Hindawi
Journal of Mathematics
Volume 2020, Article ID 6631047, 6 pages
https://doi.org/10.1155/2020/6631047

Research Article

Characterizations of a Class of Dirichlet-Type Spaces and Related Operators

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Received 28 October 2020; Revised 17 November 2020; Accepted 23 November 2020; Published 9 December 2020

Academic Editor: Xiaolong Qin

In this paper, some characterizations are given in terms of the boundary value and Poisson extension for the Dirichlet-type space $\mathscr{D}(\mu)$. The multipliers of $\mathscr{D}(\mu)$ and Hankel-type operators from $\mathscr{D}(\mu)$ to $L^2(P_\mu dA)$ are also investigated.

1. Introduction

Let $\mathbb{D}$ be the unit disk of complex plane $\mathbb{C}$. For $0 < p < \infty$, the Hardy space, denoted by H^p, is the space consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^p d\theta < \infty. \quad (1)$$

Here, $H(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$.

Let $\partial\mathbb{D}$ denote the boundary of $\mathbb{D}$ and dA denote the normalized Lebesgue measure on $\mathbb{D}$. Let μ be a positive Borel measure on $\partial\mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the space $\mathscr{D}(\mu)$, called the Dirichlet-type space, if

$$\int_{\mathbb{D}} \left|f'(z)\right|^2 P_\mu(z) dA(z) < \infty, \quad (2)$$

where

$$P_\mu(z) = \int_0^{2\pi} \frac{1-|z|^2}{\left|e^{it}-z\right|^2} \frac{d\mu(t)}{2\pi}. \quad (3)$$

The space $\mathscr{D}(\mu)$ was introduced by Richter in [1] for studying analytic two isometrics. It was shown in [1] that $\mathscr{D}(\mu) \subset H^2$. The norm on $\mathscr{D}(\mu)$ is defined as follows:

$$\|f\|_{\mathscr{D}(\mu)}^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} \left|f'(z)\right|^2 P_\mu(z) dA(z). \quad (4)$$

The space $\mathscr{D}(\mu)$ is a Hilbert space with

$$\langle f, g\rangle_{\mathscr{D}(\mu)} = \langle f, g\rangle_{H^2} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} P_\mu(z) dA(z), \quad (5)$$

$\mathscr{D}(\mu) = H^2$ when $\mu = 0$. If $d\mu = dm$, then $\mathscr{D}(\mu)$ coincides with the Dirichlet space $\mathscr{D}$. By (Proposition 2.2 in [1]), we have

$$\int_{\partial\mathbb{D}} D_\zeta(f) d\mu(\zeta) = \int_{\mathbb{D}} \left|f'(z)\right|^2 P_\mu(z) dA(z). \quad (6)$$

Here,

$$D_\zeta(f) = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(e^{it}) - f(\zeta)}{e^{it} - \zeta} \right|^2 dt. \quad (7)$$

Let $f \in L^2(\partial\mathbb{D})$. We say that $f \in L^2(\mu)$ if

$$\int_{\partial\mathbb{D}} \int_0^{2\pi} \frac{\left|f\left(e^{i\theta}\right) - f(\zeta)\right|^2}{\left|e^{i\theta} - \zeta\right|^2} d\theta \, d\mu(\zeta) < \infty. \quad (8)$$

The norm of the space $L^2(\mu)$ is given by

$$\|f\|^2_{L^2(\mu)} = \|f\|^2_{L^2(\partial\mathbb{D})} + \int_{\partial\mathbb{D}}\int_0^{2\pi}\frac{\left|f\left(e^{i\theta}\right)-f(\zeta)\right|^2}{\left|e^{i\theta}-\zeta\right|^2}\mathrm{d}\theta\mathrm{d}\mu(\zeta). \tag{9}$$

The space $\mathscr{D}(\mu)$ has been investigated by many authors. In [2], Richter and Sundberg studied the cyclic vectors of $\mathscr{D}(\mu)$. Shimorin studied the reproducing kernels and extremal functions of $\mathscr{D}(\mu)$ in [3], see [4–6], for the study of Carleson measure for $\mathscr{D}(\mu)$. The study of composition operators and Toeplitz operators on $\mathscr{D}(\mu)$ can be found in [7, 8], respectively, see [9–11], for more study of the space $\mathscr{D}(\mu)$.

In this paper, we provided some characterizations for the space $\mathscr{D}(\mu)$ by the boundary value and Poisson extension. Moreover, we study the multipliers of $\mathscr{D}(\mu)$ and the Hankel-type operator from $\mathscr{D}(\mu)$ to $L^2(P_\mu dA)$.

In this paper, we always assume that μ is a positive Borel measure on $\partial\mathbb{D}$ and C is a positive constant that may differ from one occurrence to the other. The notation $F \lesssim G$ means that there exists a C such that $F \le CG$. The notation $F \asymp G$ indicates that $G \lesssim F$ and also $F \lesssim G$.

2. Characterizations of the Space $\mathscr{D}(\mu)$

Let $f \in L^1(\partial\mathbb{D})$. The Poisson extension of f, denoted by $\widehat{f}$, is

$$\widehat{f}(z) = \int_0^{2\pi} f\left(e^{\mathrm{it}}\right)\frac{1-|z|^2}{\left|e^{\mathrm{it}}-z\right|^2}\frac{\mathrm{d}t}{2\pi}, \quad z\in\mathbb{D}. \tag{10}$$

It is well known that $\widehat{f}$ is a harmonic function on $\mathbb{D}$.

Let $C^1(\mathbb{D})$ denote the space of all functions on $\mathbb{D}$ with continuous partial derivatives. For $f \in C^1(\mathbb{D})$, the gradient of f is defined by

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right). \tag{11}$$

First, we state some lemmas.

Lemma 1 (see [6, 8]). *Let $f \in L^2(\partial\mathbb{D})$. Then,*

$$\int_{\partial\mathbb{D}} D_\zeta(f)\mathrm{d}\mu(\zeta) < \infty, \tag{12}$$

if and only if

$$\int_{\mathbb{D}} |\nabla\widehat{f}(z)|^2 P_\mu(z)\mathrm{d}A(z) < \infty. \tag{13}$$

Remark 1. Let $f \in L^2(\partial\mathbb{D})$ and $F \in C^1(\mathbb{D})$ such that $\lim_{r\longrightarrow 1}F(re^{i\theta}) = f(e^{i\theta})$(a.e.) for $e^{i\theta} \in \partial\mathbb{D}$. Then,

$$\int_{\partial\mathbb{D}} D_\zeta(f)\mathrm{d}\mu(\zeta) \lesssim \|f\|^2_{L^2(\partial\mathbb{D})} + \int_{\mathbb{D}}|\nabla F(z)|^2 P_\mu(z)\mathrm{d}A(z). \tag{14}$$

For $f \in H^2$, let f_b denote the boundary value of f.

Corollary 1. *Let $f \in H^2$. Then, $f \in \mathscr{D}(\mu)$ if and only if $f_b \in L^2(\mu)$.*

Proof. Since $f \in H^2$, then $f = \widehat{f_b}$. The desired result follows from Lemma 1. ☐

Lemma 2. *Let $f \in L^2(\partial\mathbb{D})$. Then, the following statements are equivalent:*

(a) $\int_{\partial\mathbb{D}} D_\zeta(f)d\mu(\zeta) < \infty$.
(b) $\int_{\mathbb{D}} |\nabla\widehat{f}(z)|^2 P_\mu(z)dA(z) < \infty$.
(c) $\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}(\widehat{|f|^2}(z) - |\widehat{f}(z)|^2)d\mu_r(z) < \infty$*, where*

$$\mathrm{d}\mu_r(z) = \int_{\partial\mathbb{D}}\frac{r^2\left(1-r^2\right)}{|\zeta - rz|}\mathrm{d}\mu(\zeta)\mathrm{d}A(z). \tag{15}$$

Proof. $(a)\Longleftrightarrow(b)$ This implication follows by Lemma 1. ☐

Proof. $(b)\Longleftrightarrow(c)$ For $z \in \mathbb{D}, r \in (0,1)$, set

$$P_{\mu_r}(z) = \int_{\partial\mathbb{D}}\frac{r^2\left(1-|z|^2\right)}{|\zeta - rz|^2}\mathrm{d}\mu(\zeta). \tag{16}$$

From [11], we see that $P_{\mu_r}(z)$ is subharmonic with

$$\lim_{r\longrightarrow 1^-} P_{\mu_r}(z) = P_\mu(z). \tag{17}$$

By Green's formula, we obtain

$$\begin{aligned} P_{\mu_r}(z) &= \frac{2}{\pi}\int_{\mathbb{D}}\left(\frac{\partial^2}{\partial w\partial\overline{w}}P_{\mu_r}(w)\right)\log\left|\frac{1-\overline{w}z}{w-z}\right|\mathrm{d}A(w) \\ &\asymp \int_{\mathbb{D}}\int_{\partial\mathbb{D}}\frac{r^2\left(1-r^2\right)}{|\zeta - rw|}\mathrm{d}\mu(\zeta)\log\left|\frac{1-\overline{w}z}{w-z}\right|\mathrm{d}A(w). \end{aligned} \tag{18}$$

According to (17) and (18) and Hardy-Littlewood's identity (see page 238 in [12]), we have

$$\begin{aligned}\int_{\mathbb{D}}|\nabla \widehat{f}(z)|^2P_{\mu}(z)\mathrm{d}A(z) &= \lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}|\nabla \widehat{f}(z)|^2P_{\mu_r}(z)\mathrm{d}A(z)\\ &\asymp \lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}|\nabla \widehat{f}(z)|^2\left(\int_{\mathbb{D}}\int_{\partial\mathbb{D}}\frac{r^2(1-r^2)}{|\zeta - rw|}\mathrm{d}\mu(\zeta)\log\left|\frac{1-\overline{w}z}{w-z}\right|\mathrm{d}A(w)\right)\mathrm{d}A(z)\\ &= \lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left(\int_{\mathbb{D}}|\nabla \widehat{f}(z)|^2\log\left|\frac{1-\overline{w}z}{w-z}\right|\mathrm{d}A(z)\right)\int_{\partial\mathbb{D}}\frac{r^2(1-r^2)}{|\zeta - rw|}\mathrm{d}\mu(\zeta)\mathrm{d}A(w)\\ &= \lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left(\widehat{|f|^2}(w) - |\widehat{f}(w)|^2\right)\mathrm{d}\mu_r(w).\end{aligned} \tag{19}$$

The proof is complete. □

Theorem 1. *Let $f \in H^2$. Then, the following statements are equivalent:*

(a) $f \in \mathscr{D}(\mu)$.

(b) $\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}|f - \widehat{f}(z)|^2(z)d\mu_r(z) < \infty$.

(c) $\int_{\partial\mathbb{D}}D_{\zeta}(|f|)d\mu(\zeta) < \infty$ and

$$\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left(\widehat{|f|}^2(z) - |f(z)|^2\right)\mathrm{d}\mu_r(z) < \infty. \tag{20}$$

(d) $\int_{\partial\mathbb{D}}D_{\zeta}(|f|)d\mu(\zeta) < \infty$ and there exists a harmonic function g such that $|f| \le g$ on $\mathbb{D}$ and

$$\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left(g^2(z) - |f(z)|^2\right)\mathrm{d}\mu_r(z) < \infty. \tag{21}$$

Proof. $(a)\Leftrightarrow(b)$ This implication follows by Lemma 2 and

$$\widehat{|f|^2}(z) - |f(z)|^2 = |f - \widehat{f}(z)|^2(z). \tag{22}$$

$(a)\Rightarrow(c)$ If $f \in \mathscr{D}(\mu)$, then $\int_{\partial\mathbb{D}}D_{\zeta}(|f|)\mathrm{d}\mu(\zeta) < \infty$. Since

$$\begin{aligned}(\widehat{|f|}(z))^2 &= \left(\int_{\partial\mathbb{D}}|f(\zeta)|\frac{1-|z|^2}{|\zeta - z|^2}\frac{|\mathrm{d}\zeta|}{2\pi}\right)^2\\ &\le \int_{\partial\mathbb{D}}|f(\zeta)|^2\frac{1-|z|^2}{|\zeta - z|^2}\frac{|\mathrm{d}\zeta|}{2\pi}\int_{\partial\mathbb{D}}\frac{1-|z|^2}{|\zeta - z|^2}\frac{|\mathrm{d}\zeta|}{2\pi}\\ &= \widehat{|f|^2}(z).\end{aligned} \tag{23}$$

We get (c) from Lemma 2 and Corollary 1.

$(c)\Rightarrow(d)$ Inequality (20) implies

$$\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left((\widehat{|f|(z)})^2 - |f(z)|^2\right)\mathrm{d}\mu_r(z) < \infty. \tag{24}$$

Let $g = \widehat{|f|}$. Then, $g^2 \le (\widehat{|f|})^2$. Thus,

$$\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left((g(z))^2 - |f(z)|^2\right)\mathrm{d}\mu_r(z) < \infty. \tag{25}$$

$(d)\Rightarrow(a)$ By Lemma 2,

$$\lim_{r\longrightarrow 1^-}\int_{\mathbb{D}}\left(\widehat{|f|^2}(z) - (\widehat{|f|}(z))^2\right)\mathrm{d}\mu_r(z) < \infty. \tag{26}$$

Assume that g is a harmonic function such that $|f| \le g$. Note that $\widehat{|f|}$ is the least harmonic function equal to or greater than $|f|$ (see [12]); hence, $\widehat{|f|} \le g$. By Lemmas 1 and 2 and Corollary 1, $f \in \mathscr{D}(\mu)$. The proof is complete. □

3. Multipliers of $\mathscr{D}(\mu)$

Let $I \subset \partial\mathbb{D}$. The Carleson box $S(I)$ is

$$S(I) = \{r\zeta \in \mathbb{D}\colon 1 - |I| < r < 1; \zeta \in I\}. \tag{27}$$

Assume that ν is a positive Borel measure on $\mathbb{D}$. If $\sup_{I\subset\partial\mathbb{D}}(\nu(S(I))/|I|) < \infty$, then we say that ν is a Carleson measure.

If there exists a constant $C > 0$ (see [4, 5])

$$\int_{\mathbb{D}}|f(z)|^2\mathrm{d}\nu(z) \le C\|f\|^2_{\mathscr{D}(\mu)}, \quad \text{for all } f \in \mathscr{D}(\mu), \tag{28}$$

then we call that ν is a μ-Carleson measure.

Let $g \in L^{\infty}(\partial\mathbb{D})$ and $f \in L^2(\mu)$. g is called the pointwise multipliers of $L^2(\mu)$ if $gf \in L^2(\mu)$. We denote the space of all pointwise multipliers of $L^2(\mu)$ by $M(L^2(\mu))$.

Lemma 3. *Let ν be a positive Borel measure on $\mathbb{D}$. Then, ν is a μ-Carleson measure if and only if*

$$\int_{\mathbb{D}}|\widehat{g}(z)|^2\mathrm{d}\nu(z) \lesssim \|g\|^2_{L^2(\mu)}, \tag{29}$$

for all $g \in L^2(\mu)$.

Proof. First, we assume that ν is a μ-Carleson measure. Suppose that $g \in L^2(\mu)$. Without loss of generality, let g be a real-valued function. Suppose that $\widetilde{g}$ is the harmonic conjugate of $\widehat{g}$. Set $f = \widehat{g} + i\widetilde{g}$. Then, $|\nabla \widehat{f}(z)| \asymp |f'(z)|$ by the Cauchy–Riemann equation. From Lemma 2.3 in [7] and Lemma 1, we obtain

$$\begin{aligned}\int_{\mathbb{D}}|\widehat{g}(z)|^2 d\nu(z) &\leq \int_{\mathbb{D}}|f(z)|^2 d\nu(z)\\ &\lesssim \|f\|^2_{\mathscr{D}(\mu)}\\ &= \|f\|^2_{H^2} + \int_{\mathbb{D}}\left|f'(z)\right|^2 P_\mu(z) dA(z)\\ &\asymp |f(0)|^2 + \int_{\mathbb{D}}|\nabla\widehat{g}(z)|^2 P_\mu(z) dA(z)\\ &\lesssim \|g\|^2_{L^2(\mu)}.\end{aligned} \tag{30}$$

Conversely, for $f \in \mathscr{D}(\mu)$, by Corollary 1, $f_b \in L^2(\mu)$ and $f = \widehat{f}_b$. Then,

$$\int_{\mathbb{D}}|f(z)|^2 d\nu(z) \lesssim \left\|f_b\right\|^2_{L^2(\mu)} \lesssim \|f\|^2_{\mathscr{D}(\mu)}, \tag{31}$$

which implies that ν is a μ-Carleson measure. □

Theorem 2. *$g \in M(L^2(\mu))$ if and only if $g \in L^\infty(\partial\mathbb{D})$ and $|\nabla\widehat{g}|^2 P_\mu dA$ is a μ-Carleson measure.*

Proof. Assume that $g \in L^\infty(\partial\mathbb{D})$ and $|\nabla\widehat{g}|^2 P_\mu dA$ is a μ-Carleson measure. Let $f \in L^2(\mu)$. By Remark 1, we obtain

$$\begin{aligned}\|fg\|^2_{L^2(\mu)} &\lesssim \|fg\|^2_{L^2(\partial\mathbb{D})} + \int_{\mathbb{D}}|\nabla(\widehat{f}\widehat{g})(z)|^2 P_\mu(z) dA(z)\\ &\leq \|fg\|^2_{L^2(\partial\mathbb{D})} + \int_{\mathbb{D}}|\widehat{g}(z)|^2|\nabla\widehat{f}(z)|^2 P_\mu(z) dA(z)\\ &\quad + \int_{\mathbb{D}}|\widehat{f}(z)|^2|\nabla\widehat{g}(z)|^2 P_\mu(z) dA(z).\end{aligned} \tag{32}$$

By Lemma 1 and Corollary 1, we obtain

$$\int_{\mathbb{D}}|\widehat{g}(z)|^2|\nabla\widehat{f}(z)|^2 P_\mu(z) dA(z) \leq C\|\widehat{g}\|^2_{L^\infty(\mathbb{D})}\|f\|^2_{L^2(\mu)}. \tag{33}$$

In addition, since $|\nabla\widehat{g}|^2 P_\mu dA$ is a μ-Carleson measure, by Lemma 3, we have

$$\int_{\mathbb{D}}|\widehat{f}(z)|^2|\nabla\widehat{g}(z)|^2 P_\mu(z) dA(z) \leq C\|f\|^2_{L^2(\mu)}. \tag{34}$$

Combining (32)–(34), we obtain that $g \in M(L^2(\mu))$.

Conversely, assume that $g \in M(L^2(\mu))$. Then, by Theorem 2.7 in [6], we see that $g \in L^\infty(\partial\mathbb{D})$. For $f \in \mathscr{D}(\mu)$, by the Closed Graph Theorem, Lemma 1, and Corollary 1, we obtain

$$\begin{aligned}\int_{\mathbb{D}}|\nabla(f\widehat{g})(z)|^2 P_\mu(z) dA(z) &\leq C\|fg\|^2_{L^2(\mu)} \leq C\|f\|^2_{L^2(\mu)}\\ &\leq C\|f\|^2_{\mathscr{D}(\mu)}.\end{aligned} \tag{35}$$

Next, we show that $|\nabla\widehat{g}|^2 P_\mu dA$ is a μ-Carleson measure. From the fact that $|\nabla f| \asymp |f'(z)|$, we obtain

$$\begin{aligned}\int_{\mathbb{D}}|\widehat{g}(z)|^2|\nabla f(z)|^2 P_\mu(z) dA(z) &\leq C\int_{\mathbb{D}}|\nabla f(z)|^2 P_\mu(z) dA(z)\\ &\asymp C\int_{\mathbb{D}}\left|f'(z)\right|^2 P_\mu(z) dA(z)\\ &\leq C\|f\|^2_{\mathscr{D}(\mu)}.\end{aligned} \tag{36}$$

Then, by (35) and (36),

$$\begin{aligned}\int_{\mathbb{D}}|f(z)|^2|\nabla\widehat{g}(z)|^2 P_\mu(z) dA(z) &\leq C\int_{\mathbb{D}}\left(|\nabla(f\widehat{g})(z)|^2 + |\widehat{g}(z)|^2|\nabla f(z)|^2\right) P_\mu(z) dA(z)\\ &\leq C\|f\|^2_{\mathscr{D}(\mu)},\end{aligned} \tag{37}$$

which implies that $|\nabla\widehat{g}|^2 P_\mu dA$ is a μ-Carleson measure.

By Theorem 2, we obtain the following result. □

Corollary 2. *Let $f \in H^2$. Then, $f \in M(\mathscr{D}(\mu))$ if and only if $f_b \in M(L^2(\mu))$.*

4. Hankel-Type Operators on $\mathscr{D}(\mu)$

Let $\mathscr{P}$ denote the set of all polynomials on $\mathbb{D}$. From [1, 2], we see that $\mathscr{P}$ is dense in $\mathscr{D}(\mu)$. Let

$$Pf(z) = \int_D \frac{f(w)}{(1-\overline{w}z)^2} dA(w). \tag{38}$$

From Theorem 1.10 in [13], we see that $P: L^2(\mathbb{D}) \longrightarrow A^2$ is a bounded projection. Here, A^2 is the Bergman space which consists of all $f \in H(\mathbb{D})$ such that $\int_{\mathbb{D}}|f(z)|^2 dA(z) < \infty$. For $f \in A^2$, we define a Hankel-type operator h_f on $\mathscr{P}$ by

$$h_f(g) = \overline{P(f\overline{g})}, \quad g \in \mathscr{P}. \tag{39}$$

Lemma 4 (see Theorem 2.3 in [10]). *Let $\tau, \sigma > -1$. Then, $f \in \mathscr{D}(\mu)$ if and only if*

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z)-f(w)|^2}{|1-\overline{z}w|^{4+\sigma+\tau}} P_\mu(z) dA_\sigma(z) dA_\tau(w) < \infty, \tag{40}$$

where $dA_\sigma(z) = (1-|z|^2)^\sigma dA(z)$.

Lemma 5 (see Theorem 3.4 in [10]). *Let T be the operator defined by*

$$Tg(z) = \int_{\mathbb{D}} \frac{|g(w)|}{|1-\overline{w}z|^2} dA(w), \quad g \in L^2(\mathbb{D}). \tag{41}$$

Then, $T: L^2(P_\mu dA) \longrightarrow L^2(P_\mu dA)$ is bounded.

Theorem 3. *Let $g \in L^2(\mathbb{D})$ such that $|g|^2 P_\mu dA$ is a μ-Carleson measure. Then, $|Tg|^2 P_\mu dA$ is a μ-Carleson measure.*

Proof. Suppose that $|g|^2 P_\mu dA$ is a μ-Carleson measure. Then, by Lemma 5,

$$\int_{\mathbb{D}} |T(fg)(z)|^2 P_\mu(z) \mathrm{d}A(z) \leq C \int_{\mathbb{D}} |f(z)g(z)|^2 P_\mu(z) \mathrm{d}A(z) \leq C\|f\|^2_{\mathscr{D}(\mu)}, \tag{42}$$

for all $f \in \mathscr{D}(\mu)$. So, it is enough to show that

$$\int_{\mathbb{D}} |f(z)Tg(z) - T(fg)(z)|^2 P_\mu(z) \mathrm{d}A(z) \leq C\|f\|^2_{\mathscr{D}(\mu)}, \tag{43}$$

for every $f \in \mathscr{D}(\mu)$.

By Hölder's inequality, we have

$$\begin{aligned} |f(z)Tg(z) - T(fg)(z)|^2 &\leq \left(\int_{\mathbb{D}} \frac{|f(z) - f(w)|}{|1 - \overline{w}z|^2} |g(w)| \mathrm{d}A(w) \right)^2 \\ &\lesssim \int_{\mathbb{D}} |g(w)|^2 \mathrm{d}A(w) \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - \overline{w}z|^4} \mathrm{d}A(w) \\ &= \|g\|^2_{L^2(\mathbb{D})} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - \overline{w}z|^4} \mathrm{d}A(w). \end{aligned} \tag{44}$$

Consequently, by Lemma 4, we obtain

$$\begin{aligned} &\int_{\mathbb{D}} |f(z)Tg(z) - T(fg)(z)|^2 P_\mu(z) \mathrm{d}A(z) \\ &\leq \|g\|^2_{L^2(\mathbb{D})} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - \overline{w}z|^4} \mathrm{d}A(w) P_\mu(z) \mathrm{d}A(z) \\ &\leq \|g\|^2_{L^2(\mathbb{D})} \|f\|^2_{\mathscr{D}(\mu)}. \end{aligned} \tag{45}$$

The desired result follows. □

Theorem 4. *Let $u \in A^2$. Then, the operator $h_u: \mathscr{D}(\mu) \longrightarrow L^2(P_\mu dA)$ is bounded if and only if $|u|^2 P_\mu dA$ is a μ-Carlson measure.*

Proof. Suppose that $|u|^2 P_\mu \mathrm{d}A$ is a μ-Carlson measure. Let $g \in \mathscr{D}(\mu)$. Then, $u\overline{g} \in L^2(P_\mu \mathrm{d}A)$. By Lemma 4, we get that $h_u(g) \in L^2(P_\mu \mathrm{d}A)$ and

$$\begin{aligned} \left\|h_u(g)\right\|_{L^2(P_\mu \mathrm{d}A)} &\leq \|T(u\overline{g})\|_{L^2(P_\mu \mathrm{d}A)} \leq C\|u\overline{g}\|_{L^2(P_\mu \mathrm{d}A)} \\ &\leq C\|g\|_{\mathscr{D}(\mu)}. \end{aligned} \tag{46}$$

So, $h_u: \mathscr{D}(\mu) \longrightarrow L^2(P_\mu \mathrm{d}A)$ is bounded.

Conversely, assume that $h_u: \mathscr{D}(\mu) \longrightarrow L^2(P_\mu \mathrm{d}A)$ is bounded. We need to prove that

$$\|u\overline{g}\|_{L^2(P_\mu \mathrm{d}A)} \leq C\|g\|_{\mathscr{D}(\mu)}, \quad \text{for any } g \in \mathscr{D}(\mu). \tag{47}$$

By Hölder's inequality we have

$$\begin{aligned} \left| \int_{\mathbb{D}} \frac{u(w)\left(\overline{g(z) - g(w)}\right)}{(1 - \overline{w}z)^2} \mathrm{d}A(w) \right|^2 &\leq \int_{\mathbb{D}} |u(w)|^2 \mathrm{d}A(w) \int_{\mathbb{D}} \frac{|g(z) - g(w)|^2}{|1 - \overline{w}z|^4} \mathrm{d}A(w) \\ &= \|u\|^2_{A^2} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^2}{|1 - \overline{w}z|^4} \mathrm{d}A(w). \end{aligned} \tag{48}$$

Since

$$u(z)\overline{g(z)} - \overline{h_u(g)(z)} = \int_{\mathbb{D}} \frac{u(w)\left(\overline{g(z) - g(w)}\right)}{(1 - \overline{w}z)^2} \mathrm{d}A(w), \tag{49}$$

by Lemma 4 and the fact that $h_u: \mathscr{D}(\mu) \longrightarrow L^2(P_\mu \mathrm{d}A)$ is bounded, we obtain

$$\begin{aligned} \|u\overline{g}\|^2_{L^2(P_\mu \mathrm{d}A)} &\leq \|u\|^2_{A^2} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^2}{|1 - \overline{w}z|^4} \mathrm{d}A(w) P_\mu(z) \mathrm{d}A(z) \\ &\quad + \left\|h_u\right\|^2 \|g\|^2_{\mathscr{D}(\mu)} \\ &\lesssim \left(\|u\|^2_{A^2} + \left\|h_u\right\|^2 \right) \|g\|^2_{\mathscr{D}(\mu)}. \end{aligned} \tag{50}$$

The proof is complete. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The first author was supported by NNSF of China (nos. 11701222 and 11801347), China Postdoctoral Science Foundation (no. 2018M633090), and Key Projects of Fundamental Research in Universities of Guangdong Province (no. 2018KZDXM034).

References

[1] S. Richter, "A representation theorem for cyclic analytic two-isometries," *Transactions of the American Mathematical Society*, vol. 328, no. 1, pp. 325–349, 1991.
[2] S. Richter and C. Sundberg, "A formula for the local Dirichlet integral," *The Michigan Mathematical Journal*, vol. 38, no. 3, pp. 355–379, 1991.
[3] S. M. Shimorin, "Reproducing kernels and extremal functions in Dirichlet-type spaces," *Journal of Mathematical Sciences*, vol. 107, no. 4, pp. 4108–4124, 2001.
[4] G. R. Chacón, "Carleson measures on Dirichlet-type spaces," *Proceedings of the American Mathematical Society*, vol. 139, no. 5, p. 1605, 2011.
[5] G. R. Chacón, E. Fricain, and M. Shabankhah, "Carleson measures and reproducing kernel thesis in Dirichlet-type spaces," *St. Petersburg Mathematical Journal*, vol. 24, no. 6, pp. 847–861, 2013.
[6] R. Chartrand, "Multipliers and Carleson measures for $ D (\mu) $," *Integral Equations and Operator Theory*, vol. 45, no. 3, pp. 309–318, 2003.
[7] G. R. Chacón, "Closed-range composition operators on Dirichlet-type spaces," *Complex Analysis and Operator Theory*, vol. 7, no. 4, pp. 909–926, 2013.
[8] R. Chartrand, "Toeplitz operator on Dirichlet-type spaces," *The Journal of Operator Theory*, vol. 48, pp. 3–13, 2002.
[9] G. R. Chacón, "Interpolating sequences in harmonically weighted Dirichlet spaces," *Integral Equations and Operator Theory*, vol. 69, no. 1, pp. 73–85, 2011.
[10] X. Liu, G. R. Chacón, and Z. Lou, "Characterizations of the dirichlet-type space," *Complex Analysis and Operator Theory*, vol. 9, no. 6, pp. 1269–1286, 2015.
[11] S. Shimorin, "Complete nevanlinna-pick property of dirichlet-type spaces," *Journal of Functional Analysis*, vol. 191, no. 2, pp. 276–296, 2002.
[12] J. Garnett, *Bounded Analytic Functions*, Academic Press, Cambridge, MA, USA, 1981.
[13] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Springer, New York, NY, USA, 2000.