Research Article

Orlicz Generalized Difference Sequence Space and the Linked Pre-Quasi Operator Ideal

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1. Introduction

The operator ideals have a wide field of mathematics in functional analysis, for instance, in eigenvalue distribution theorem, geometric structure of Banach spaces, and theory of fixed point. By $c^0, c, \ell_\infty, \ell_1$, and $c_0$, we denote the spaces of all, convergent, bounded, $r$-absolutely sumable and null sequences of complex numbers, respectively. $\mathbb{N}$ indicates the set of nonnegative integers. Tripathy et al. [1] introduced and studied the forward and backward generalized difference sequence spaces:

$$\begin{align*}
U(\Delta_{n}^{(m)}) &= \{(w_k) \in c^0 : (\Delta_{n}^{(m)}w_k) \in U\}, \\
U(\Delta_{n}^{m}) &= \{(w_k) \in c^0 : (\Delta_{n}^{m}w_k) \in U\},
\end{align*}$$

where $m, n \in \mathbb{N}$, $U = c^0, c$ or $c_0$, with

$$\begin{align*}
\Delta_{n}^{(m)}w_k &= \sum_{y=0}^{m} (-1)^y C_s^{m} w_{k+m}, \\
\Delta_{n}^{m}w_k &= \sum_{y=0}^{m} (-1)^y C_s^{m} w_{k-m},
\end{align*}$$

respectively. When $n = 1$, the generalized difference sequence spaces reduced to $U(\Delta_{n}^{(m)})$ defined and investigated by Et and Çolak [2]. If $m = 1$, the generalized difference sequence spaces reduced to $U(\Delta_{n})$ defined and investigated by Tripathy and Esi [3]. While if $n = 1$ and $m = 1$, the generalized difference sequence spaces reduced to $U(\Delta)$ defined and studied by Kizmaz [4].

Definition 1 (see [5]). The backward generalized difference $\Delta_{n+1}^{m}$ is called an absolute nondecreasing if $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$, then $|\Delta_{n+1}^{m}||x_i| \leq |\Delta_{n+1}^{m}||y_i|$.

An Orlicz function $\psi$ [6] is a function $\psi : [0, \infty) \rightarrow [0, \infty)$, which is convex, continuous, and nondecreasing with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$ and $\psi(u) \rightarrow \infty$, as $u \rightarrow \infty$. An Orlicz function $\psi$ [7] is said to satisfy $\delta_2$-condition for all values of $x \geq 0$, if there exists a constant $k > 0$, such that $\psi(2x) \leq k\psi(x)$. The $\delta_2$-condition is equivalent to $\psi(lx) \leq kl\psi(x)$, for all values of $x$ and for $l > 1$.

Lindentraus and Tzafriri [8] utilized the idea of an Orlicz function to define Orlicz sequence space:
\[ \ell_{\psi} = \{ u \in C^n : \rho(\beta u) < \infty, \text{ for some } \beta > 0 \}, \quad \text{where } \rho(u) = \sum_{k=0}^{\infty} \psi(|u_k|), \]

where \((\ell_{\psi}, \| \cdot \|)\) is a Banach space with the Luxemburg norm:

\[ \| u \| = \inf \left\{ \beta > 0 : \rho \left( \frac{|u|}{\beta} \right) \leq 1 \right\}. \]

Every Orlicz sequence space contains a subspace that is isomorphic to \(c_0\) or \(\ell^q\), for some \(1 \leq q < \infty\).

Let \(r = (r_j) \in \mathbb{R}^{+\mathbb{N}}\), where \(\mathbb{R}^{+\mathbb{N}}\) indicates the space of sequences with positive real numbers, and we define the Orlicz backward generalized difference sequence space as follows:

\[ (\ell_{\psi}(\Delta^m_{n+1})) = \left\{ w = (w_j) \in C^n : \exists \sigma > 0, \text{ with } \tau(\sigma w) < \infty \right\}, \]

where \(\tau(w) = \sum_{j=0}^{\infty} \psi(|\Delta^m_{n+1}|w_j|), \quad w_j = 0 \quad \text{for } j \leq 0, \quad \Delta^m_{n+1}|w_j| = \Delta^m_{n+1}|w_j| - \Delta^m_{n+1}|w_{j-1}|, \quad \text{and } \Delta^m w_j = w_j, \text{ for all } j, n, m \in \mathbb{N}. \)

It is a Banach space, with

\[ X^S = \left\{ X^S(W, Z) : \right\}, \quad \text{where } X^S(W, Z) = \left\{ V \in \mathcal{B}(W, Z) : \left( (s_j(V))_{j=0}^{\infty} \in X \right) \right\}, \]

\[ X^{app} = \left\{ X^{app}(W, Z) : \right\}, \quad \text{where } X^{app}(W, Z) = \left\{ V \in \mathcal{B}(W, Z) : \left( (\alpha_j(V))_{j=0}^{\infty} \in X \right) \right\}, \]

\[ X^{Kol} = \left\{ X^{Kol}(W, Z) : \right\}, \quad \text{where } X^{Kol}(W, Z) = \left\{ V \in \mathcal{B}(W, Z) : \left( (d_j(V))_{j=0}^{\infty} \in X \right) \right\}, \]

\[ X^\tau = \left\{ X^\tau(W, Z) : \right\}, \quad \text{where } X^\tau(W, Z) = \left\{ V \in \mathcal{B}(W, Z) : \left( (\varphi_j(V))_{j=0}^{\infty} \in X, \| V - \varphi_j(V) \| \leq 0, \text{ for all } j \in \mathbb{N} \right) \right\}. \]

A few of operator ideals in the class of Hilbert spaces or Banach spaces are defined by distinct scalar sequence spaces, such as the ideal of compact operators \(\mathcal{K}_{\tau}\) formed by \((d_\tau(V))\) and \(c_0\). Pietsch [11] studied the quasi-ideals \((\ell_{\tau})^{app}\) for \(r \in (0, \infty)\), the ideals of Hilbert Schmidt operators between Hilbert spaces constructed by \(\ell_2\), and the ideals of nuclear operators generated by \(\ell_1\). He explained that \(\mathcal{S} = (\ell_{\tau})^{app}\) for \(r \in [1, \infty)\), where \(\mathcal{S}\) is the closed class of all finite rank operators, and the class \((\ell_{\tau})^{app}\) became simple Banach and small [12].

The strictly inclusion \((\ell_{\tau})^{PP}(W, Z) \subset (\ell_{\tau})^{PP}(W, Z) \mathcal{B}(W, Z)\), whenever \(j > r > 0\), \(W\) and \(Z\) are infinite dimensional Banach spaces investigated through Makarov and Faried [13]. Faried and Bakery [14] gave a generalization of the class of quasi-operator ideal which is the pre-quasi operator ideal, and they examined several geometric and topological structures of \((\ell_{\psi})^S\) and \((ces(r))^S\). Başarır and Kara [15] studied the compact operators on some Euler \(B(m)\)-difference sequence spaces. İlhan et al. [16] investigated the multiplication operators on Cesáro second-order function spaces. The point of this article to explain some results of \((\ell_{\psi}(\Delta^m_{n+1}))\), equipped with a pre-quasi norm \(\tau\). Firstly, we give the necessary conditions on any \(s\)-type \((\ell_{\psi}(\Delta^m_{n+1}))\), to give an operator ideal. Secondly, some geometric and topological structures of \((\ell_{\psi}(\Delta^m_{n+1}))^S\) have been studied, such as closed, small, simple Banach and \((\ell_{\psi}(\Delta^m_{n+1}))^S = (\ell_{\psi}(\Delta^m_{n+1}))^{app}\). A strictly inclusion relation of \((\ell_{\psi}(\Delta^m_{n+1}))^{app}\) has been determined for different Orlicz functions and \(\Delta^m_{n+1}\).

2. Preliminaries and Definitions

Definition 2 (see [11]). An operator \(V \in \mathcal{B}(W, Z)\) is called approximable if there are \(a_j(V)\), \(\psi_j(V)\) for all \(j \in \mathbb{N}\), such that \(\lim_{\| V \| \to 0} \| V - a_j(V) \| = 0\).

By \(Y(W, Z)\), we will indicate the space of all approximable operators from \(W\) to \(Z\). The sequence \(\epsilon_j = (0, 0, \ldots, 1, 0, 0, \ldots)\) with 1 in the \(j\)-th coordinate, for every \(j \in \mathbb{N}\), will be used in the sequel.

Lemma 1 (see [11]). Let \(V \in \mathcal{B}(W, Z)\). If \(V \neq Y(W, Z)\), then there are \(G \in \mathcal{B}(W)\) and \(B \in \mathcal{B}(Z)\) such that \(BVG_x = \epsilon_x\), for all \(x \in \mathbb{N}\).

Definition 3 (see [11]). A Banach space \(W\) is called simple if \(\mathcal{B}(W)\) includes one and only one nontrivial closed ideal.
Theorem 1 (see [11]). If $W$ is Banach space with $\dim(W) = \infty$, then
\[
\mathcal{S}(W) \subseteq Y(W) \subseteq \mathcal{R}_r(W) \subseteq \mathcal{B}(W).
\] (8)

Definition 4 (see[14]). The space of linear sequence spaces $\mathcal{V}$ is called a special space of sequences (ss) if

(1) $e_r \in \mathcal{V}$ with $r \in \mathbb{N}$,
(2) let $u = (u_r) \in \mathbb{C}^\mathbb{N}$, $v = (v_r) \in \mathcal{V}$, and $|u_r| \leq |v_r|$, for every $r \in \mathbb{N}$, then $u \in \mathcal{V}$. This means $\mathcal{V}$ be solid.
(3) if $(u_r)_{r=0}^\infty \in \mathcal{V}$, then $(u_{[r/2]})_{r=0}^\infty \in \mathcal{V}$, wherever $[r/2]$ means the integral part of $r/2$.

Definition 5 (see [5]). A subspace of the (ss) $\mathcal{V}_r$, is called a premodular (ss) if there is a function $\tau: \mathcal{V} \rightarrow [0, \infty)$ verifying the following conditions:

(i) $\tau(y) \geq 0$ for each $y \in \mathcal{V}$ and $\tau(\theta) = 0$, where $\theta$ is the zero element of $\mathcal{V}$
(ii) There exists $a \geq 0$ such that $\tau(a|\theta|) \leq a|\theta|\tau(y)$, for all $y \in \mathcal{V}$ and $\eta \in \mathcal{C}$
(iii) For some $b \geq 1$, $\tau(y + z) \leq b(\tau(y) + \tau(z))$, for every $y, z \in \mathcal{V}$
(iv) $|u_r| \leq |z_r|$ with $r \in \mathbb{N}$, which implies that $\tau(|y_r|) \leq \tau(|z_r|)$
(v) For some $b_0 \geq 1$, $\tau(|(y_r)|) \leq \tau(|(y_{[r/2]})|) \leq b_0 \tau(|y_r|)$
(vi) If $y = (y_r)_{r=0}^\infty \in \mathcal{V}$ and $d > 0$, then there is $r_0 \in \mathbb{N}$ with $\tau((y_r)_{r=r_0}^\infty) < d$
(vii) There is $t > 0$ with $\tau(y, 0, 0, 0, \ldots, 0) \leq t|\tau(1, 0, 0, 0, \ldots, 0)|$, for any $0 \in \mathcal{C}$

The (ss) $\mathcal{V}_r$, is called pre- quasi normed (ss) if $\tau$ satisfies Parts (i)–(iii) of Definition 5 and when the space $\mathcal{V}$ is complete under $\tau$, then $\mathcal{V}_r$, is called a pre- quasi Banach (ss).

Theorem 2 (see [5]). A pre- quasi norm (ss) $\mathcal{V}_r$, whenever it is premodular (ss).

By $\mathfrak{R}$, we will denote the class of all bounded linear operators between any pair of Banach spaces.

Definition 6 (see [5]). A class $\mathfrak{R} \subseteq \mathcal{R}$ is called an operator ideal if every $\mathfrak{R}(W, Z) = \mathfrak{R} \cap \mathcal{R}(W, Z)$ satisfies the following conditions:

(i) $\mathfrak{R} \supseteq \mathcal{S}$
(ii) The space $\mathfrak{R}(W, Z)$ is linear over $\mathbb{C}$
(iii) If $V \in \mathcal{R}(W_0, W)$, $G \in \mathfrak{R}(W, Z)$, and $Q \in \mathfrak{R}(Z, Z_0)$, then $QGV \in \mathfrak{R}(W_0, Z_0)$, where $W_0$ and $Z_0$ are Banach spaces

Definition 7 (see [5]). A pre- quasi norm on the ideal $B$ is a function $\zeta: B \rightarrow [0, \infty)$ which satisfies the following conditions:

(1) For all $V \in B(W, Z)$, $\zeta(V) \geq 0$ and $\zeta(V) = 0$ if and only if $V = 0$
(2) There is $H \geq 1$ such that $\zeta(\eta V) \leq H|\theta|\zeta(V)$, for all $V \in B(W, Z)$ and $\eta \in \mathcal{C}$
(3) There is $b \geq 1$ such that $\zeta(V_1 + V_2) \leq b(\zeta(V_1) + \zeta(V_2))$, for all $V_1, V_2 \in B(W, Z)$
(4) There is $D \geq 1$ such that if $U \in \mathfrak{R}(W_0, W)$, $T \in B(W, Z)$, and $V \in \mathfrak{R}(Z, Z_0)$, then $\zeta(VTU) \leq D\|V\|\zeta(T)||U||$

Theorem 3 (see [14]). The class $X_\sigma$ is an operator ideal, if $X_\sigma$ is a (ss).

Theorem 4 (see [14]). The function $\zeta(V) = \tau(s_r(V))_{r=0}^\infty$ forms a pre- quasi norm on $X_\sigma$, whenever $X_\sigma$ be a premodular (ss).

The inequality [17] $|a_i + b_i| \leq H(|a_i| + |b_i|)$, where $q_i \geq 0$ for all $i \in \mathbb{N}$, $H = \max[1, 2^{m-1}]$ and $h = \sup q_i$, will be used in the sequel.

3. Main Results

We give the necessary conditions on $s$-type $\ell_p(\Delta_m^{\infty})$ under $\tau: \ell_p(\Delta_m^{\infty}) \rightarrow [0, \infty)$ such that $(\ell_p(\Delta_m^{\infty}))_{\tau}$ forms an operator ideal.

Theorem 5. For $(\ell_p(\Delta_m^{\infty}))_{\tau} = \{x = (s_p(V)) \in \mathbb{C}^\mathbb{N}; V \in \mathfrak{R}(W, Z) \text{ and } \tau(x) < \infty\}$. If $(\ell_p(\Delta_m^{\infty}))_{\tau}$ is an operator ideal, then the following conditions are satisfied:

(1) The set $(\ell_p(\Delta_m^{\infty}))_{\tau}$ contains $F$, the space of all the sequences with finite nonzero numbers
(2) If $(s_p(V_1))_{r=0}^\infty \in (\ell_p(\Delta_m^{\infty}))_{\tau}$ and $(s_p(V_2))_{r=0}^\infty \in (\ell_p(\Delta_m^{\infty}))_{\tau}$, then $(s_p(V_1 + V_2))_{r=0}^\infty \in (\ell_p(\Delta_m^{\infty}))_{\tau}$
(3) For all $\lambda \in \mathbb{C}$ and $(s_p(V))_{r=0}^\infty \in (\ell_p(\Delta_m^{\infty}))_{\tau}$, then $|\lambda|(s_p(V))_{r=0}^\infty \in (\ell_p(\Delta_m^{\infty}))_{\tau}$
(4) The sequence space $(\ell_p(\Delta_m^{\infty}))_{\tau}$ is solid

Proof. let $(\ell_p(\Delta_m^{\infty}))_{\tau}$ be an operator ideal.

(i) We have $\mathcal{S}(W, Z) \subseteq (\ell_p(\Delta_m^{\infty}))_{\tau}$. Hence, for all $T \in \mathcal{S}(W, Z)$, we have $(s_p(V))_{r=0}^\infty \in F$. This gives that $(s_p(V))_{r=0}^\infty \in (\ell_p(\Delta_m^{\infty}))_{\tau}$. Hence, $F \subseteq (\ell_p(\Delta_m^{\infty}))_{\tau}$.
(ii) The space $(\ell_p(\Delta_m^{\infty}))_{\tau}$ is linear over $\mathbb{C}$. Hence, for each $\lambda \in \mathbb{C}$ and $V_1, V_2 \in (\ell_p(\Delta_m^{\infty}))_{\tau}$, we have $V_1 + V_2 \in (\ell_p(\Delta_m^{\infty}))_{\tau}$. This implies that
\[
(s_r(V_i))_{r=0}^{\infty} \in (\ell_\psi(\Delta^m_{n+1}))_
(s_r(V_2))_{r=0}^{\infty} \in (\ell_\psi(\Delta^m_{n+1})), \quad (s_r(V_1 + V_2))_{r=0}^{\infty} \in (\ell_\psi(\Delta^m_{n+1})),
\]
\[
\lambda \in \mathbb{C},
\]
\[
(s_r(V_i))_{r=0}^{\infty} \in (\ell_\psi(\Delta^m_{n+1})), \quad \tau(|s_r(V_i)|_{r=0}^{\infty} \in (\ell_\psi(\Delta^m_{n+1}))).
\] 

(iii) If \( A \in \mathcal{B}(W_0, W) \), \( B \in (\ell_\psi(\Delta^m_{n+1}))_r \), and \( D \in \mathcal{B}(Z, Z_0) \), then \( DBA \in (\ell_\psi(\Delta^m_{n+1}))_r \), where \( W_0 \) and \( Z_0 \) are arbitrary Banach spaces. Therefore, if \( A \in \mathcal{B}(W_0, W) \), \( s_r(B) \in (\ell_\psi(\Delta^m_{n+1}))_r \), and \( D \in \mathcal{B}(Z, Z_0) \), then \( (s_r(DBA))_{r=0}^{\infty} \in (\ell_\psi(\Delta^m_{n+1}))_r \). In addition, \( \|s_r(DBA)\|_r \leq \|\|s_r(B)\|A\| \). By using condition 3, \( (\|\|DBA\|s_r(B)\|A\|)_r \). This means that \( (\ell_\psi(\Delta^m_{n+1}))_r \) is solid.

We explain that for any backward generalized difference \( \Delta^m_{n+1} \), the space \( (\ell_\psi(\Delta^m_{n+1}))_r \) is not operator ideal.

Theorem 6. The space \( (\ell_\psi(\Delta^m_{n+1}))_r \) is not operator ideal, where \( \psi \) is an Orlicz function satisfying \( \delta_2 \)-condition and \( \tau(w) = \sum_{j=0}^{\infty} \psi(\Delta^m_{n+1}|w|) \), for all \( w \in \ell_\psi(\Delta^m_{n+1}) \).

Proof. if we choose \( m = 2, q = 1, \) and \( \nu = \psi \), for \( k = 3 \) or \( \nu = 2 \), otherwise, for all \( s, k \in \mathbb{N} \). We have \( |\psi_k| \leq |w_k| \), for all \( k \in \mathbb{N} \), \( w \in (\ell_\psi(\Delta^m_{n+1}))_r \), and \( \nu \neq (\ell_\psi(\Delta^m_{n+1})). \) Hence, the space \( (\ell_\psi(\Delta^m_{n+1}))_r \) is not solid. This finishes the proof.

\[ \text{Theorem 7. If } \psi \text{ is an Orlicz function satisfying } \delta_2 \text{-condition and } \Delta^m_{n+1} \text{ is an absolute nondecreasing, then the space } (\ell_\psi(\Delta^m_{n+1}))_r \text{ is a premodular Banach (ss)).} \]

\[ \tau(\nu + \omega) = \sum_{j=0}^{\infty} \psi(\Delta^m_{n+1}|\nu_j + \omega_j|) \leq \sum_{j=0}^{\infty} \psi(\Delta^m_{n+1}|\nu_j| + \Delta^m_{n+1}|\omega_j|) \]
\[ \leq \frac{1}{2} \left( \sum_{j=0}^{\infty} \psi(2\Delta^m_{n+1}|\nu_j|) + \sum_{j=0}^{\infty} \psi(2\Delta^m_{n+1}|\omega_j|) \right) \leq b \left( \tau(\nu) + \tau(\omega) \right) \leq B(\tau(\nu) + \tau(\omega)) < \infty, \]
for some \( B = \max\{1, b/2\} \). Then, \( \nu + \omega \in \ell_\psi(\Delta^m_{n+1}) \).

(ii) Assume \( \lambda \in \mathbb{C} \) and \( \nu \in \ell_\psi(\Delta^m_{n+1}) \). Since \( \psi \) is satisfying \( \delta_2 \)-condition, we have

\[ \tau(\lambda \nu) = \sum_{j=0}^{\infty} \psi(\Delta^m_{n+1}|\lambda \nu_j|) \leq d|\lambda| \sum_{j=0}^{\infty} \psi(\Delta^m_{n+1}|\nu_j|) \leq D|\lambda| \tau(\nu) < \infty, \]
where \( D = \max\{1, d\} \). Then, \( \lambda \nu \in \ell_\psi(\Delta^m_{n+1}) \). Hence, from Parts (i) and (iii), the space \( \ell_\psi(\Delta^m_{n+1}) \) is linear. Therefore, \( e_r \in \ell_\psi(\Delta^m_{n+1}) \), for all \( r \in \mathbb{N} \) and \( q \geq 1 \). Therefore, \( e_r \in \ell_\psi(\Delta^m_{n+1}) \) for all \( r \in \mathbb{N} \).

(2) Suppose \( |x_i| \leq |y_i| \), for all \( i \in \mathbb{N} \) and \( y \in \ell_\psi(\Delta^m_{n+1}) \). \( \psi \) is nondecreasing, and \( \Delta^m_{n+1} \) is an absolute nondecreasing. Hence, we have

\[ \tau(x) = \sum_{i=0}^{\infty} \psi(\Delta^m_{n+1}|x_i|) \leq \sum_{i=0}^{\infty} \psi(\Delta^m_{n+1}|y_i|) = \tau(y) < \infty, \]
so \( x \in \ell_\psi(\Delta_{n+1}^m) \).

(3) Assume that \( (v_i) \in \ell_\psi(\Delta_{n+1}^m) \). We have

\[
\tau\left( \left( \frac{v_i}{\sqrt{2}} \right) \right) = \sum_{i=0}^{\infty} \psi\left( \left\| \Delta_{n+1}^m \left| v_i \right| \right\| \right) \leq 2 \sum_{i=0}^{\infty} \psi\left( \left\| \Delta_{n+1}^m \left| v_i \right| \right\| \right) = 2 \tau(v),
\]

Since \( \psi \) is nondecreasing, hence, for \( i, j \geq i_0 \) and \( k \in \mathbb{N} \), we conclude

\[
\left| \Delta_{n+1}^m \left| x_i^k \right| - \Delta_{n+1}^m \left| x_j^k \right| \right| \leq \varepsilon.
\]

Therefore, \( (\Delta_{n+1}^m|x_i^k|) \) is a Cauchy sequence in \( C \) for fixed \( k \in \mathbb{N} \), so \( \lim_{j \to \infty} \Delta_{n+1}^m \left| x_j^k \right| = \Delta_{n+1}^m \left| x_i^k \right| \) for fixed \( k \in \mathbb{N} \). Hence, \( \tau(x^0 - x^i) < \psi(\varepsilon) \), for all \( i \geq i_0 \). Finally, to show that \( x^0 \in \ell_\psi(\Delta_{n+1}^m) \), we have

\[
\tau(x^0) = \tau(x^0 - x^i + x^i) \leq \tau\left( x^0 - x^i \right) + \tau(x^i) < \infty.
\]

Therefore, \( x^0 \in \ell_\psi(\Delta_{n+1}^m) \). This gives that \( (\ell_\psi(\Delta_{n+1}^m))_c \) is a premodular Banach (sss).

In view of Theorem 2, we conclude the following theorem.

Theorem 8. If \( \psi \) is an Orlicz function satisfying \( \delta_2 \)-condition and \( \Delta_{n+1}^m \) is an absolute nondecreasing, then the space \( (\ell_\psi(\Delta_{n+1}^m))_c \) is pre-qui Banach (sss), where

\[
\tau(x) = \sum_{j=0}^{\infty} \psi\left( \left\| \Delta_{n+1}^m \left| x_j \right| \right\| \right), \text{ for all } x \in \ell_\psi(\Delta_{n+1}^m).
\]

Corollary 1. If \( 0 < p < \infty \) and \( \Delta_{n+1}^m \) is an absolute nondecreasing, then \( (\ell_\psi(\Delta_{n+1}^m))_c \) is a premodular Banach (sss), where \( \tau(x) = \sum_{j=0}^{\infty} \left\| \Delta_{n+1}^m \left| x_j \right| \right\|^p \), for all \( x \in \ell_\psi(\Delta_{n+1}^m) \).

4. Pre-Quasi Banach Closed Ideal

We introduce the sufficient conditions on \( \ell_\psi(\Delta_{n+1}^m) \) such that the class \( (\ell_\psi(\Delta_{n+1}^m))^S \) is Banach and closed.

Theorem 9. If \( \psi \) is an Orlicz function satisfying \( \delta_2 \)-condition and \( \Delta_{n+1}^m \) is an absolute nondecreasing, then \( (\ell_\psi(\Delta_{n+1}^m))^S, \zeta \)

\[
\zeta(V_i - V_j) = \sum_{k=0}^{\infty} \psi\left( \left\| \Delta_{n+1}^m \left( \delta_k (V_i - V_j) \right) \right\| \right) \geq \psi\left( \left\| \Delta_{n+1}^m \left( \left\| V_i - V_j \right\| \right) \right\| \right).
\]

Proof. Let the conditions be satisfied. Hence, from Theorems 3, 4, and 7, the function \( \zeta \) is a pre-qui norm on the ideal \( (\ell_\psi(\Delta_{n+1}^m))^S \). Let \( (V_i) \) be a Cauchy sequence in \( (\ell_\psi(\Delta_{n+1}^m))^S(W, Z) \). Since \( \mathcal{B}(W, Z) \supseteq (\ell_\psi(\Delta_{n+1}^m))^S(W, Z) \), we have

\[
\zeta(V_i - V_j) = \sum_{k=0}^{\infty} \psi\left( \left\| \Delta_{n+1}^m \delta_k (V_i - V_j) \right\| \right) \geq \psi\left( \left\| \Delta_{n+1}^m \left( \left\| V_i - V_j \right\| \right) \right\| \right).
\]
Therefore, $(V_j)_{j \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}(W, Z)$. Since $\mathcal{B}(W, Z)$ is a Banach space, hence $T \in \mathcal{B}(W, Z)$ with $\lim_{j \to \infty} \|V_j - V\| = 0$ and while $(s_n(V_j))_{n=0}^{\infty} = (\ell^{\infty}m_1 v_{\tau})_{j=0}^{\infty}$.

Theorem 10. If $\psi$ is an Orlicz function satisfying $\delta_2$-condition and $\Delta^m_{\infty}$ is an absolute nondecreasing, then $((\ell^{\infty}m_1 v_{\tau})_{j=0}^{\infty})$, $\zeta_t$ is a pre-quasi closed operator ideal, with $\beta(w) = \sum_{n=0}^{\infty} \psi(\Delta^m_{\infty}(w))$, for all $w \in (\ell^{\infty}m_1 v_{\tau})$ and $\zeta_t(V) = \tau(\psi(V))_{n=0}^{\infty}$. Hence, $V \in (\ell^{\infty}m_1 v_{\tau})_{j=0}^{\infty}$.

Proof. Let the conditions be satisfied. Therefore, by using Theorems 3, 4, and 7, the function $\zeta_t$ is a pre-quasi norm on $\Delta^m_{\infty}$.

Therefore, $(s_n(V_j))_{n=0}^{\infty} = (\ell^{\infty}m_1 v_{\tau})_{j=0}^{\infty}$. Hence gives $V \in (\ell^{\infty}m_1 v_{\tau})_{j=0}^{\infty}$.

Corollary 2. $((\ell^{\infty}m_1 v_{\tau})_{j=0}^{\infty})$, $\zeta_t$ is a pre-quasi closed and Banach, with $\beta(w) = \sum_{n=0}^{\infty} |\Delta^m_{\infty}(w)|^p$, for all $w \in (\ell^{\infty}m_1 v_{\tau})$ and $\zeta_t(V) = \tau(\psi(V))_{n=0}^{\infty}$ if $0 < p < \infty$ and $\Delta^m_{\infty}$ is an absolute nondecreasing.

5. Small and Simple Pre-Quasi Operator Ideal

We explain the sufficient conditions on $\ell^{\infty}m_1 v_{\tau}$ such that the strictly inclusion relation of $(\ell^{\infty}m_1 v_{\tau})^p$, for different $p$ and $\Delta^m_{\infty}$ has been happened.

Theorem 11. For any infinite dimensional Banach spaces $W$ and $Z$. Let $\psi_1$ and $\psi_2$ be two Orlicz functions satisfying $\delta_2$-condition with $\psi_1(t) < \psi_2(t)$ for all $t \in (0, \infty)$ and $\Delta^m_{\infty}$ is an absolute nondecreasing, for all $n, m \in \mathbb{N}$ then

$$(\ell^{\infty}m_1 v_{\tau})^p(W, Z) \subset (\ell^{\infty}m_1 v_{\tau})^p(W, Z) \subset (\mathcal{B}(W, Z)).$$

Corollary 3. For any infinite dimensional Banach spaces $W$ and $Z$, $j \geq r > 0$ and absolute nondecreasing $\Delta^m_{n}$ for every $n, m \in \mathbb{N}$, then
imasurable. By Lemma 1, we have
\[ G \in \Upsilon \]
\[ \ell_\psi \bigl( \Delta_{n+1}^m \bigr) \sim (W, Z) \subset \ell_\psi \bigl( \Delta_{n+1}^m \bigr) [W, Z] \subset \mathcal{R}(W, Z). \]

(25)

We study the conditions such that the class \((\ell_\psi(\Delta_{n+1}^m))^{\text{app}}\) is small.

**Theorem 12.** For any Banach spaces \(W\) and \(Z\) with \(\dim(W) = \dim(Z) = \infty\). Let \(\psi\) be an Orlicz function satisfying \(\delta_2\)-condition and \(\Delta_{n+1}^m\) be an absolute nondecreasing, then the class \((\ell_\psi(\Delta_{n+1}^m))^{\text{app}}\) is small.

\[ 1 = u_k(I_j) = u_k(A_j A_j^{-1} I_j H_j H_j^{-1}) \leq \|A_j\| \|u_k(A_j^{-1} I_j H_j)\| H_j^{-1} = \|A_j\| \|u_k(I_j A_j^{-1} I_j H_j)\| H_j^{-1} \]
\[ \leq \|A_j\| \|d_k(I_j A_j^{-1} I_j H_j, Q_j)\| H_j^{-1} \leq \|A_j\| \|u_k(I_j A_j^{-1} I_j H_j Q_j)\| H_j^{-1}. \]

(26)

for \(0 \leq k \leq 1\). Since \(\psi\) is an Orlicz function satisfying \(\delta_2\)-condition, we have

\[ (i + 1) \psi(1) \leq a \|A_j\| H_j^{-1} \sum_{k=0}^i \psi \left( |\Delta_{n+1}^m A_k(I_j A_j^{-1} I_j H_j Q_j)| \right) \]
\[ (i + 1) \psi(1) \leq a \|A_j\| H_j^{-1} \|g(I_j A_j^{-1} I_j H_j Q_j)\| \]
\[ (i + 1) \psi(1) \leq a \delta \|A_j\| H_j^{-1} \|I_j H_j Q_j\| = \delta \|A_j\| H_j^{-1} \|I_j A_j^{-1} H_j Q_j\| \]
\[ (i + 1) \psi(1) \leq 4a \delta, \]

for some \(a \geq 1\). Since \(i\) is an arbitrary, we have a contradiction. So, \(W\) and \(Z\) cannot be infinite dimensional while \((\ell_\psi(\Delta_{n+1}^m))^{\text{app}}(W, Z) = \mathcal{R}(W, Z)\).

By the same manner, one can prove that the class \((\ell_\psi(\Delta_{n+1}^m))^{\text{Kol}}\) is small. \(\Box\)

**Theorem 13.** Hold any Banach spaces \(W\) and \(Z\) with \(\dim(W) = \dim(Z) = \infty\). Let \(\nu\) be an Orlicz function \(\psi\) satisfying \(\delta_2\)-condition and \(\Delta_{n+1}^m\) is an absolute nondecreasing, then the class \((\ell_\psi(\Delta_{n+1}^m))^{\text{Kol}}\) is small.

Proof. Assume there is \(V \in \mathcal{R}((\ell_\psi(\Delta_{n+1}^m))^S, (\ell_\psi(\Delta_{n+2}^m))^S)\) which is not approximable. By Lemma 1, we have \(G \in \mathcal{R}((\ell_\psi(\Delta_{n+1}^m))^S)\) and

\[ B \in \mathcal{R}((\ell_\psi(\Delta_{n+2}^m))^S) \text{ with } BVG_{k} = I_k. \]

Therefore, for all \(k \in \mathbb{N}\), we get

\[ \|I_k\| (\ell_\psi(\Delta_{n+2}^m))^S = \sum_{n=0}^{\infty} \psi_1 \left( |\Delta_{n+2}^m S_n(I_k)| \right) \leq \|BVG\| \|I_k\| (\ell_\psi(\Delta_{n+2}^m))^S \leq \sum_{n=0}^{\infty} \psi_2 \left( |\Delta_{n+1}^m S_n(I_k)| \right). \]

(29)

**Corollary 4.** For any infinite dimensional Banach spaces \(W\) and \(Z\). Let \(\psi_1\) and \(\psi_2\) be two Orlicz functions satisfying
\[ \delta_{2}\text{-condition with } \psi_{s}(t) < \psi_{t}(t) \text{ for all } t \in (0, \infty) \text{ and } \Delta_{m}^{n} \text{ is an absolute nondecreasing, for all } n, m \in \mathbb{N}, \text{ then} \]

\[ \mathcal{B}((\ell_{\psi}(\Delta_{m}^{n+1})), \ell_{\psi}(\Delta_{m}^{n+1})) = \mathcal{B}((\ell_{\psi}(\Delta_{m}^{n+1})), \ell_{\psi}(\Delta_{m}^{n+1})). \]  

(30)

**Proof.** Clearly, since each approximable operator is compact. \( \square \)

**Theorem 15.** Pick up any Banach spaces \( W \) and \( Z \) with \( \dim(W) = \dim(Z) = \infty \). If \( \psi \) is an Orlicz function satisfying \( \delta_{2}\text{-condition and } \Delta_{m}^{n} \) is an absolute nondecreasing, then the class \( (\ell_{\psi}(\Delta_{m}^{n+1})) \) is simple.

**Proof.** Suppose there is \( V \in \mathcal{B}((\ell_{\psi}(\Delta_{m}^{n+1})), \ell_{\psi}(\Delta_{m}^{n+1})) \) and \( V \neq \mathcal{Y}(\ell_{\psi}(\Delta_{m}^{n+1})). \) Therefore, from Lemma 1, one can find that \( A, B \in \mathcal{B}((\ell_{\psi}(\Delta_{m}^{n+1})), \ell_{\psi}(\Delta_{m}^{n+1})) \) with \( B\mathcal{V}I = I_{k} \). This means that \( I_{k}((\ell_{\psi}(\Delta_{m}^{n+1})) \in \mathcal{B}((\ell_{\psi}(\Delta_{m}^{n+1})), \ell_{\psi}(\Delta_{m}^{n+1})). \) Consequently, \( \mathcal{B}((\ell_{\psi}(\Delta_{m}^{n+1}))) \) \( \mathcal{Y}(\ell_{\psi}(\Delta_{m}^{n+1})). \) Therefore, \( \mathcal{B}((\ell_{\psi}(\Delta_{m}^{n+1}))) \) includes one and only one nontrivial closed ideal \( \mathcal{Y}(\ell_{\psi}(\Delta_{m}^{n+1})). \) \( \square \)

6. Eigenvalues of \( s\text{-Type Orlicz Generalized Difference Sequence Space} \)

We explain here the sufficient conditions on \( \ell_{\psi}(\Delta_{m}^{n+1}) \) such that \( (\ell_{\psi}(\Delta_{m}^{n+1})) \) equals \( (\ell_{\psi}(\Delta_{m}^{n+1})). \)

**Theorem 16.** Pick up any Banach spaces \( W \) and \( Z \) with \( \dim(W) = \dim(Z) = \infty \). If \( \psi \) is an Orlicz function satisfying \( \delta_{2}\text{-condition and } \Delta_{m}^{n} \) is an absolute nondecreasing, then

\[ (\ell_{\psi}(\Delta_{m}^{n+1})) \subseteq (W, Z) = (\ell_{\psi}(\Delta_{m}^{n+1})). \]  

(31)

**Proof.** Suppose \( V \in \mathcal{B}((\ell_{\psi}(\Delta_{m}^{n+1})), (W, Z)) \) and then \( (s_{r}(V))_{r=0}^{\infty} \in \ell_{\psi}(\Delta_{m}^{n+1}), \) we have \( \sum_{r=0}^{\infty} \psi((\Delta_{m}^{n+1}s_{r}(V))) < \infty. \) Since \( \Delta_{m}^{n} \) is continuous, so \( \lim_{r \to \infty} s_{r}(V) = 0. \) Let \( \|V - s_{r}(V)\| \) be an invertible, for all \( r \in \mathbb{N}, \) then \( \|V - s_{r}(V)\|^{-1} \) exists and bounded, for each \( r \in \mathbb{N}. \) Therefore, \( \lim_{r \to \infty} \|V - s_{r}(V)\|^{-1} = \|V\|^{-1} \) with \( V^{-1} \in \mathcal{B}(Z, W). \)

From the pre- quasi operator ideal of \( \mathcal{B}((\ell_{\psi}(\Delta_{m}^{n+1})), \zeta), \) one has

\[ I = V^{-1} \in (\ell_{\psi}(\Delta_{m}^{n+1})), (Z) \implies (s_{r}(I))_{r=0}^{\infty} \in (\ell_{\psi}(\Delta_{m}^{n+1})), \]

\[ \implies \lim_{r \to \infty} s_{r}(I) = 0. \]  

(32)

Therefore, \( \lim_{r \to \infty} s_{r}(I) = 1. \) We have a contradiction, and then \( \|V - s_{r}(V)\| \) is not invertible, for all \( r \in \mathbb{N}. \) Hence, \( \|s_{r}(V))_{r=0}^{\infty} \) is the eigenvalues of \( V. \) Conversely, if \( V \in (\ell_{\psi}(\Delta_{m}^{n+1})), (W, Z) \) then \( (v_{r}(V))_{r=0}^{\infty} \in \ell_{\psi}(\Delta_{m}^{n+1}) \) and \( \|V - v_{r}(V)\| = 0, \) for all \( n \in \mathbb{N}. \) This gives that \( V = v_{r}(V), \) for all \( r \in \mathbb{N}, \) and then \( s_{r}(V) = s_{r}(v_{r}(V)) = (v_{r}(V)), \) for all \( r \in \mathbb{N}. \)

Therefore, \( \|s_{r}(V))_{r=0}^{\infty} \in \ell_{\psi}(\Delta_{m}^{n+1}) \), so \( V \in (\ell_{\psi}(\Delta_{m}^{n+1})), (W, Z). \) This completes the proof. \( \square \)

7. Conclusion

We have introduced the concept of the pre-quasi norm on the new sequence space generated by the domain of generalized backward difference operator in Orlicz sequence space. This space is not operator ideal since it is not solid. However, if the generalized backward difference operator is an absolute nondecreasing and Orlicz function satisfies \( \delta_{2}\text{-condition, then the operator ideal constructed by this sequence space and } s \text{-numbers will be Banach, closed, small, and simple. Finally, we have found the spectrum of all operators contained in this operator ideal.}

Data Availability

The data used to support the findings of the study are available from the corresponding author upon request.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All the authors contributed equally to the writing of this paper and read and approved the final manuscript.

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