\tau_1, \tau_2)\alpha$ -continuous
- (2) For each $x \in X$ and each $\sigma_1\sigma_2$ -open subset V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $U \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$
- (3) $F^-(V) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^-(\sigma_1\sigma_2\text{-Cl}(V)))))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (4) $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Int}(K))))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y
- (5) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$ -closed subset K of Y
- (6) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y
- (7) $F^-(\sigma_1\sigma_2\text{-Int}(B)) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$ for every subset B of Y
- (8) $F^-(V) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(F^-(\sigma_1\sigma_2\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (9) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^+(K)$ for every $(\sigma_1, \sigma_2)r$ -closed subset K of Y
- (10) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (11) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ for every subset B of Y
- (12) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$ -open subset V of Y

Proof. The proof is similar to that of Theorem 20. □

Lemma 12. If $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is lower weakly $(\tau_1, \tau_2)\alpha$ -continuous, then for each $x \in X$ and each subset B of Y such that $(\tau_1, \tau_2)\theta\text{-Int}(B) \cap F(x) \neq \emptyset$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $U \subseteq F^-(B)$.

Proof. Suppose that $(\tau_1, \tau_2)\theta\text{-Int}(B) \cap F(x) \neq \emptyset$. Then, there exists a nonempty $\sigma_1\sigma_2$ -open subset V of Y such that $V \subseteq \sigma_1\sigma_2\text{-Cl}(B)$ and $F(x) \cap V \neq \emptyset$. Since F is lower weakly $(\tau_1, \tau_2)\alpha$ -continuous, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X

containing x such that $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for every $z \in U$, and hence $U \subseteq F^-(B)$. □

Theorem 22. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is lower weakly $(\tau_1, \tau_2)\alpha$ -continuous
- (2) $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(B)) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ for every subset B of Y
- (3) $F((\tau_1, \tau_2)\alpha\text{-Cl}(A)) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(F(A))$ for every subset A of X

Proof

(1) $\implies$ (2): Let B be any subset of Y . Suppose that $x \notin F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Then, we have

$$\begin{aligned} x &\in X - F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B)) \\ &= F^-(Y - ((\sigma_1, \sigma_2)\theta\text{-Cl}(B))) \\ &= F^-(Y - ((\sigma_1, \sigma_2)\theta\text{-Int}(Y - B))). \end{aligned} \quad (50)$$

Therefore, $F(x) \in (\sigma_1, \sigma_2)\theta\text{-Int}(Y - B)$. By Lemma 12, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $U \subseteq F^-(Y - B) = X - F^+(B)$, and hence $U \cap F^+(B) = \emptyset$. Thus, $x \notin (\tau_1, \tau_2)\alpha\text{-Cl}(F^+(B))$. Consequently, we obtain $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(B)) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$.

(2) $\implies$ (1): Let V be any $\sigma_1\sigma_2$ -open subset of Y . By (2) and Lemma 11 (1), we have $(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(V)) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(V)) = F^+(\sigma_1\sigma_2\text{-Cl}(V))$, and by Theorem 21, F is lower weakly $(\tau_1, \tau_2)\alpha$ -continuous.

(2) $\implies$ (3): Let A be any subset of X . By (2), we have $(\tau_1, \tau_2)\alpha\text{-Cl}(A) \subseteq (\tau_1, \tau_2)\alpha\text{-Cl}(F^+(F(A))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(F(A)))$. Thus, $F((\tau_1, \tau_2)\alpha\text{-Cl}(A)) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(F(A))$.

(3) $\implies$ (2): Let B be any subset of Y . By (3), we have $F((\tau_1, \tau_2)\alpha\text{-Cl}(F^+(B))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(F(F^+(B))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(B)$, and hence

$$(\tau_1, \tau_2)\alpha\text{-Cl}(F^+(B)) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B)). \quad (51)$$

□

Definition 16. A function $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be weakly $(\tau_1, \tau_2)\alpha$ -continuous if for each $x \in X$ and each $\sigma_1\sigma_2$ -open subset V containing $f(x)$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$.

Corollary 3. For a function $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) f is weakly $(\tau_1, \tau_2)\alpha$ -continuous
- (2) $f^1(V) \subseteq (\tau_1, \tau_2)\alpha\text{-Int}(f^1(\sigma_1\sigma_2\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (3) $(\tau_1, \tau_2)\alpha\text{-Cl}(f^1(\sigma_1\sigma_2\text{-Int}(K))) \subseteq f^1(K)$ for every $(\sigma_1, \sigma_2)r$ -closed subset K of Y
- (4) $(\tau_1, \tau_2)\alpha\text{-Cl}(f^1(V)) \subseteq f^1(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$ -open subset V of Y

- (5) $(\tau_1, \tau_2)\alpha\text{-Cl}(f^1(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq f^1((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ for every subset B of Y
- (6) $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(f^1(V)))) \subseteq f^1(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (7) $f^1(V) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(f^1(\sigma_1\sigma_2\text{-Cl}(V)))))$ for every $\sigma_1\sigma_2$ -open subset V of Y
- (8) $f(\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(f(A))$ for every subset A of X
- (9) $\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(f^1(B)))) \subseteq f^1((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ for every subset B of Y

Definition 17. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$ -S-closed relative to X if for every cover $\{V_\gamma | \gamma \in \Gamma\}$ of A by $\tau_1\tau_2$ -open sets of X , there exists a finite subset Γ_0 of Γ such that $A \subseteq \cup\{\tau_1\tau_2\text{-Cl}(V_\gamma) | \gamma \in \Gamma_0\}$. If X is $\tau_1\tau_2$ -S-closed relative to X , then the space (X, τ_1, τ_2) is called $\tau_1\tau_2$ -S-closed.

Theorem 23. Let $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be an upper weakly $(\tau_1, \tau_2)\alpha$ -continuous surjective multifunction such that $F(x)$ is $\tau_1\tau_2$ -compact for each $x \in X$. If (X, τ_1, τ_2) is $(\tau_1, \tau_2)\alpha$ -compact, then (Y, σ_1, σ_2) is $\sigma_1\sigma_2$ -S-closed.

Proof. Let $\{V_\gamma | \gamma \in \Gamma\}$ be any $\sigma_1\sigma_2$ -open cover of Y . For each $x \in X$, $F(x)$ is $\sigma_1\sigma_2$ -compact, and hence there exists a finite subset $\Gamma(x)$ of Γ such that

$$F(x) \subseteq \left\{ V_\gamma \mid \gamma \in \Gamma(x) \right\}. \quad (52)$$

Since F is upper weakly $(\tau_1, \tau_2)\alpha$ -continuous, there exists a $(\tau_1, \tau_2)\alpha$ -open subset of X such that $F(U(x)) \subseteq \cup\{\sigma_1\sigma_2\text{-Cl}(V_\gamma) | \gamma \in \Gamma(x)\}$. Since (X, τ_1, τ_2) is $(\tau_1, \tau_2)\alpha$ -compact, there exists a finite number of points, say, $x_1, x_2, \dots, x_n$ in X such that $X = \cup\{U(x_i) | 1 \leq i \leq n\}$. Thus, $Y = F(X) = F(\cup\{U(x_i) | 1 \leq i \leq n\}) \subseteq \cup\{\sigma_1\sigma_2\text{-Cl}(V_\gamma) | \gamma \in \Gamma(x_i); 1 \leq i \leq n\}$. This shows that (Y, σ_1, σ_2) is $\sigma_1\sigma_2$ -S-closed. $\square$

Theorem 24. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ such that $F(x)$ is $\sigma_1\sigma_2$ -regular $\sigma_1\sigma_2$ -paracompact for each $x \in X$, the following properties are equivalent:

- (1) F is upper $(\tau_1, \tau_2)\alpha$ -continuous
- (2) F is upper almost $(\tau_1, \tau_2)\alpha$ -continuous
- (3) F is upper weakly $(\tau_1, \tau_2)\alpha$ -continuous

Proof. We prove only the implication (3) $\implies$ (1) since the others are obvious. Suppose that F is upper weakly $(\tau_1, \tau_2)\alpha$ -continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$ -open subset of Y such that $F(x) \subseteq V$. Since $F(x)$ is $\sigma_1\sigma_2$ -regular $\sigma_1\sigma_2$ -paracompact, there exists a $\sigma_1\sigma_2$ -open subset G of Y such that $F(x) \subseteq G \subseteq \sigma_1\sigma_2\text{-Cl}(G) \subseteq V$. Since F is upper weakly $(\tau_1, \tau_2)\alpha$ -continuous at x and $F(x) \subseteq G$, there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(G)$, and hence $F(U) \subseteq V$. This shows that F is upper $(\tau_1, \tau_2)\alpha$ -continuous. $\square$

Lemma 13. Let A be a subset of a bitopological space (X, τ_1, τ_2) . If A is $\tau_1\tau_2$ -regular, then for every $\tau_1\tau_2$ -open set G which

intersects A , there exists a $\tau_1\tau_2$ -open set W such that $A \cap W \neq \emptyset$ and $\tau_1\tau_2\text{-Cl}(W) \subseteq G$.

Proof. Suppose that A is a $\tau_1\tau_2$ -regular subset of X and G a $\tau_1\tau_2$ -open set such that $A \cap G \neq \emptyset$. Let $x \in A \cap G$. Then, we have $x \notin X - G$ and $X - G$ is $\tau_1\tau_2$ -closed. There exist disjoint $\tau_1\tau_2$ -open subsets U and V of X such that $x \in U$ and $X - G \subseteq V$, which implies $\tau_1\tau_2\text{-Cl}(V) \cap U = \emptyset$, and hence $x \notin \tau_1\tau_2\text{-Cl}(V)$. Let $W = X - \tau_1\tau_2\text{-Cl}(V)$. Thus, $x \in W$ and $A \cap W \neq \emptyset$. Consequently, we obtain $\tau_1\tau_2\text{-Cl}(W) = \tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(V)) \subseteq \tau_1\tau_2\text{-Cl}(X - V) = X - V \subseteq G$. $\square$

Theorem 25. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ such that $F(x)$ is $\sigma_1\sigma_2$ -regular for each $x \in X$, the following properties are equivalent:

- (1) F is lower $(\tau_1, \tau_2)\alpha$ -continuous
- (2) F is lower almost $(\tau_1, \tau_2)\alpha$ -continuous
- (3) F is lower weakly $(\tau_1, \tau_2)\alpha$ -continuous

Proof. We prove only the implication (3) $\implies$ (1) since the others are obvious. Suppose that F is lower weakly $(\tau_1, \tau_2)\alpha$ -continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$ -open subset of Y such that $F(x) \cap V \neq \emptyset$. Since $F(x)$ is $\sigma_1\sigma_2$ -regular, by Lemma 13, there exists a $\sigma_1\sigma_2$ -open subset G of Y such that $F(x) \cap G \neq \emptyset$ and $\sigma_1\sigma_2\text{-Cl}(G) \subseteq V$. Since F is lower weakly $(\tau_1, \tau_2)\alpha$ -continuous at x , there exists a $(\tau_1, \tau_2)\alpha$ -open subset U of X containing x such that $\sigma_1\sigma_2\text{-Cl}(G) \cap F(z) \neq \emptyset$ for every $z \in U$ and hence $V \cap F(z) \neq \emptyset$ for every $z \in U$ as well, and this shows that F is lower $(\tau_1, \tau_2)\alpha$ -continuous at x . This shows that F is lower $(\tau_1, \tau_2)\alpha$ -continuous. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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