Research Article

An Analytic Characterization of (p, q) -White Noise Functionals

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In this paper, a characterization theorem for the $\mathcal{S}$ -transform of infinite dimensional distributions of noncommutative white noise corresponding to the (p, q) -deformed quantum oscillator algebra is investigated. We derive a unitary operator U between the noncommutative L^2 -space and the (p, q) -Fock space which serves to give the construction of a white noise Gel'fand triple. Next, a general characterization theorem is proven for the space of (p, q) -Gaussian white noise distributions in terms of new spaces of (p, q) -entire functions with certain growth rates determined by Young functions and a suitable (p, q) -exponential map.

1. Introduction

The white noise distribution theory originally aiming to extend Itô theory keeping contact with Lévy's stochastic variational calculus [1] has been developed to an efficient infinite dimensional calculus with considerable applications to quantum physics, infinite dimensional harmonic analysis, infinite dimensional differential equations, quantum stochastic calculus, and mathematical finance, see, e.g., [2–8] and the references therein. This theory is based on the quantum decomposition of the Gaussian random variable $\langle \omega, \xi \rangle$ given as the sum of creation and annihilation operators which satisfies the canonical commutative commutation relation. As a generalization by replacing the classical commutative notion of independence by some other type in a noncommutative probability space, we conclude that the noncommutative white noise theory is a generalization of classical white noise theory to the description of quantum systems. In the framework of the free probability, Alpay and Salomon [9] (see also [10]) constructed a noncommutative analog of the Kondratiev space. For $q \in (-1, 1)$, Bożejko et al. introduced q -analogs of Brownian motions and Gaussian processes in [11, 12], which

are governed by classical independence for $q = 1$ and free independence for $q = 0$ introduced by Voiculescu et al. in [13].

The aim of the present paper is to introduce a proper mathematical framework of (p, q) -white noise calculus based on the noncommutative white noise corresponding to the (p, q) -deformed oscillator algebra [14]. More precisely, as a generalization by using the second-parameter refinement of the q -Fock space, formulated as the (p, q) -Fock space $\mathcal{F}_{p,q}(\mathcal{H})$ which is constructed via a direct generalization of Bożejko and Speicher's framework, yielding the q -Fock space when $p = 1$, we introduce the noncommutative analogs of Gaussian processes (white noise measure) for the relation of the (p, q) -deformed quantum oscillator algebra. Next, we construct a white noise Gel'fand triple, and we derive the characterization of the space of generalized functions in terms of new spaces of (p, q) -entire functions with certain growth.

Our paper is organized as follows. Section 2 is devoted to study the (p, q) -white noise functionals with special emphasis on the chaos decomposition of the noncommutative L^2 -space with respect to the vacuum expectation τ based on orthogonalization of polynomials of (p, q) -white noise. In

Section 3, firstly for a fixed Young function θ with particular condition, we construct a nuclear Gel'fand triple

$$\mathcal{W}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}) \subset L^2(\tau) \subset \mathcal{W}_{p,q,\theta}^*(\mathcal{E}_{\mathbb{C}}), \quad (1)$$

of test and generalized functions, and we introduce the $\mathcal{S}$ -transform which is our main analytical tool in working with these spaces and serves to prove a characterization of white noise functionals.

2. Noncommutative Orthogonal Polynomials of (p, q) -White Noise

We start with the real Gel'fand triple:

$$\mathcal{E} := \mathcal{S}(\mathbb{R}) \subset H = L^2(\mathbb{R}, dt) \subset \mathcal{E}' := \mathcal{S}'(\mathbb{R}), \quad (2)$$

where $\mathcal{S}(\mathbb{R})$ is the space of rapidly decreasing functions and $\mathcal{S}'(\mathbb{R})$ is the dual space, i.e., the space of tempered distributions. We denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear form on $\mathcal{E}' \times \mathcal{E}$ and by $\|\cdot\|_0$ the norm of H . For notational convenience, the $\mathbb{C}$ -bilinear form on $\mathcal{E}_{\mathbb{C}} \times \mathcal{E}_{\mathbb{C}}$ is denoted by the same symbol so that $|\xi|_0^2 = \langle \xi, t\xi \rangle$ holds for $\xi \in \mathcal{H} := H_{\mathbb{C}}$ (in general, the complexification of a real vector space X is denoted by $X_{\mathbb{C}}$). In [15], Simon has proved that the space $\mathcal{E}$ is a nuclear Fréchet space constructed from the Hilbert space H and the Harmonic oscillator operator $A = 1 + t - (d^2/dt^2)$, i.e., $\mathcal{E} = \text{proj} \lim_{s \rightarrow \infty} \mathcal{E}_s = \bigcap_{s \geq 0} \mathcal{E}_s$, where for $s \geq 0$, $\mathcal{E}_s$ is the Hilbert space corresponding to the domain of A^s , i.e.,

$$\mathcal{E}_s = \{ \xi \in H; |\xi|_s = |A^s \xi|_0 < \infty \}. \quad (3)$$

We define $\mathcal{E}_{-s}$ to be the completion of H with respect to $\|\cdot\|_{-s} = |A^{-s} \cdot|_0$, and hence we obtain a chain of Hilbert space $\{\mathcal{E}_s, s \in \mathbb{R}\}$, and one can see that

$$\mathcal{E}' = \text{ind} \lim_{s \rightarrow \infty} \mathcal{E}_{-s} = \bigcup_{s \geq 0} \mathcal{E}_{-s}. \quad (4)$$

Let S_n denote the symmetric group of all permutations on $1, n := \{1, \dots, n\}$ and $I(\sigma)$ denote the number of inversions of the permutation $\sigma \in S_n$ defined by

$$I(\sigma) = \#(\text{Inv}(\sigma)) := \#\{(i, j) | 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}, \quad (5)$$

where $\#(A)$ denotes the cardinality of the set A . Analogously, the pair (i, j) with $i < j$ is called a coinversion in σ if $\sigma(i) < \sigma(j)$. The corresponding coinversion is encoded by (i, j) and contained in the set

$$\text{Cinv}(\sigma) := \{(i, j) | 1 \leq i < j \leq n, \sigma(i) < \sigma(j)\}, \quad (6)$$

with cardinality $C(\sigma) := \#(\text{Cinv}(\sigma))$. Denote $\mathcal{F}_0(\mathcal{H}) = \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}$ the full Fock space over $\mathcal{H}$ with the inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{F}_0^{\text{fin}}(\mathcal{H})$ the linear span of vectors of the form $\xi_1 \otimes \dots \otimes \xi_n \in \mathcal{H}^{\otimes n}$, $n \in \mathbb{N}$, where $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ for the vacuum vector $\Omega = (1, 0, 0, \dots) \in \mathcal{F}_0(\mathcal{H})$. We equip $\mathcal{F}_0^{\text{fin}}(\mathcal{H})$ with the inner product

$$\langle \xi_1 \otimes \dots \otimes \xi_n, \eta_1 \otimes \dots \otimes \eta_m \rangle = \delta_{n,m} \prod_{k=1}^n \langle \xi_k, \eta_k \rangle. \quad (7)$$

Recall that for p and q , two real numbers such that $0 < q < p \leq 1$, the (p, q) -factorial is defined by

$$[n]_{p,q}! = \prod_{k=1}^n [k]_{p,q}, \quad \text{with } [0]_{p,q}! = 1, \quad (8)$$

where $[n]_{p,q}$ is the (p, q) -deformation of the natural number n given by

$$[n]_{p,q} = \sum_{i=1}^n q^{i-1} p^{n-i} = \frac{p^n - q^n}{p - q}, \quad (9)$$

$$[0]_{p,q} = 0.$$

Define the operator $\mathcal{T}_{p,q}$ on $\mathcal{F}_0^{\text{fin}}(\mathcal{H})$ by a linear extension of

$$\begin{aligned} \mathcal{T}_{p,q} \Omega &= \Omega, \\ \mathcal{T}_{p,q}(\xi_1 \otimes \dots \otimes \xi_n) &= \sum_{\sigma \in S_n} q^{I(\sigma)} p^{C(\sigma)} \xi_{\sigma(1)} \otimes \dots \otimes \xi_{\sigma(n)}, \end{aligned} \quad (10)$$

and put

$$\xi_1 \otimes_{p,q} \dots \otimes_{p,q} \xi_n := \mathcal{T}_{p,q}(\xi_1 \otimes \dots \otimes \xi_n), \quad \xi_i \in \mathcal{H}, i \in 1, n. \quad (11)$$

Define $\mathcal{F}_{p,q}^{(n)}(\mathcal{H})$ as the separable Hilbert space which coincides with $\mathcal{H}^{\otimes n}$ as a set and has a scalar product:

$$\langle f^{(n)}, g^{(n)} \rangle_{p,q} := \langle f^{(n)}, g^{(n)} \rangle_{\mathcal{F}_{p,q}^{(n)}(\mathcal{H})} = \langle \mathcal{T}_{p,q} f^{(n)}, g^{(n)} \rangle. \quad (12)$$

Hence, the (p, q) -Fock space denoted $\mathcal{F}_{p,q}(\mathcal{H})$ is defined by

$$\mathcal{F}_{p,q}(\mathcal{H}) = \bigoplus_{n=0}^{+\infty} \mathcal{F}_{p,q}^{(n)}(\mathcal{H}), \quad (13)$$

and if we denote $\mathcal{F}_{p,q}^{\text{fin}}(\mathcal{H})$ the linear span of vectors of the form

$$\xi_1 \otimes_{p,q} \dots \otimes_{p,q} \xi_n \in \mathcal{F}_{p,q}^{(n)}(\mathcal{H}), \quad n \in \mathbb{N}, \quad (14)$$

one can see that $\langle \cdot, \cdot \rangle_{p,q}$ on $\mathcal{F}_{p,q}^{\text{fin}}(\mathcal{H})$ satisfies the following useful relation:

$$\langle f^{(n)}, \xi_1 \otimes_{p,q} \dots \otimes_{p,q} \xi_m \rangle_{p,q} = \delta_{n,m} [n]_{p,q}! \langle f^{(n)}, \xi_1 \otimes \dots \otimes \xi_m \rangle. \quad (15)$$

For more details about the properties of the operator $\mathcal{T}_{p,q}$ and the construction of the (p, q) -Fock space, see [16].

Definition 1. For each $\xi \in \mathcal{H}$, we define the (p, q) -creation operator $a^*(\xi)$ and the (p, q) -annihilation operator $a(\xi)$ on the dense subspace $\mathcal{F}_{p,q}^{\text{fin}}(\mathcal{H})$ as follows:

$$\begin{aligned}
a^*(\xi)\Omega &= \xi, \\
a^*(\xi)f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n &= \xi \otimes_{p,q} f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n, \\
a(\xi)\Omega &= 0, \\
a(\xi)f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n &= \sum_{i=1}^n q^{i-1} p^{n-i} \langle \xi, f_i \rangle f_1 \otimes_{p,q} \cdots \otimes_{p,q} \widetilde{f_i} \otimes_{p,q} \cdots \otimes_{p,q} f_n,
\end{aligned} \tag{16}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$ and the symbol $\widetilde{f_i}$ means that f_i has to be deleted in the tensor product.

The (p, q) -creation and (p, q) -annihilation operators fulfill the (p, q) -commutation relations of the (p, q) -deformed quantum oscillator algebra, i.e.,

$$\begin{aligned}
a(\xi)a^*(\eta) - pa^*(\eta)a(\xi) &= q^N \langle \xi, \eta \rangle 1, \\
a(\xi)a^*(\eta) - qa^*(\eta)a(\xi) &= p^N \langle \xi, \eta \rangle 1, \quad \forall \xi, \eta \in \mathcal{H}, \\
[N, a^*(\xi)] &= a^*(\xi), \\
[N, a(\xi)] &= -a(\xi),
\end{aligned} \tag{17}$$

where

$$q^N f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n = q^n f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n, \tag{18}$$

such that N is the standard number operator defined by

$$N f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n = n f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n, \tag{19}$$

and the commutator $[\cdot, \cdot]$ is defined by $[B, C] = BC - CB$. For more details, one can see [16].

Now, we will introduce noncommutative analogs of Gaussian processes (white noise measure) for the relation of the (p, q) -deformed quantum oscillator algebra. For $t \in \mathbb{R}$, if we denote by b_t and b_t^* the standard pointwise annihilation and creation operators on $\mathcal{F}_{p,q}(\mathcal{H})$ defined by

$$\begin{aligned}
(b_t f^{(n)})(t_1, \dots, t_{n-1}) &= \sum_{i=1}^n q^{i-1} p^{n-i} f^{(n)}(t_1, \dots, t_{i-1}, t, t_i, \dots, t_{n-1}), \\
b_t^* f^{(n)} &= \delta_t \widehat{\otimes} f^{(n)},
\end{aligned} \tag{20}$$

where δ_t is the delta function at t and $\widehat{\otimes}$ stands for the symmetric tensor product, then one can see that the (p, q) -creation and (p, q) -annihilation operators are given as the smeared operators in terms of b_t and b_t^* , i.e.,

$$\begin{aligned}
a(\xi) &= \int_{\mathbb{R}} \xi(t) b_t dt, \\
a^*(\xi) &= \int_{\mathbb{R}} \xi(t) b_t^* dt, \quad \forall \xi \in \mathcal{H}.
\end{aligned} \tag{21}$$

Now, the (p, q) -white noise is defined by

$$\omega(t) := b_t + b_t^*. \tag{22}$$

Thus, by using (21), we deduce that $\omega(t)$ is an operator-valued distribution which satisfies

$$\langle \omega, \xi \rangle := \int_{\mathbb{R}} \omega(t) \xi(t) dt = a^*(\xi) + a(\xi). \tag{23}$$

Moreover, for each $\xi^{(n)} \in \mathcal{F}_{p,q}^{\text{fin}}(\mathcal{H})$, we define a monomial of ω by

$$\begin{aligned}
\langle \omega^{\otimes n}, \xi^{(n)} \rangle &= \int_{\mathbb{R}^n} \xi^{(n)}(t_1, \dots, t_n) \omega(t_1) \cdots \omega(t_n) dt_1, \dots, dt_n \\
&= \int_{\mathbb{R}^n} \xi^{(n)}(t_1, \dots, t_n) (b_{t_1}^* + b_{t_1}) \cdots (b_{t_n}^* + b_{t_n}) dt_1, \dots, dt_n.
\end{aligned} \tag{24}$$

Using the Cauchy-Schwarz inequality, we easily conclude that (24) indeed identifies a bounded linear operator in $\mathcal{F}_{p,q}(\mathcal{H})$.

Let $\mathcal{P}$ denote the complex unital $*$ -algebra generated by $\{\langle \omega, \xi \rangle, \xi \in \mathcal{H}\}$, i.e., the algebra of noncommutative polynomials in the variables $\langle \omega, \xi \rangle$. Evidently, $\mathcal{P}$ consists of all noncommutative polynomials in ω which are of the form:

$$T = \xi^{(0)} + \sum_{i=1}^k \langle \omega^{\otimes i}, \xi^{(i)} \rangle, \quad \xi^{(0)} \in \mathbb{C}, \xi^{(i)} \in \mathcal{F}_{p,q}^{(i)}(\mathcal{H}). \tag{25}$$

In particular, elements of $\mathcal{P}$ are linear operators acting on $\mathcal{F}_{p,q}(\mathcal{H})$.

Definition 2. Let τ be a vacuum state on $\mathcal{P}$ defined by

$$\tau(T) = \langle T\Omega, \Omega \rangle_{\mathcal{F}_{p,q}(\mathcal{H})}, \quad T \in \mathcal{P}, \tag{26}$$

where Ω is the vacuum vector in $\mathcal{F}_{p,q}(\mathcal{H})$. The inner product $\langle \cdot, \cdot \rangle_{L^2(\tau)}$ on $\mathcal{P}$ is defined by

$$\langle T_1, T_2 \rangle_{L^2(\tau)} := \tau(T_2^* T_1) = \langle T_1 \Omega, T_2 \Omega \rangle_{\mathcal{F}_{p,q}(\mathcal{H})}, \quad T_1, T_2 \in \mathcal{P}, \tag{27}$$

where T_2^* is the adjoint operator of T_2 in $\mathcal{F}_{p,q}(\mathcal{H})$. Moreover, the noncommutative L^2 -space $L^2(\tau)$ is the Hilbert space obtained as the closure of $\mathcal{P}$ with respect to the norm induced by the scalar product $\langle \cdot, \cdot \rangle_{L^2(\tau)}$.

For $n \in \mathbb{N} \setminus \{0\}$, we denote by $\mathcal{P}_n$ the subset of $\mathcal{P}$ consisting of all noncommutative polynomials of order $\leq n$, i.e., all $T \in \mathcal{P}$ given as in (25) with $k \leq n$. Let $\tilde{\mathcal{P}}_n$ denote the closure of $\mathcal{P}_n$ in $L^2(\tau)$, and let $\mathcal{OP}_n$ be the set of orthogonal polynomials of order n defined by

$$\mathcal{OP}_n = \tilde{\mathcal{P}}_n \ominus \tilde{\mathcal{P}}_{n-1}, \quad (28)$$

where $\ominus$ denotes the orthogonal difference in $L^2(\tau)$.

$$\begin{aligned} \langle \cdot : \omega : \xi \rangle &= \langle \omega, \xi \rangle, \quad \xi \in \mathcal{H}, \\ \langle \cdot : \omega^{\otimes(n+1)} : \xi^{\otimes_{p,q}(n+1)} \rangle &= \langle \omega, \xi \rangle \langle \cdot : \omega^{\otimes n} : \xi^{\otimes_{p,q} n} \rangle - [n]_{p,q} \langle \xi, \xi \rangle \langle \cdot : \omega^{\otimes(n-1)} : \xi^{\otimes_{p,q}(n-1)} \rangle. \end{aligned} \quad (30)$$

Furthermore, under the action of U , the operator of the left multiplication by the monomial $\langle \omega, \xi \rangle$ in $L^2(\tau)$ (denoted by $M_{\langle \omega, \xi \rangle}$) becomes $\langle \omega, \xi \rangle$, i.e.,

$$UM_{\langle \omega, \xi \rangle}U^{-1} = \langle \omega, \xi \rangle. \quad (31)$$

Proof. Firstly from equation (27), it is clear that $U: L^2(\tau) \rightarrow \mathcal{F}_{p,q}(\mathcal{H})$ is extended by continuity to a unitary operator. Moreover, since $\mathcal{P}$ is dense in $L^2(\tau)$, we get the orthogonal decomposition

$$L^2(\tau) = \bigoplus_{n=0}^{+\infty} \mathcal{OP}_n, \quad (32)$$

and for each n , one can see that $U(\mathcal{OP}_n) = \mathcal{F}_{p,q}^{(n)}(\mathcal{H})$. Hence, using the fact that $U(\langle \cdot : \omega^{\otimes n} : \xi^{\otimes(n)} \rangle)$ is the orthogonal projection in $\mathcal{F}_{p,q}(\mathcal{H})$ of

$$U(\langle \omega^{\otimes n}, \xi^{\otimes(n)} \rangle) = \langle \omega^{\otimes n}, \xi^{\otimes(n)} \rangle \Omega = \xi^{\otimes(n)}, \quad (33)$$

on $\mathcal{F}_{p,q}^{(n)}(\mathcal{H})$, we obtain

$$U(\langle \cdot : \omega^{\otimes n} : \xi^{\otimes(n)} \rangle) = \xi^{\otimes(n)}. \quad (34)$$

On the other hand, we have

$$\begin{aligned} U(\langle \cdot : \omega : \xi \rangle) &= \xi = (a^*(\xi) + a(\xi))\Omega = \langle \omega, \xi \rangle \Omega = U(\langle \omega, \xi \rangle), \\ &\quad \forall \xi \in \mathcal{H}, \end{aligned} \quad (35)$$

and this gives $\langle \cdot : \omega : \xi \rangle = \langle \omega, \xi \rangle$. Therefore, (23) and (34) yield

$$\begin{aligned} U(\langle \omega, \xi \rangle \langle \cdot : \omega^{\otimes n} : \xi^{\otimes_{p,q} n} \rangle) &= \langle \omega, \xi \rangle \xi^{\otimes_{p,q} n} \\ &= (a^*(\xi) + a(\xi)) \xi^{\otimes_{p,q} n} \\ &= \xi^{\otimes_{p,q}(n+1)} + [n]_{p,q} \langle \xi, \xi \rangle \xi^{\otimes_{p,q}(n-1)} \\ &= U(\langle \cdot : \omega^{\otimes(n+1)} : \xi^{\otimes_{p,q}(n+1)} \rangle \\ &\quad + [n]_{p,q} \langle \xi, \xi \rangle \langle \cdot : \omega^{\otimes(n-1)} : \xi^{\otimes_{p,q}(n-1)} \rangle), \end{aligned} \quad (36)$$

Theorem 1. For each $T \in \mathcal{P}$, define $UT := T\Omega$. Then, U is extended by continuity to a unitary operator $U: L^2(\tau) \rightarrow \mathcal{F}_{p,q}(\mathcal{H})$ defined as follows:

$$U(\langle \cdot : \omega^{\otimes n} : \xi^{\otimes(n)} \rangle) = \xi^{\otimes(n)}, \quad \xi^{\otimes(n)} \in \mathcal{F}_{p,q}^{\text{fin}}(\mathcal{H}), \quad (29)$$

where $\langle \cdot : \omega^{\otimes n} : \xi^{\otimes(n)} \rangle$ is the orthogonal projection of the monomial $\langle \omega^{\otimes n}, \xi^{\otimes(n)} \rangle$ onto $\mathcal{OP}_n$ and given recursively by

from which (30) follows.

3. (p, q) White Noise Gel'fand Triple and Characterization Theorem

Recall that a Young function is a continuous, convex, and increasing function

$$\theta: [0, +\infty[\rightarrow [0, +\infty[, \quad (37)$$

such that

$$\begin{aligned} \theta(0) &= 0, \\ \lim_{x \rightarrow +\infty} \frac{\theta(x)}{x} &= +\infty. \end{aligned} \quad (38)$$

Define a weight sequence $\{\theta_{n,p,q}\}_{n=0}^{\infty}$ by

$$\theta_{n,p,q} = \inf_{x>0} \frac{E_{p,q}(\theta(x))}{x^n}, \quad n = 0, 1, 2, \dots, \quad (39)$$

where θ is a Young function and $E_{p,q}$ is the (p, q) -exponential function defined by

$$E_{p,q}(x) = \sum_{n=0}^{+\infty} \frac{q^{(n(n-1)/2)}}{[n]_{p,q}!} x^n, \quad x \in \mathbb{R}. \quad (40)$$

Let $\theta_{n,p,q}^*$ be the sequence associated with the (p, q) -polar function θ^* of θ , defined by

$$\theta^*(x) = E_{p,q}^{-1} \left(\sup_{t \geq 0} \frac{E_{p,q}(xt)}{E_{p,q}(\theta(t))} \right), \quad x \geq 0. \quad (41)$$

For simplicity of notation, we denote

$$|\varphi^{(n)}|_{s,p,q} = |\varphi^{(n)}|_{\mathcal{F}_{p,q}^{(n)}(\mathcal{E}_{s,\mathbb{C}})}, \quad s \in \mathbb{Z}, \quad (42)$$

where $\mathcal{F}_{p,q}^{(n)}(\mathcal{E}_{s,\mathbb{C}})$ is constructed as for $\mathcal{F}_{p,q}^{(n)}(\mathcal{H})$ by replacing $\mathcal{H}$ by the space $\mathcal{E}_{s,\mathbb{C}}$ given by equation (3). Now, suppose a pair $s \geq 0$ and $\gamma > 0$ is given, then for $\vec{\varphi} = (\varphi^{(n)})_{n=0}^{\infty}$ with $\varphi^{(n)} \in \mathcal{F}_{p,q}^{(n)}(\mathcal{E}_{s,\mathbb{C}})$, we put

$$\|\vec{\varphi}\|_{s,p,q,\theta,\gamma}^2 := \sum_{n=0}^{+\infty} \theta_{n,p,q}^{-2} \gamma^{-n} |\varphi^{(n)}|_{s,p,q}^2. \quad (43)$$

Hence, we obtain a projective system of Hilbert spaces

$$\mathcal{F}_{p,q}(\mathcal{E}_{s,\mathbb{C}}, \gamma, \theta) = \{\vec{\varphi} = t(\varphi_n)_{n=0}^{\infty}; q\varphi_n h \in \mathcal{F}_{p,q}^{(n)}(\mathcal{E}_{s,\mathbb{C}}), C\|\vec{\varphi}\|_{s,p,q,\theta,\gamma} < \infty\}. \quad (45)$$

Finally, we define the nuclear space $\mathcal{F}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}})$ by

$$\mathcal{F}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}) = \text{proj} \lim_{s \rightarrow +\infty; \gamma \downarrow 0} \mathcal{F}_{p,q}(\mathcal{E}_{s,\mathbb{C}}, \gamma, \theta). \quad (46)$$

Definition 3. The space of (p, q) -white noise test functions $\mathcal{W}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}})$ is defined as a projective system of Hilbert space $\{\mathcal{W}_{p,q,\theta,\gamma}(\mathcal{E}_{s,\mathbb{C}}); s \geq 0, \gamma > 0\}$, where $\mathcal{W}_{p,q,\theta,\gamma}(\mathcal{E}_{s,\mathbb{C}})$ is the set of function φ of the form

$$\varphi(\omega) = \sum_{n=0}^{+\infty} \langle \omega^{\otimes n}; \varphi^{(n)} \rangle, \quad \vec{\varphi} = (\varphi^{(n)}) \in \mathcal{F}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}), \quad (47)$$

such that

$$\|\varphi\|_{s,p,q,\theta,\gamma} \equiv \|\vec{\varphi}\|_{s,p,q,\theta,\gamma} < \infty. \quad (48)$$

Moreover, if $\mathcal{W}_{p,q,\theta,\gamma}(\mathcal{E}_{-s,\mathbb{C}})$ is the set of functions Φ of the form

$$\Phi(\omega) = \sum_{n=0}^{+\infty} \langle \omega^{\otimes n}; \Phi^{(n)} \rangle, \quad \vec{\Phi} = (\Phi^{(n)}) \in \mathcal{F}_{p,q,\theta}^*(\mathcal{E}_{\mathbb{C}}), \quad (49)$$

equipped with Hilbertian norm

$$\|\Phi\|_{-s,p,q,\theta,\gamma}^2 := \sum_{n=0}^{+\infty} \theta_{n,p,q}^2 \gamma^n |\Phi^{(n)}|_{-s,p,q}^2, \quad (50)$$

the space of (p, q) -white noise generalized functions is defined by

$$\mathcal{W}_{p,q,\theta}^*(\mathcal{E}_{\mathbb{C}}) \equiv \bigcup_{s \geq 0, \gamma > 0} \mathcal{W}_{p,q,\theta,\gamma}(\mathcal{E}_{-s,\mathbb{C}}). \quad (51)$$

Theorem 2. Assume that the Young function θ satisfies the following condition:

$$E_{p,q}(r) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} r^k}{\sqrt{k!}} \frac{\sqrt{k!}}{[k]_{p,q}!} \leq \left(\sum_{k=0}^{\infty} \frac{q^{k(k-1)} k!}{[k]_{p,q}^2} \right)^{(1/2)} e^{(r^2/2)}, \quad r > 0, \quad (58)$$

$$\{\mathcal{F}_{p,q}(\mathcal{E}_{s,\mathbb{C}}, \gamma, \theta); s \geq 0, \gamma > 0\}, \quad (44)$$

where

$$\lim_{x \rightarrow \infty} \sup \frac{\theta(x)}{x^2} < \infty. \quad (52)$$

Then, we obtain the so-called (p, q) -white noise Gelfand triple of Hilbert spaces

$$\mathcal{W}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}) \longrightarrow L^2(\tau) \longrightarrow \mathcal{W}_{p,q,\theta}^*(\mathcal{E}_{\mathbb{C}}), \quad (53)$$

with the $\mathbb{C}$ -bilinear form on $\mathcal{W}_{p,q,\theta}^*(\mathcal{E}_{\mathbb{C}}) \times \mathcal{W}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}})$ given by

$$\langle \langle \Phi, \varphi \rangle \rangle = \sum_{n=0}^{+\infty} \langle \Phi^{(n)}, \varphi^{(n)} \rangle_{p,q}, \quad (54)$$

where $\langle \cdot, \cdot \rangle_{p,q}$ is the canonical $\mathbb{C}$ -bilinear form on $\mathcal{E}_{\mathbb{C}}'^{\otimes p,q n} \times \mathcal{E}_{\mathbb{C}}^{\otimes p,q n}$ which is compatible with the inner product of $\mathcal{H}^{\otimes p,q n}$ defined by equation (12).

Proof. Let $\vec{\varphi} = (\varphi^{(n)})_{n=0}^{\infty} \in \mathcal{F}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}})$. By definition, we have

$$\|\vec{\varphi}\|_{\mathcal{F}_{p,q}}^2 = \sum_{n=0}^{\infty} \langle \varphi^{(n)}, \varphi^{(n)} \rangle_{p,q} = \sum_{n=0}^{\infty} \theta_{n,p,q}^2 \gamma^n \theta_{n,p,q}^{-2} \gamma^{-n} |\varphi^{(n)}|_{p,q}^2. \quad (55)$$

On the other hand, condition (52) guarantees the existence of two constant numbers $a > 0$ and $b > 0$ such that

$$e^{\theta(r)} \leq a e^{br^2}, \quad r > 0. \quad (56)$$

Then, by a simple calculus, one can see that

$$\inf_{r>0} \frac{e^{\theta(r)}}{r^n} \leq a \left(\frac{2be}{n} \right)^{(n/2)}. \quad (57)$$

Hence, by using the fact that

and the inequality (57), we obtain

$$\theta_{n,p,q}^2 \gamma^n \leq a^2 \left(\frac{2be\gamma}{n} \right)^n \sum_{k=0}^{\infty} \frac{q^{k(k-1)} k!}{[k]_{p,q}^2}. \quad (59)$$

Therefore, for $\gamma < (2be)^{-1}$, we have $C \equiv \sup_{n \geq 0} \theta_{n,p,q}^2 \gamma^n < \infty$. Thus, (55) becomes

$$\|\vec{\varphi}\|_{\mathcal{F}_{p,q}(\mathcal{H})}^2 \leq C \sum_{n=0}^{\infty} \theta_{n,p,q}^{-2} \gamma^{-n} |\varphi^{(n)}|_{\mathcal{F}_{p,q}^{(n)}(\mathcal{H})}^2 = C \|\vec{\varphi}\|_{0,p,q,\theta,\gamma}^2 \leq C \|\vec{\varphi}\|_{s,p,q,\theta,\gamma}^2, \quad (60)$$

which means that $\mathcal{F}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}) \subset \mathcal{F}_{p,q}(\mathcal{H})$ and the inclusion is continuous. On the other hand, if we put $\mathcal{W}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}) := U^{-1}(\mathcal{F}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}))$, where U is the isomorphism given in Theorem 1, we obtain the following diagram:

$$\begin{array}{ccc} \mathcal{F}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}) & \longrightarrow & \mathcal{F}_{p,q}(\mathcal{H}) \\ \downarrow & & \downarrow \\ \mathcal{W}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}) & \longrightarrow & L^2(\tau) \end{array} \quad (61)$$

Moreover, one can see that $\mathcal{W}_{p,q,\theta,\gamma}^*(\mathcal{E}_{-s,\mathbb{C}})$ is the dual of $\mathcal{W}_{p,q,\theta,\gamma}^*(\mathcal{E}_{s,\mathbb{C}})$ with respect to $L^2(\tau)$, and we obtain the nuclear Gel'fand triple given by (53). From here the statement follows.

Now our goal is to derive a characterization of the space of (p, q) -white noise generalized functions by using a suitable space of (p, q) -entire functions with certain growth determined by using the Young functions and a suitable (p, q) -exponential map.

Let $(\mathcal{B}, \|\cdot\|)$ be a complex Banach space. Define the space $\mathcal{B}^{\infty}$ by

$$\mathcal{B}^{\infty} := \left\{ \vec{\xi} = t(\xi_k)_{k \in \mathbb{N}}; q\xi_k h \in \mathcal{B}, \forall \vec{\xi} \in \mathcal{B}^{\infty}, \|\vec{\xi}\|_{\infty} C := \sup_{k \in \mathbb{N}} \|\xi_k\| < \infty \right\}. \quad (62)$$

Then, $(\mathcal{B}^{\infty}, \|\cdot\|_{\infty})$ becomes a Banach space.

Definition 4. Let $\mathcal{H}$ be a fixed Hilbert space. A $\mathbb{C}$ -valued function F is said to be (p, q) -entire function on $\mathcal{H}^{\infty}$, if there exists $(F_n)_{n=0}^{\infty}$ with $F_n \in \mathcal{H}'^{\otimes p,q,n}$ such that

$$F(\vec{\xi}) = \sum_{n=0}^{\infty} \langle F_n, \xi_1 \otimes_{p,q} \cdots \otimes_{p,q} \xi_n \rangle, \quad \vec{\xi} = (\xi_k)_{k \in \mathbb{N}} \in \mathcal{H}^{\infty}, \quad (63)$$

where the series in the right hand side of (63) converges uniformly on every bounded subset of $\mathcal{H}^{\infty}$.

For $s \in \mathbb{R}$ and $\gamma > 0$, let $\Gamma_{p,q,\theta,\gamma}(\mathcal{E}_{s,\mathbb{C}})$ be the space of (p, q) -entire functions g on the complex Hilbert space $\mathcal{E}_{s,\mathbb{C}}^{\infty}$ such that

$$\|g\|_{s,p,q,\theta,\gamma} := \sup \{ |g(z)| E_{p,q}^{-1}[\theta(\gamma|z|_s)], z \in \mathcal{E}_{s,\mathbb{C}} \} < +\infty. \quad (64)$$

Note that $\{\Gamma_{p,q,\theta,\gamma}(\mathcal{E}_{-s,\mathbb{C}}); s \in \mathbb{N}, \gamma > 0\}$ becomes a projective system of Banach spaces as $s \rightarrow \infty$ and $\gamma \downarrow 0$. Then, we can define

$$\mathcal{N}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}') := \text{proj} \lim_{s \rightarrow \infty; \gamma \downarrow 0} \Gamma_{p,q,\theta,\gamma}(\mathcal{E}_{-s,\mathbb{C}}). \quad (65)$$

This is called the space of (p, q) -entire functions on $\mathcal{E}_{\mathbb{C}}'$ with (θ, p, q) -exponential growth of minimal type. Similarly, $\{\Gamma_{p,q,\theta,\gamma}(\mathcal{E}_{s,\mathbb{C}}); s \in \mathbb{N}, \gamma > 0\}$ becomes an inductive system of Banach spaces as $s \rightarrow \infty$ and $\gamma \downarrow 0$. $\mathcal{G}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}})$ the space of (p, q) -entire functions on $\mathcal{E}_{s,\mathbb{C}}$ with (θ, p, q) -exponential growth of finite type is defined by

$$\mathcal{G}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}) = \bigcup_{s \geq 0, \gamma > 0} \Gamma_{p,q,\theta,\gamma}(\mathcal{E}_{s,\mathbb{C}}). \quad (66)$$

Lemma 1. Let $F \in \mathcal{N}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}')$ be given by

$$F(\vec{z}) = \sum_{n=0}^{\infty} \langle z_1 \otimes_{p,q} \cdots \otimes_{p,q} z_n, F_n \rangle, \quad (67)$$

where $F_n \in \mathcal{E}_{\mathbb{C}}^{\otimes p,q,n}$ and $\vec{z} = (z_k)_{k \in \mathbb{N}} \in \mathcal{E}_{\mathbb{C}}^{\infty}$. Then, for any $s \geq 0$, there exists $s' > s$ such that the canonical embedding $i_{s',s}^*: \mathcal{E}_{s',\mathbb{C}} \rightarrow \mathcal{E}_{s,\mathbb{C}}$ is of the Hilbert-Schmidt type, and for $\gamma > 0$, we get

$$|F_n|_{s,p,q} \leq \|F\|_{-s',p,q,\theta,\gamma} \theta_{n,p,q} \gamma^n \|i_{s',s}^*\|_{HS}^n. \quad (68)$$

Proof. Fixing $s' > s \geq 0$ and $\vec{z} = (z_k)_{k \in \mathbb{N}} \in \mathcal{E}_{-s',\mathbb{C}}^{\infty}$ such that

$$|z_k|_{-s'} = 1, \quad \forall k \in \mathbb{N}. \quad (69)$$

By definition, the series in the right hand side of (67) converges uniformly on every bounded subset of $\mathcal{E}_{-s',\mathbb{C}}^{\infty}$. Then, for every $R > 0$, we have the following Cauchy's integral formula:

$$\langle z_1 \otimes_{p,q} \cdots \otimes_{p,q} z_n, F_n \rangle = \frac{1}{2i\pi} \int_{|\lambda|=R} \frac{F(\lambda \vec{z})}{\lambda^{n+1}} d\lambda, \quad n \in \mathbb{N}. \quad (70)$$

Therefore, by using the fact that $F \in \mathcal{N}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}')$, for $\gamma > 0$, we get

$$|\langle z_1 \otimes_{p,q} \cdots \otimes_{p,q} z_n, F_n \rangle| \leq \|F\|_{-s',p,q,\theta,\gamma} \frac{E_{p,q}[\theta(\gamma R)]}{(\gamma R)^n} \gamma^n, \quad \forall R > 0, \quad (71)$$

which gives

$$|\langle z_1 \otimes_{p,q} \cdots \otimes_{p,q} z_n, F_n \rangle| \leq \|F\|_{-s',p,q,\theta,\gamma} \theta_{n,p,q} \gamma^n. \quad (72)$$

Let now $\vec{e} = (e_k)_{k \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{E}_{s',\mathbb{C}}$. Then, we get

$$\begin{aligned}
|F_n|_{s,p,q}^2 &\leq \sum_{k_1, \dots, k_n \geq 0} \left| \langle F_n, e_{k_1} \otimes_{p,q} \dots \otimes_{p,q} e_{k_n} \rangle \right|^2 \\
&\leq \sum_{k_1, \dots, k_n \geq 0} |F|_{-s', p, q, \theta, \gamma}^2 \theta_{n,p,q}^2 \gamma^{2n} \left(\prod_{j=1}^n |e_{k_j}|_{s'}^2 \right) \\
&\leq |F|_{-s', p, q, \theta, \gamma}^2 \theta_{n,p,q}^2 \gamma^{2n} \|i_{s',s}^{\otimes n}\|_{HS}^2 \\
&\leq |F|_{-s', p, q, \theta, \gamma}^2 \theta_{n,p,q}^2 \gamma^{2n} \|i_{s',s}^{\otimes n}\|_{HS}^{2n}.
\end{aligned} \tag{73}$$

This provides the desired inequality.

Lemma 2. For each $\xi \in \mathcal{E}_{\mathbb{C}}$, the generating function of the noncommutative polynomials $\langle : \omega^{\otimes n} : \xi^{\otimes_{p,q} n} \rangle$ defined by

$$: E_{p,q}(\langle \omega, \xi \rangle) := \sum_{n=0}^{\infty} \frac{q^{(n(n-1)/2)}}{[n]_{p,q}!} \langle : \omega^{\otimes n} : \xi^{\otimes_{p,q} n} \rangle, \tag{74}$$

is an element in $\mathcal{W}_{p,q,\theta}^*(\mathcal{E}_{\mathbb{C}})$. Moreover, for all $s \geq 0$ and $\gamma > 0$, there exist $c, \gamma' > 0$, and $s' > s$ such that

$$\| : E_{p,q}(\langle \omega, \xi \rangle) : \|_{s,p,q,\theta,\gamma} \leq c E_{p,q} \left[\theta^* \left(\frac{|\xi|_{s'}}{\gamma'} \right) \right]. \tag{75}$$

Proof. Let $s \geq 0$ and $\gamma > 0$, and then for any $\xi \in \mathcal{E}_{s,\mathbb{C}}$, we have

$$\begin{aligned}
|E_{p,q}(\langle z, \xi \rangle)| E_{p,q}^{-1}[\theta(\gamma|z|_{-s})] &\leq E_{p,q}(|z|_{-s} |\xi|_s) E_{p,q}^{-1}[\theta(\gamma|z|_{-s})] \\
&\leq E_{p,q} \left[\theta^* \left(\frac{|\xi|_s}{\gamma} \right) \right].
\end{aligned} \tag{76}$$

This proves that $E_{p,q}(\langle \cdot, \xi \rangle) \in \mathcal{N}_{p,q,\theta}(\mathcal{E}_{\mathbb{C}}')$, and we obtain

$$\begin{aligned}
\|E_{p,q}(\langle \cdot, \xi \rangle)\|_{-s,p,q,\theta,\gamma} &= \sup_{z \in \mathcal{E}_{-s}} \left\{ |E_{p,q}(\langle z, \xi \rangle)| E_{p,q}^{-1}[\theta(\gamma|z|_{-s})] \right\} \\
&\leq E_{p,q} \left[\theta^* \left(\frac{|\xi|_s}{\gamma} \right) \right].
\end{aligned} \tag{77}$$

On the other hand, if we choose $s' > s$ such that the embedding $i_{s',s} : \mathcal{E}_{s',\mathbb{C}} \longrightarrow \mathcal{E}_{s,\mathbb{C}}$ is of the Hilbert-Schmidt type and $\gamma' > 0$ such that

$$\frac{\gamma'}{\sqrt{\gamma}} \|i_{s',s}\|_{HS} < 1, \tag{78}$$

and by using Lemma 1 and (77), we get

$$\begin{aligned}
\| : E_{p,q}(\langle \omega, \xi \rangle) : \|_{s,p,q,\theta,\gamma}^2 &= \sum_{n=0}^{\infty} \theta_{n,p,q}^{-2} \gamma^{-n} \frac{q^{n(n-1)}}{[n]_{p,q}^2} |\xi^{\otimes_{p,q} n}|_{s,p,q}^2 \\
&\leq \|E_{p,q}(\langle \cdot, \xi \rangle)\|_{-s', p, q, \theta, \gamma'}^2 \sum_{n=0}^{\infty} q^{n(n-1)} \left(\frac{\gamma'}{\sqrt{\gamma}} \|i_{s',s}\|_{HS} \right)^{2n} \\
&\leq c^2 E_{p,q}^2 \left[\theta^* \left(\frac{|\xi|_{s'}}{\gamma'} \right) \right].
\end{aligned} \tag{79}$$

This implies that $E_{p,q}(\langle \omega, \xi \rangle) : \in \mathcal{W}_{p,q,\theta}^*(\mathcal{E}_{\mathbb{C}})$.

As a consequence, we can define the $\mathcal{S}$ -transform of a distribution $\Phi \in \mathcal{W}_{p,q,\theta}^*(\mathcal{E}_{\mathbb{C}})$, at $\xi \in \mathcal{E}_{\mathbb{C}}$, as follows:

$$\mathcal{S}\Phi(\xi) := \langle \langle \Phi, E_{p,q}(\langle \cdot, \xi \rangle) \rangle \rangle. \tag{80}$$

Moreover, by using (15) and (49), we get

$$\begin{aligned}
\mathcal{S}\Phi(\xi) &= \sum_{n=0}^{+\infty} \frac{q^{(n(n-1)/2)}}{[n]_{p,q}!} \langle \Phi^{(n)}, \xi^{\otimes_{p,q} n} \rangle_{p,q} \\
&= \sum_{n=0}^{+\infty} q^{(n(n-1)/2)} \langle \Phi^{(n)}, \xi^{\otimes n} \rangle.
\end{aligned} \tag{81}$$

Theorem 3. Assume that the sequences $\theta_{n,p,q}$ and $\theta_{n,p,q}^*$ satisfy

$$\theta_{n,p,q} \theta_{n,p,q}^* \leq \beta \frac{q^{(n(n-1)/2)}}{[n]_{p,q}!}, \tag{82}$$

for some constant $\beta > 0$. Then, the $\mathcal{S}$ -transform realizes a topological isomorphism from $\mathcal{W}_{p,q,\theta}^*(\mathcal{E}_{\mathbb{C}})$ onto the space $\mathcal{G}_{p,q,\theta^*}(\mathcal{E}_{\mathbb{C}})$.

Proof. Let $\Phi \in \mathcal{W}_{p,q,\theta}^*(\mathcal{E}_{\mathbb{C}})$, and then there exist $s \geq 0$ and $\gamma > 0$ such that $\Phi \in \mathcal{W}_{p,q,\theta,\gamma}^*(\mathcal{E}_{-s,\mathbb{C}})$, and we have

$$\begin{aligned}
|\mathcal{S}\Phi(\xi)| &\leq \sum_{n=0}^{+\infty} \theta_{n,p,q} \gamma^{(n/2)} |\Phi^{(n)}|_{-s,p,q} \theta_{n,p,q}^{-1} \gamma^{(-n/2)} \frac{q^{(n(n-1)/2)}}{[n]_{p,q}!} |\xi^{\otimes n}|_s \\
&\leq \left(\sum_{n=0}^{+\infty} \theta_{n,p,q}^2 \gamma^n |\Phi^{(n)}|_{-s,p,q}^2 \right)^{(1/2)} \left(\sum_{n=0}^{+\infty} \frac{q^n \theta_{n,p,q}^{-2}}{\gamma^n ([n]_{p,q}!)^2} |\xi|_s^{2n} \right)^{(1/2)}.
\end{aligned} \tag{83}$$

On the other hand, by inequality (75), there exist $c, \gamma' > 0$, and $s' > s$ such that

$$|\mathcal{S}\Phi(\xi)| \leq c \|\Phi\|_{-s,p,q,\theta,\gamma} E_{p,q} \left[\theta^* \left(\frac{|\xi|_{s'}}{\gamma'} \right) \right], \tag{84}$$

which yields

$$|\mathcal{S}\Phi|_{s',p,q,\theta^*,\gamma'^{-1}} \leq c \|\Phi\|_{-s,p,q,\theta,\gamma}. \tag{85}$$

This proves the continuity and injectivity of the $\mathcal{S}$ -transform.

Conversely, given $g \in \mathcal{G}_{p,q,\theta^*}(\mathcal{E}_{\mathbb{C}})$, then there exist $s \geq 0$ and $\gamma > 0$ such that $g \in \mathcal{G}_{p,q,\theta^*,\gamma}(\mathcal{E}_{s,\mathbb{C}})$ with Taylor expansion

$$g(\xi) = \sum_{n=0}^{+\infty} \langle G_n, \xi^{\otimes_{p,q} n} \rangle, \quad G_n \in \mathcal{E}_{\mathbb{C}}^{\otimes_{p,q} n}. \tag{86}$$

Put $\Phi(\omega) = \sum_{n=0}^{+\infty} q^{(n(1-n)/2)} [n]_{p,q}! \langle : \omega^{\otimes n} : G_n \rangle$, and then (15) and (81) yield

$$\mathcal{S}\Phi(\xi) = \sum_{n=0}^{+\infty} [n]_{p,q}! \langle G_n, \xi^{\otimes n} \rangle = \sum_{n=0}^{+\infty} \langle G_n, \xi^{\otimes_{p,q} n} \rangle = g(\xi). \tag{87}$$

Using the same technics as in Lemma 1, we immediately prove that for all $s' > s$ such that $i_{s',s}$ is of the Hilbert-Schmidt type the following inequality holds:

$$|G_n|_{-s',p,q} \leq \|g\|_{s,p,q,\theta^*,\gamma} \theta_{n,p,q}^* \gamma^n \|i_{s',s}\|_{HS}^n. \quad (88)$$

Thus, under condition (82), we obtain

$$\begin{aligned} \|\Phi\|_{-s',p,q,\theta,\gamma'}^2 &= \sum_{n=0}^{+\infty} q^{n(1-n)} [n]_{p,q}^2 \theta_{n,p,q}^2 \gamma'^n |G_n|_{-s',p,q}^2 \\ &\leq \sum_{n=0}^{+\infty} q^{n(1-n)} [n]_{p,q}^2 \theta_{n,p,q}^2 \gamma'^n \theta_{n,p,q}^{*2} \gamma^{2n} \|g\|_{s,p,q,\theta^*,\gamma}^2 \|i_{s',s}\|_{HS}^{2n} \\ &\leq \|g\|_{s,p,q,\theta^*,\gamma}^2 \sum_{n=0}^{+\infty} \left(\beta \gamma \sqrt{\gamma'} \|i_{s',s}\|_{HS} \right)^{2n}. \end{aligned} \quad (89)$$

On the other hand, for all $\gamma' > 0$ such that $\beta \gamma \sqrt{\gamma'} \|i_{s',s}\|_{HS} < 1$, one can see that the series $\sum_{n=0}^{+\infty} (\beta \gamma \sqrt{\gamma'} \|i_{s',s}\|_{HS})^{2n}$ converges. This proves that $\mathcal{S}$ acts surjectively and that $\mathcal{S}^{-1}$ is continuous.

Data Availability

All data required for this paper are included within this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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