Hindawi
Journal of Mathematics
Volume 2020, Article ID 6636105, 7 pages
https://doi.org/10.1155/2020/6636105

Research Article

Periodic Solutions of a System of Nonlinear Difference Equations with Periodic Coefficients

Durhasan Turgut Tollu

Necmettin Erbakan University, Faculty of Science, Department of Mathematics and Computer Sciences, Konya, Turkey

Correspondence should be addressed to Durhasan Turgut Tollu; dttollu@erbakan.edu.tr

Received 6 October 2020; Revised 18 November 2020; Accepted 26 November 2020; Published 16 December 2020

Academic Editor: Hijaz Ahmad

This paper is dealt with the following system of difference equations $x_{n+1} = (a_n/x_n) + (b_n/y_n)$, $y_{n+1} = (c_n/x_n) + (d_n/y_n)$, where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the initial values x_0 and y_0 are the positive real numbers, and the sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$, and $(d_n)_{n\geq 0}$ are two-periodic and positive. The system is an extension of a system where every positive solution is two-periodic or converges to a two-periodic solution. Here, the long-term behavior of positive solutions of the system is examined by using a new method to solve the system.

1. Introduction

Studying concrete nonlinear difference equations and systems has attracted a great recent interest. Many studies have been published on this topic in the last twenty years (see, e.g., [1–15]). Particularly, there has been a renewed interest in solvable nonlinear difference equations and systems for fifteen years (see, e.g., [5, 16–21], and the related references therein). Solvable difference equations are not interesting for themselves only, but they can be also applied in other areas of mathematics, as well as other areas of science (see, e.g., [22, 23]).

One of the first examples of solvable nonlinear difference equations is presented in note [24] where Brand solves the nonlinear difference equation:

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \quad n \in \mathbb{N}_0, \tag{1}$$

where the initial value x_0 is a real number and the parameters a, b, c, and d are the real numbers with the restrictions $c \neq 0$ and $ad - bc \neq 0$ and studies long-term behavior of solutions of the equation. The note presents a transformation which transforms the nonlinear equation into a linear one. The idea has been used many times in showing solvability of some difference equations, as well as of some systems of difference equations (see, e.g., [5, 18, 20, 21, 25, 26]). Another example of solvable nonlinear difference equations is the following system of nonlinear difference equation:

$$x_{n+1} = \frac{a}{x_n} + \frac{b}{y_n}, y_{n+1} = \frac{c}{x_n} + \frac{d}{y_n}, \quad n \in \mathbb{N}_0, \tag{2}$$

where the initial values x_0 and y_0 are the positive real numbers and the parameters a, b, c, and d are the positive real numbers. System (2) can be transformed into an equation of form (1) by dividing the first equation of (2) by its second one. So, the results on equation (1) can be used to obtain the results on system (2). System (2) was studied for the first time in [17] by using the method described above. Also, in [17], it is shown that every positive solution of the system (2) is two-periodic or converges to a two-periodic solution. For more results on system (2), see [19, 22, 27].

System (2) can be extended by interchanging the parameters a, b, c, and d with the sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$, and $(d_n)_{n\geq 0}$. More concretely, another extension of (2) is the following system of difference equations:

$$x_{n+1} = \frac{a_n}{x_n} + \frac{b_n}{y_n}, y_{n+1} = \frac{c_n}{x_n} + \frac{d_n}{y_n}, \quad n \in \mathbb{N}_0, \tag{3}$$

where the initial values x_0 and y_0 are the positive real numbers and $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$, and $(d_n)_{n\geq 0}$ are the two-periodic sequences of positive real numbers. For extensions with periodic sequences of some difference equations and systems, see [2, 28, 29].

Our main purpose in this paper is to determine the long-term behavior of positive solutions of system (3). We also use a new method to solve the system without needing some other nonlinear difference equations such as (1). Throughout this paper, we assume that $a_{2n} = a_0$, $a_{2n+1} = a_1$; $b_{2n} = b_0$, $b_{2n+1} = b_1$; $c_{2n} = c_0$, $c_{2n+1} = c_1$; and $d_{2n} = d_0$, $d_{2n+1} = d_1$ with $a_0 \neq a_1$, $b_0 \neq b_1$, $c_0 \neq c_1$, and $d_0 \neq d_1$. We also adopt the conventions:

$$\sum_{k=m}^{m-l} s_k = 0, \ \prod_{k=m}^{m-l} s_k = 1, \quad l \in \mathbb{N}, \tag{4}$$

where (s_k) is any sequence and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1. A solution $(x_n, y_n)_{n\geq 0}$ of the system

$$x_{n+1} = f(x_n, y_n), y_{n+1} = g(x_n, y_n), \quad n \in \mathbb{N}_0, \tag{5}$$

is eventually periodic with period p, if there is $n_1 > 0$ such that $(x_{n+p}, y_{n+p}) = (x_n, y_n)$ for $n \geq n_1$. If $n_1 = 0$, then the solution is periodic with period p.

The following result is extracted from [30].

Remark 1. A product $\prod_{k=0}^{\infty}(1+\alpha_k)$ with positive terms α_k is convergent if and only if $\sum_{k=0}^{\infty}\alpha_k$ converges.

2. Main Results

In this section, we formulate and prove our main results.

Theorem 1. *Assume that $x_0, y_0 > 0$ and $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$, and $(d_n)_{n\geq 0}$ are the two-periodic sequences of positive real numbers. Then, system (3) can be solved in closed form.*

Proof. First, it is easy to show by induction that $x_n, y_n > 0$, for all $n \in \mathbb{N}_0$. Multiplying both equations in (3) by the following positive product:

$$\prod_{k=0}^{n} x_k y_k, \tag{6}$$

we obtain

$$\prod_{k=0}^{n+1} x_k \prod_{k=0}^{n} y_k = a_n \prod_{k=0}^{n-1} x_k \prod_{k=0}^{n} y_k + b_n \prod_{k=0}^{n} x_k \prod_{k=0}^{n-1} y_k, \tag{7}$$

$$\prod_{k=0}^{n} x_k \prod_{k=0}^{n+1} y_k = c_n \prod_{k=0}^{n-1} x_k \prod_{k=0}^{n} y_k + d_n \prod_{k=0}^{n} x_k \prod_{k=0}^{n-1} y_k, \tag{8}$$

for all $n \in \mathbb{N}_0$. Note that the equalities (7) and (8) constitute a linear system with respect to the following products:

$$\begin{aligned} u_n &= \prod_{k=0}^{n} x_k \prod_{k=0}^{n-1} y_k, \\ v_n &= \prod_{k=0}^{n-1} x_k \prod_{k=0}^{n} y_k. \end{aligned} \tag{9}$$

Therefore, we can write this system in the vector form:

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} b_n & a_n \\ d_n & c_n \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \tag{10}$$

where $u_0 = x_0$ and $v_0 = y_0$, which is simpler, for all $n \in \mathbb{N}_0$. Let

$$A_n = \begin{pmatrix} b_n & a_n \\ d_n & c_n \end{pmatrix}. \tag{11}$$

Then, since the sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$, and $(d_n)_{n\geq 0}$ are two-periodic, the matrix A_n becomes

$$A_n = \begin{cases} A_0 = \begin{pmatrix} b_0 & a_0 \\ d_0 & c_0 \end{pmatrix}, & \text{if } n \text{ is even}, \\ A_1 = \begin{pmatrix} b_1 & a_1 \\ d_1 & c_1 \end{pmatrix}, & \text{if } n \text{ is odd}. \end{cases} \tag{12}$$

Now, we decompose (10) with respect to even-subscript and odd-subscript terms as follows:

$$\begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} = \begin{pmatrix} b_0 & a_0 \\ d_0 & c_0 \end{pmatrix} \begin{pmatrix} u_{2n} \\ v_{2n} \end{pmatrix}, \tag{13}$$

$$\begin{pmatrix} u_{2n+2} \\ v_{2n+2} \end{pmatrix} = \begin{pmatrix} b_1 & a_1 \\ d_1 & c_1 \end{pmatrix} \begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix}, \tag{14}$$

for all $n \in \mathbb{N}_0$. From which (13) and (14) follows that

$$\begin{pmatrix} u_{2n+2} \\ v_{2n+2} \end{pmatrix} = \begin{pmatrix} b_1 & a_1 \\ d_1 & c_1 \end{pmatrix} \begin{pmatrix} b_0 & a_0 \\ d_0 & c_0 \end{pmatrix} \begin{pmatrix} u_{2n} \\ v_{2n} \end{pmatrix}, \tag{15}$$

or

$$\begin{pmatrix} u_{2n+2} \\ v_{2n+2} \end{pmatrix} = \begin{pmatrix} a_1 d_0 + b_0 b_1 & a_0 b_1 + a_1 c_0 \\ b_0 d_1 + c_1 d_0 & a_0 d_1 + c_0 c_1 \end{pmatrix} \begin{pmatrix} u_{2n} \\ v_{2n} \end{pmatrix}. \tag{16}$$

Let $A_1 A_0 = A$. Then, we consider two cases of the matrix A as the following:

Case 1: $\operatorname{rank}(A) = 1$. In this case, the first row in the matrix A is linearly dependent on the second one. Without loss of generality, we may assume that

$$(b_0 d_1 + c_1 d_0, a_0 d_1 + c_0 c_1) = K(a_1 d_0 + b_0 b_1, a_0 b_1 + a_1 c_0), \tag{17}$$

where K is a positive constant such that

$$K = \frac{b_0 d_1 + c_1 d_0}{a_1 d_0 + b_0 b_1} = \frac{a_0 d_1 + c_0 c_1}{a_0 b_1 + a_1 c_0}. \tag{18}$$

Using (17) in system (16), we have

$$u_{2n+2} = (a_1 d_0 + b_0 b_1) u_{2n} + (a_0 b_1 + a_1 c_0) v_{2n},$$
$$v_{2n+2} = K((a_1 d_0 + b_0 b_1) u_{2n} + (a_0 b_1 + a_1 c_0) v_{2n}), \quad (19)$$

which implies the relation

$$v_{2n+2} = K u_{2n+2}, \quad (20)$$

for all $n \in \mathbb{N}_0$. By the last three relations, we have

$$u_{2n+2} = (a_1 d_0 + b_0 b_1 + K(a_0 b_1 + a_1 c_0)) u_{2n}, \quad (21)$$

from which it follows that

$$u_{2n} = (a_1 d_0 + b_0 b_1 + K(a_0 b_1 + a_1 c_0))^{n-1} u_2, \quad (22)$$

$$v_{2n} = K(a_1 d_0 + b_0 b_1 + K(a_0 b_1 + a_1 c_0))^{n-1} u_2, \quad (23)$$

for all $n \in \mathbb{N}$. Using (22) and (23) in (13), we obtain

$$u_{2n+1} = (b_0 + K a_0)(a_1 d_0 + b_0 b_1 + K(a_0 b_1 + a_1 c_0))^{n-1} u_2,$$
$$v_{2n+1} = (d_0 + K c_0)(a_1 d_0 + b_0 b_1 + K(a_0 b_1 + a_1 c_0))^{n-1} u_2, \quad (24)$$

for all $n \in \mathbb{N}$. On the other hand, the changes of variables in (9) yield

$$x_{n+1} = \frac{u_{n+1} v_{n-1}}{u_n v_n} x_{n-1}, \quad (25)$$

$$y_{n+1} = \frac{v_{n+1} u_{n-1}}{v_n u_n} y_{n-1}, \quad (26)$$

for all $n \in \mathbb{N}$. Hence, from (25) and (26), we obtain

$$x_{2n} = x_0 \prod_{k=1}^{n} \frac{u_{2k} v_{2k-2}}{u_{2k-1} v_{2k-1}}, \quad (27)$$

$$x_{2n+1} = x_1 \prod_{k=1}^{n} \frac{u_{2k+1} v_{2k-1}}{u_{2k} v_{2k}}, \quad (28)$$

$$y_{2n} = y_0 \prod_{k=0}^{n} \frac{v_{2k} u_{2k-2}}{v_{2k-1} u_{2k-1}}, \quad (29)$$

$$y_{2n+1} = y_1 \prod_{k=1}^{n} \frac{v_{2k+1} u_{2k-1}}{v_{2k} u_{2k}}. \quad (30)$$

By employing (13) in (27)–(30), we have the following closed formulas:

$$x_{2n} = x_0 \prod_{k=1}^{n} \frac{u_{2k} v_{2k-2}}{(b_0 u_{2k-2} + a_0 v_{2k-2})(d_0 u_{2k-2} + c_0 v_{2k-2})}, \quad (31)$$

$$x_{2n+1} = x_1 \prod_{k=1}^{n} \frac{(b_0 u_{2k} + a_0 v_{2k})(d_0 u_{2k-2} + c_0 v_{2k-2})}{u_{2k} v_{2k}}, \quad (32)$$

$$y_{2n} = y_0 \prod_{k=1}^{n} \frac{v_{2k} u_{2k-2}}{(d_0 u_{2k-2} + c_0 v_{2k-2})(b_0 u_{2k-2} + a_0 v_{2k-2})}, \quad (33)$$

$$y_{2n+1} = y_1 \prod_{k=1}^{n} \frac{(d_0 u_{2k} + c_0 v_{2k})(b_0 u_{2k-2} + a_0 v_{2k-2})}{v_{2k} u_{2k}}, \quad (34)$$

which is valid for all $n \in \mathbb{N}_0$, respectively. Consequently, in the case $\operatorname{rank}(A) = 1$, by using the formulas (22) and (23) in (31)–(34), we have the general solution of (3) as follows:

$$x_{2n} = x_0 \prod_{k=1}^{n} \frac{K(a_1 d_0 + b_0 b_1 + K(a_0 b_1 + a_1 c_0))}{(b_0 + K a_0)(d_0 + K c_0)}, \quad (35)$$

$$x_{2n+1} = x_1 \prod_{k=1}^{n} \frac{(b_0 + K a_0)(d_0 + K c_0)}{K(a_1 d_0 + b_0 b_1 + K(a_0 b_1 + a_1 c_0))}, \quad (36)$$

$$y_{2n} = y_0 \prod_{k=1}^{n} \frac{K(a_1 d_0 + b_0 b_1 + K(a_0 b_1 + a_1 c_0))}{(d_0 + K c_0)(b_0 + K a_0)}, \quad (37)$$

$$y_{2n+1} = y_1 \prod_{k=1}^{n} \frac{(d_0 + K c_0)(b_0 + K a_0)}{K(a_1 d_0 + b_0 b_1 + K(a_0 b_1 + a_1 c_0))}, \quad (38)$$

where

$$K = \frac{b_0 d_1 + c_1 d_0}{a_1 d_0 + b_0 b_1} = \frac{a_0 d_1 + c_0 c_1}{a_0 b_1 + a_1 c_0}, \quad (39)$$

for all $n \in \mathbb{N}_0$.

Case 2: $\operatorname{rank}(A) = 2$. In this case, both rows in the matrix A are linearly independent of each other. This case also implies that A has two different eigenvalues denoted by λ_1 and λ_2. Since these eigenvalues will correspond to two linear independent eigenvectors, we may write the matrix A as follows:

$$A = P\Lambda P^{-1}, \tag{40}$$

where

$$P = \begin{pmatrix} \dfrac{a_0b_1 + a_1c_0}{\lambda_1 - (a_1d_0 + b_0b_1)} & \dfrac{a_0b_1 + a_1c_0}{\lambda_2 - (a_1d_0 + b_0b_1)} \\ 1 & 1 \end{pmatrix},$$

$$P^{-1} = \begin{pmatrix} \dfrac{b_0d_1 + c_1d_0}{\lambda_1 - \lambda_2} & \dfrac{\lambda_1 - (a_1d_0 + b_0b_1)}{\lambda_1 - \lambda_2} \\ -\dfrac{b_0d_1 + c_1d_0}{\lambda_1 - \lambda_2} & -\dfrac{\lambda_2 - (a_1d_0 + b_0b_1)}{\lambda_1 - \lambda_2} \end{pmatrix}, \tag{41}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Therefore, we may write system (16) as the following:

$$Z_{2n+2} = P\Lambda P^{-1} Z_{2n}, \tag{42}$$

where

$$Z_{2n} = \begin{pmatrix} u_{2n} \\ v_{2n} \end{pmatrix}, \tag{43}$$

for all $n \in \mathbb{N}_0$. From (42), we have

$$P^{-1} Z_{2n+2} = \Lambda P^{-1} Z_{2n}, \tag{44}$$

from which it follows that

$$P^{-1} Z_{2n} = \Lambda^n P^{-1} Z_0, \tag{45}$$

for all $n \in \mathbb{N}_0$. Multiplying both sides of (45) by the matrix P, we have

$$Z_{2n} = P\Lambda^n P^{-1} Z_0, \tag{46}$$

or after some computations

$$\begin{pmatrix} u_{2n} \\ v_{2n} \end{pmatrix} = \begin{pmatrix} C_1\lambda_1^n - C_2\lambda_2^n \\ C_3\lambda_1^n - C_4\lambda_2^n \end{pmatrix}, \tag{47}$$

where

$$C_1 = \frac{a_0b_1 + a_1c_0}{\lambda_1 - \lambda_2}\left(\frac{b_0d_1 + c_1d_0}{\lambda_1 - (a_1d_0 + b_0b_1)} u_0 + v_0\right), \tag{48}$$

$$C_2 = \frac{a_0b_1 + a_1c_0}{\lambda_1 - \lambda_2}\left(\frac{b_0d_1 + c_1d_0}{\lambda_2 - (a_1d_0 + b_0b_1)} u_0 + v_0\right), \tag{49}$$

$$C_3 = \frac{b_0d_1 + c_1d_0}{\lambda_1 - \lambda_2} u_0 + \frac{\lambda_1 - (a_1d_0 + b_0b_1)}{\lambda_1 - \lambda_2} v_0, \tag{50}$$

$$C_4 = \frac{b_0d_1 + c_1d_0}{\lambda_1 - \lambda_2} u_0 + \frac{\lambda_2 - (a_1d_0 + b_0b_1)}{\lambda_1 - \lambda_2} v_0, \tag{51}$$

for all $n \in \mathbb{N}_0$. From the last vectorial equality, we have

$$u_{2n} = C_1\lambda_1^n - C_2\lambda_2^n, \tag{52}$$

$$v_{2n} = C_3\lambda_1^n - C_4\lambda_2^n, \tag{53}$$

for all $n \in \mathbb{N}_0$. From (13), (52), and (53), we have the formulas

$$u_{2n+1} = (b_0C_1 + a_0C_3)\lambda_1^n - (b_0C_2 + a_0C_4)\lambda_2^n, \tag{54}$$

$$v_{2n+1} = (d_0C_1 + c_0C_3)\lambda_1^n - (d_0C_2 + c_0C_4)\lambda_2^n, \tag{55}$$

for all $n \in \mathbb{N}_0$. Also, we can write the formulas (31)–(34) as the following:

$$x_{2n} = x_0 \prod_{k=1}^{n} \frac{(u_{2k}/v_{2k-2})}{(b_0(u_{2k-2}/v_{2k-2}) + a_0)(d_0(u_{2k-2}/v_{2k-2}) + c_0)}, \tag{56}$$

$$x_{2n+1} = x_1 \prod_{k=1}^{n} \frac{(b_0(u_{2k}/v_{2k}) + a_0)(d_0(u_{2k-2}/v_{2k-2}) + c_0)}{(u_{2k}/v_{2k-2})}, \tag{57}$$

$$y_{2n} = y_0 \prod_{k=1}^{n} \frac{(v_{2k}/u_{2k-2})}{(d_0 + c_0(v_{2k-2}/u_{2k-2}))(b_0 + a_0(v_{2k-2}/u_{2k-2}))}, \tag{58}$$

$$y_{2n+1} = y_1 \prod_{k=1}^{n} \frac{(d_0 + c_0(v_{2k}/u_{2k}))(b_0 + a_0(v_{2k-2}/u_{2k-2}))}{(v_{2k}/u_{2k-2})}, \tag{59}$$

for all $n \in \mathbb{N}_0$. Finally, by employing (52)–(55) in (56)–(59), we have the general solution of (3) as the following:

$$\begin{aligned}
x_{2n} &= x_0\prod_{k=1}^{n}\frac{\left(\left(C_1\lambda_1^{k}-C_2\lambda_2^{k}\right)/\left(C_3\lambda_1^{k-1}-C_4\lambda_2^{k-1}\right)\right)}{\left(b_0\left(\left(C_1\lambda_1^{k-1}-C_2\lambda_2^{k-1}\right)/\left(C_3\lambda_1^{k-1}-C_4\lambda_2^{k-1}\right)\right)+a_0\right)\left(d_0\left(\left(C_1\lambda_1^{k-1}-C_2\lambda_2^{k-1}\right)/\left(C_3\lambda_1^{k-1}-C_4\lambda_2^{k-1}\right)\right)+c_0\right)},\\
x_{2n+1} &= x_1\prod_{k=1}^{n}\frac{\left(b_0\left(\left(C_1\lambda_1^{k}-C_2\lambda_2^{k}\right)/\left(C_3\lambda_1^{k}-C_4\lambda_2^{k}\right)\right)+a_0\right)\left(d_0\left(\left(C_1\lambda_1^{k-1}-C_2\lambda_2^{k-1}\right)/\left(C_3\lambda_1^{k-1}-C_4\lambda_2^{k-1}\right)\right)+c_0\right)}{\left(\left(C_1\lambda_1^{k}-C_2\lambda_2^{k}\right)/\left(C_3\lambda_1^{k-1}-C_4\lambda_2^{k-1}\right)\right)},\\
y_{2n} &= y_0\prod_{k=1}^{n}\frac{\left(\left(C_3\lambda_1^{k}-C_4\lambda_2^{k}\right)/\left(C_1\lambda_1^{k-1}-C_2\lambda_2^{k-1}\right)\right)}{\left(d_0+c_0\left(\left(C_3\lambda_1^{k-1}-C_4\lambda_2^{k-1}\right)/\left(C_1\lambda_1^{k-1}-C_2\lambda_2^{k-1}\right)\right)\right)\left(b_0+a_0\left(\left(C_3\lambda_1^{k-1}-C_4\lambda_2^{k-1}\right)/\left(C_1\lambda_1^{k-1}-C_2\lambda_2^{k-1}\right)\right)\right)},\\
y_{2n+1} &= y_1\prod_{k=1}^{n}\frac{\left(d_0+c_0\left(\left(C_3\lambda_1^{k}-C_4\lambda_2^{k}\right)/\left(C_1\lambda_1^{k}-C_2\lambda_2^{k}\right)\right)\right)\left(b_0+a_0\left(\left(C_3\lambda_1^{k-1}-C_4\lambda_2^{k-1}\right)/\left(C_1\lambda_1^{k-1}-C_2\lambda_2^{k-1}\right)\right)\right)}{\left(\left(C_3\lambda_1^{k}-C_4\lambda_2^{k}\right)/\left(C_1\lambda_1^{k-1}-C_2\lambda_2^{k-1}\right)\right)},
\end{aligned}\tag{60}$$

for all $n \in \mathbb{N}_0$. □

The following theorem determines and characterizes the long-term behavior of positive solutions of (3) according to the parameters in the case $\operatorname{rank}(A) = 1$.

Theorem 2. *Assume that $x_0, y_0 > 0$ and $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$, and $(d_n)_{n\geq 0}$ are the two-periodic sequences of positive real numbers. If*

$$\operatorname{rank}\begin{pmatrix} a_1d_0+b_0b_1 & a_0b_1+a_1c_0 \\ b_0d_1+c_1d_0 & a_0d_1+c_0c_1\end{pmatrix} = 1, \tag{61}$$

then, for the solutions of system (3), the following is true:

(i) If $(K(a_1d_0+b_0b_1+K(a_0b_1+a_1c_0)))/((b_0+Ka_0)(d_0+Kc_0)) < 1$, then $x_{2n}\longrightarrow 0$, $x_{2n+1}\longrightarrow\infty$, $y_{2n}\longrightarrow 0$, and $y_{2n+1}\longrightarrow\infty$ as $n\longrightarrow\infty$

(ii) If $(K(a_1d_0+b_0b_1+K(a_0b_1+a_1c_0)))/((b_0+Ka_0)(d_0+Kc_0)) > 1$, then $x_{2n}\longrightarrow\infty$, $x_{2n+1}\longrightarrow 0$, $y_{2n}\longrightarrow\infty$, and $y_{2n+1}\longrightarrow 0$ as $n\longrightarrow\infty$

(iii) If $(K(a_1d_0+b_0b_1+K(a_0b_1+a_1c_0)))/((b_0+Ka_0)(d_0+Kc_0)) = 1$, then every solution of (3) is two-periodic, where K is given by (39)

Proof. The proof follows directly from formulas (35)–(38). That is to say, it is clearly seen from these formulas that if

$$\frac{K\left(a_1d_0+b_0b_1+K\left(a_0b_1+a_1c_0\right)\right)}{\left(b_0+Ka_0\right)\left(d_0+Kc_0\right)} < 1, \tag{62}$$

then $x_{2n}\longrightarrow 0$, $x_{2n+1}\longrightarrow\infty$, $y_{2n}\longrightarrow 0$, and $y_{2n+1}\longrightarrow\infty$ as $n\longrightarrow\infty$. If

$$\frac{K\left(a_1d_0+b_0b_1+K\left(a_0b_1+a_1c_0\right)\right)}{\left(b_0+Ka_0\right)\left(d_0+Kc_0\right)} > 1, \tag{63}$$

then $x_{2n}\longrightarrow\infty$, $x_{2n+1}\longrightarrow 0$, $y_{2n}\longrightarrow\infty$, and $y_{2n+1}\longrightarrow 0$ as $n\longrightarrow\infty$. If

$$\frac{K\left(a_1d_0+b_0b_1+K\left(a_0b_1+a_1c_0\right)\right)}{\left(b_0+Ka_0\right)\left(d_0+Kc_0\right)} = 1, \tag{64}$$

then every solution of (3) is two-periodic such that $x_{2n} = x_0$, $x_{2n+1} = x_1$, $y_{2n} = y_0$, and $y_{2n+1} = y_1$. □

The following theorem determines and characterizes the long-term behavior of positive solutions of (3) according to the parameters in the case $\operatorname{rank}(A) = 2$.

Theorem 3. *Assume that $x_0, y_0 > 0$ and $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$, and $(d_n)_{n\geq 0}$ are the two-periodic sequences of positive real numbers. If*

$$\operatorname{rank}\begin{pmatrix} a_1d_0+b_0b_1 & a_0b_1+a_1c_0 \\ b_0d_1+c_1d_0 & a_0d_1+c_0c_1\end{pmatrix} = 2, \tag{65}$$

then, for the solutions of system (3), the following is true:

(i) If $(C_1\lambda_1/C_3)-(b_0(C_1/C_3)+a_0)(d_0(C_1/C_3)+c_0) < 0$, then $x_{2n}\longrightarrow 0$, $x_{2n+1}\longrightarrow\infty$, $y_{2n}\longrightarrow 0$, and $y_{2n+1}\longrightarrow\infty$ as $n\longrightarrow\infty$

(ii) If $(C_1\lambda_1/C_3)-(b_0(C_1/C_3)+a_0)(d_0(C_1/C_3)+c_0) > 0$, then $x_{2n}\longrightarrow\infty$, $x_{2n+1}\longrightarrow 0$, $y_{2n}\longrightarrow\infty$, and $y_{2n+1}\longrightarrow 0$ as $n\longrightarrow\infty$

(iii) If $(C_1\lambda_1/C_3)-(b_0(C_1/C_3)+a_0)(d_0(C_1/C_3)+c_0) = 0$, then every solution of (3) converges to a two-periodic positive solution of the system, where C_1 and C_3 are given by (48) and (50)

Proof. (i)-(ii). Let

$$\begin{aligned}
p_k &= \frac{u_{2k}}{v_{2k-2}} = \frac{C_1\lambda_1^{k}-C_2\lambda_2^{k}}{C_3\lambda_1^{k-1}-C_4\lambda_2^{k-1}},\\
q_k &= \frac{u_{2k}}{v_{2k}} = \frac{C_1\lambda_1^{k}-C_2\lambda_2^{k}}{C_3\lambda_1^{k}-C_4\lambda_2^{k}},\\
r_k &= \frac{v_{2k}}{u_{2k-2}} = \frac{C_3\lambda_1^{k}-C_4\lambda_2^{k}}{C_1\lambda_1^{k-1}-C_2\lambda_2^{k-1}},\\
s_k &= \frac{v_{2k}}{u_{2k}} = \frac{C_3\lambda_1^{k}-C_4\lambda_2^{k}}{C_1\lambda_1^{k}-C_2\lambda_2^{k}}.
\end{aligned}\tag{66}$$

Then, from (56)–(59), we have

$$x_{2n} = x_0 \prod_{k=1}^{n} \left(1 + \frac{p_k - (b_0 q_{k-1} + a_0)(d_0 q_{k-1} + c_0)}{(b_0 q_{k-1} + a_0)(d_0 q_{k-1} + c_0)} \right), \tag{67}$$

$$x_{2n+1} = \frac{x_1}{\prod_{k=1}^{n} (1 + (p_k - (b_0 q_k + a_0)(d_0 q_{k-1} + c_0))/((b_0 q_k + a_0)(d_0 q_{k-1} + c_0)))}, \tag{68}$$

$$y_{2n} = y_0 \prod_{k=1}^{n} \left(1 + \frac{r_k - (d_0 + c_0 s_{k-1})(b_0 + a_0 s_{k-1})}{(d_0 + c_0 s_{k-1})(b_0 + a_0 s_{k-1})} \right), \tag{69}$$

$$y_{2n+1} = \frac{y_1}{\prod_{k=1}^{n} (1 + ((r_k - (d_0 + c_0 s_k)(b_0 + a_0 s_{k-1}))/((d_0 + c_0 s_k)(b_0 + a_0 s_{k-1}))))}. \tag{70}$$

We assume without loss of generality that $|\lambda_1| > |\lambda_2|$. Then, we have the limits

$$\begin{aligned} \lim_{k\longrightarrow\infty} p_k &= \frac{C_1\lambda_1}{C_3}, \\ \lim_{k\longrightarrow\infty} q_k &= \frac{C_1}{C_3}, \\ \lim_{k\longrightarrow\infty} r_k &= \frac{C_3\lambda_1}{C_1}, \\ \lim_{k\longrightarrow\infty} s_k &= \frac{C_3}{C_1}, \end{aligned} \tag{71}$$

and so

$$\begin{aligned} L_1 &= \lim_{k\longrightarrow\infty} \frac{p_k - (b_0 q_{k-1} + a_0)(d_0 q_{k-1} + c_0)}{(b_0 q_{k-1} + a_0)(d_0 q_{k-1} + c_0)} \\ &= \frac{(C_1\lambda_1/C_3) - (b_0(C_1/C_3) + a_0)(d_0(C_1/C_3) + c_0)}{(b_0(C_1/C_3) + a_0)(d_0(C_1/C_3) + c_0)}, \\ L_2 &= \lim_{k\longrightarrow\infty} \frac{r_k - (d_0 + c_0 s_{k-1})(b_0 + a_0 s_{k-1})}{(d_0 + c_0 s_{k-1})(b_0 + a_0 s_{k-1})} \\ &= \frac{(C_3\lambda_1/C_1) - (b_0 + a_0(C_3/C_1))(d_0 + c_0(C_3/C_1))}{(b_0 + a_0(C_3/C_1))(d_0 + c_0(C_3/C_1))}, \end{aligned} \tag{72}$$

where

$$\frac{C_1}{C_3} = \frac{a_0 b_1 + a_1 c_0}{\lambda_1 - (a_1 d_0 + b_0 b_1)}. \tag{73}$$

Note that $L_1 = L_2$. Hereby, convergence characters of the infinite series

$$\sum_{k=0}^{\infty} \frac{p_k - (b_0 q_{k-1} + a_0)(d_0 q_{k-1} + c_0)}{(b_0 q_{k-1} + a_0)(d_0 q_{k-1} + c_0)}, \tag{74}$$

$$\sum_{k=0}^{\infty} \frac{r_k - (d_0 + c_0 s_{k-1})(b_0 + a_0 s_{k-1})}{(d_0 + c_0 s_{k-1})(b_0 + a_0 s_{k-1})}, \tag{75}$$

are the same. We can say from a well-known fundamental result about infinite series that (74) and (75) are divergent, if $(C_1\lambda_1/C_3) - (b_0(C_1/C_3) + a_0)(d_0(C_1/C_3) + c_0) \neq 0$. So, the proofs of items (i)-(ii) follow from (67)–(70) and Remark 1.

(iii) From (74), for sufficiently large n_0, we have

$$\sum_{k=0}^{\infty} \frac{p_k - (b_0 q_{k-1} + a_0)(d_0 q_{k-1} + c_0)}{(b_0 q_{k-1} + a_0)(d_0 q_{k-1} + c_0)} = S_1(n_0) + K_1(n_0), \tag{76}$$

where

$$\begin{aligned} S_1(n_0) &= \sum_{k=0}^{n_0} \frac{p_k - (b_0 q_{k-1} + a_0)(d_0 q_{k-1} + c_0)}{(b_0 q_{k-1} + a_0)(d_0 q_{k-1} + c_0)}, \\ K_1(n_0) &= \sum_{k=n_0}^{\infty} \frac{(C_1\lambda_1/C_3) - (b_0(C_1/C_3) + a_0)(d_0(C_1/C_3) + c_0)}{(b_0(C_1/C_3) + a_0)(d_0(C_1/C_3) + c_0)}. \end{aligned} \tag{77}$$

Note that if $(C_1\lambda_1/C_3) - (b_0(C_1/C_3) + a_0)(d_0(C_1/C_3) + c_0) = 0$, then $K_1(n_0) \longrightarrow 0$ as $n \longrightarrow \infty$. That is to say, (74) is convergent. Since $L_1 = L_2$, (75) is convergent too. In this case, the proof of item (iii) follows from (67)–(70) and Remark 1. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Authors' Contributions

The author has contributed solely to the writing of this paper and has read and approved the manuscript.

References

[1] N. Akgüneş and A. Gurbanlyyev, "On the system of rational difference equations $x_n = f(x_{n-a_1}, y_{n-b_1})$, $y_n = g(y_{n-b_2}, z_{n-c_1})$, $z_n = h(z_{n-c_2}, x_{n-a_2})$," *Selcuk Journal of Applied Mathematics*, vol. 15, no. 1, pp. 1–8, 2014.

[2] C. Çinar, "On the periodic cycle of $x_{n+1} = (a_n+b_nx_n)/c_nx_n-1$," *Applied Mathematics and Computation*, vol. 150, no. 1, pp. 1–4, 2004.

[3] Y. Halim, N. Touafek, and E. M. Elsayed, "Closed form solution of some systems of rational difference equations in terms of Fibonacci numbers," *Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 21, no. 6, pp. 473–486, 2014.

[4] M. Kara and Y. Yazlik, "On the system of difference equations $x_n = (x_{n-2}y_{n-3}/y_{n-1}(a_n+b_nx_{n-2}y_{n-3}))$, $y_n = (y_{n-2}x_{n-3}/x_{n-1}(\alpha_n+\beta_ny_{n-2}x_{n-3}))$," *Journal of Mathematical Extension*, vol. 14, no. 1, pp. 41–59, 2020.

[5] M. Kara and Y. Yazlik, "Solvability of a system of nonlinear difference equations of higher order," *Turkish Journal of Mathematics*, vol. 43, no. 3, pp. 1533–1565, 2019.

[6] M. Kara, D. T. Tollu, and Y. Yazlik, "Global behavior of two-dimensional difference equations system with two periodic coefficients," *Tbilisi Mathematical Journal*, vol. 13, no. 4, pp. 49–64, 2020.

[7] A. S. Kurbanli, C. Cinar, and I. Yalcinkaya, "On the behavior of positive solutions of the system of rational difference equations," *Mathematical and Computer Modelling*, vol. 53, pp. 1261–1267, 2011.

[8] N. Touafek and Y. Halim, "On max type difference equations: expressions of solutions," *International Journal of Applied Nonlinear Science*, vol. 11, no. 4, pp. 396–402, 2011.

[9] N. Touafek and E. M. Elsayed, "On a third order rational systems of difference equations," *Analele Stiintifice ale Universitatii Al I Cuza din Iasi*, vol. 61, no. 2, pp. 367–380, 2015.

[10] N. Touafek and E. M. Elsayed, "On a second order rational systems of difference equations," *Hokkaido Mathematical Journal*, vol. 44, no. 1, pp. 29–45, 2015.

[11] I. Yalcinkaya, C. Cinar, and D. Simsek, "Global asymptotic stability of a system of difference equations," *Analysis and Applications*, vol. 87, no. 6, pp. 689–699, 2008.

[12] I. Yalcinkaya, "On the global asymptotic stability of a second-order system of difference equations," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 860152, 12 pages, 2008.

[13] I. Yalcinkaya, "On the global asymptotic behavior of a system of two nonlinear difference equations," *ARS Comb*, vol. 95, pp. 151–159, 2010.

[14] I. Yalçınkaya, H. El-Metwally, and A. E. Hamza, "Periodic solutions for some systems of difference equations," *Konuralp Journal of Mathematics*, vol. 8, no. 1, pp. 114–121, 2020.

[15] Y. Yazlik and M. Kara, "On a solvable system of difference equations of higher-order with period two coefficients," *Communications Faculty Of Science University of Ankara Series A1Mathematics and Statistics*, vol. 68, no. 2, pp. 1675–1693, 2019.

[16] Y. Akrour, N. Touafek, and Y. Halim, "On a system of difference equations of second order solved in closed form," *Miskolc Mathematical Notes*, vol. 20, no. 2, pp. 701–717, 2019.

[17] E. A. Grove, G. Ladas, L. C. McGrath, and C. T. Teixeira, "Existence and behaviour of solutions of a rational system," *Communications on Applied Nonlinear Analysis*, vol. 8, no. 1, pp. 1–25, 2001.

[18] N. Haddad, N. Touafek, and J. F. T. Rabago, "Well-defined solutions of a system of difference equations," *Journal of Applied Mathematics and Computing*, vol. 56, no. 1-2, pp. 439–458, 2018.

[19] S. Stević, "New class of solvable systems of difference equations," *Applied Mathematics Letters*, vol. 63, pp. 137–144, 2017.

[20] D. T. Tollu, Y. Yazlik, and N. Taskara, "On fourteen solvable systems of difference equations," *Applied Mathematics and Computation*, vol. 233, pp. 310–319, 2014.

[21] Y. Yazlik, D. T. Tollu, and N. Taskara, "On the solutions of a three-dimensional system of difference equations," *Kuwait Journal of Science and Engineering*, vol. 43, no. 1, pp. 95–111, 2016.

[22] E. A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman and Hall/CRC, Boca Raton, FL, USA, 2004.

[23] H. Levy and F. Lessman, *Finite Difference Equations*, Dover Publications, Inc., New York, NY, USA, 1992.

[24] L. Brand, "A sequence defined by a difference equation," *The American Mathematical Monthly*, vol. 62, no. 7, pp. 489–492, 1955.

[25] H. Sedaghat, "Global behaviours of rational difference equations of orders two and three with quadratic terms," *Journal of Difference Equations and Applications*, vol. 15, no. 3, pp. 215–224, 2009.

[26] D. T. Tollu, Y. Yazlik, and N. Taşkara, "On a solvable nonlinear difference equation of higher order," *Turkish Journal of Mathematics*, vol. 42, no. 4, pp. 1765–1778, 2018.

[27] S. Stević, "On a system of difference equations which can be solved in closed form," *Applied Mathematics and Computation*, vol. 219, pp. 9223–9228, 2013.

[28] N. Taskara, K. Uslu, and D. T. Tollu, "The periodicity and solutions of the rational difference equation with periodic coefficients," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1807–1813, 2011.

[29] İ. Yalçinkaya, C. Çinar, and M. Atalay, "On the solutions of systems of difference equations," *Advances in Difference Equations*, vol. 2008, Article ID 143943, 10 pages, 2008.

[30] K. Knopp, *Theory and Application of Infinite Series*, Blackie & Son, Glasgow, Scotland, 1954.

[31] L. C. McGrath and C. Teixeira, "Existence and behavior of solutions of the rational," *Rocky Mountain Journal of Mathematics*, vol. 36, no. 2, pp. 649–674, 2006.