Research Article

Randomised Pseudolikelihood Ratio Change Point Estimator in Garch Models

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In this paper, a randomised pseudolikelihood ratio change point estimator for GARCH model is presented. Derivation of a randomised change point estimator for the GARCH model and its consistency are given. Simulation results that support the validity of the estimator are also presented. It was observed that the randomised estimator outperforms the ordinary CUSUM of squares test, and it is optimal with large variance change ratios.

1. Introduction

Volatility plays a very important role in financial derivatives; hence, a good prediction of volatility will provide a more accurate pricing model for financial assets. The autoregressive integrated moving average (ARIMA) models developed by Box and Jenkins [1] assume a constant conditional variance of the errors; however, financial data do not obey this assumption. For this reason, Engle [2] proposed the Autoregressive Conditional Heteroscedastic (ARCH) model; this model has found wide applications in finance modeling. Bollerslev [3] generalised the ARCH model and called it the GARCH (Generalised ARCH) model. Other variations of the GARCH model [4, 5], among others referred to as Asymmetric GARCH, have been developed to address some limitations of the standard GARCH model. However, due to structural changes, modeling economic processes over long periods of time may undermine the important assumption of stationarity in these models. In fact, Lamoureux and Lastrapes and Hillebrand [6, 7] reported that the application of GARCH models to long time series of stock returns yields a high measure of persistence because of the presence of shifts in the data generating

parameters of these models, leading to false results. The problem of structural changes prompted the idea of change point detection and estimation and has found applications in quality control, climate change, finance, etc. For example, Hsu [8] studied a single change in variance in a sequence of independent random variables. Later, Inclan and Tiao [9] explored variance change for independent observations to detect multiple changes. Chen and Gupta [10], among several other authors, have explored variance change in independent variables and counts [11]. For early works and in-depth reviews, we refer to [12]. For the GARCH models, Kim et al. [13] proposed an analogy of [9] test statistic and derived its limiting distribution as a supremum of a Brownian bridge. Kokoszka and Leipus [14] proposed a similar test for which the main difference was to analyse the existence of structural break in the unconditional variance of an ARCH(p) model. To address the problem of size distortions as reported by Lee et al. [15] on Kim et al. [13], the authors based their test on standardised residuals instead of residuals. Later, Lee et al. [16] studied variance change based on Inclan and Tiao [9] test for errors in AR(p) models and kernel-type estimator regression model. Lee and Lee [17] considered parameter change problem in nonlinear time

series models with GARCH errors via [9] test statistic. Zhou and Liu [18] proposed a weighted CUSUM test statistic to test for mean change in an AR(p) process. The change point problem for variance change has usually been viewed as deterministic. Contrary to this notion, we study a randomised change point estimator in GARCH models, where we allow the test statistic to be data driven.

2. Methodology

We derive the test statistic for change point detection in GARCH processes, under the null hypothesis of no change, and the alternative that the variance has changed at some unknown time, k . Let the observation $\{Y_t\}$, $t = 1, \dots, n$, be a time series fulfilling $Y_t^1, t \leq k$, and $Y_t^2, t > k$, and the two series are based on stationary time series. Consider the model

$$Y_t = f(Y_{t-1}, Y_{t-2}, \dots, Y_{t-p}; \theta_t) + \varepsilon_t, \quad (1)$$

where the errors $\varepsilon_t = \sigma_t z_t$ and z_t has zero mean and finite second moment. In our study, we consider an autoregressive model for the conditional mean of Y_t , $f(Y_{t-1}, Y_{t-2}, \dots, Y_{t-p}; \theta_t)$, and σ_t^2 is approximated by the standard GARCH (p, q) model as

$$\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^p \beta_j \sigma_{t-j}^2, \quad t \in \mathbb{Z}, \quad (2)$$

where $\omega > 0$, $\alpha \geq 0$, and $\beta \geq 0$. For mean estimates, we focus on procedures based on M-procedures of [19], where the parameter estimator $\hat{\theta}_t$ is a solution to the equation

$$\sum_{t=1}^n \psi(Y_t, \theta_t) = 0, \quad (3)$$

such that $\mathbb{E}[\psi^2(Y_t, \theta)] < \infty$ and $\psi(Y_t, \theta_t)$ is nondecreasing in θ_t . Consider $\psi(x, \theta) = h(y)(y - \theta)$ and also define

$$F_h(n, \theta) := \sum_{t=1}^n h(Y_t)(Y_t - \theta_n). \quad (4)$$

Proposition 1. *To obtain the residuals, ε_t , we consider a weighted sum of least-squares estimate for the parameter, θ_n , under H_0 as follows:*

$$\begin{aligned} \hat{\sigma}_n^2(h) &= \frac{1}{\sum_{t=1}^n h(Y_t)} \sum_{t=1}^n h(Y_t)(Y_t - \hat{\theta}_n(h))^2, \\ \hat{\sigma}_{k*}^2(h) &= \frac{1}{\sum_{t=k+1}^n h(Y_t)} \sum_{t=k+1}^n h(Y_t)(Y_t - \hat{\theta}_{k*}(h))^2, \\ \hat{\sigma}_k^2(h) &= \frac{1}{\sum_{t=1}^n h(Y_t)} \left(\sum_{t=1}^k h(Y_t)(Y_t - \hat{\theta}_n(h))^2 + \sum_{t=k+1}^n h(Y_t)(Y_t - \hat{\theta}_{k*}(h))^2 \right). \end{aligned} \quad (11)$$

Simplifying H_k , we have

$$\sum_{t=1}^n h(Y_t)(Y_t - \theta_n)^2 = \min_{\theta} \left(\sum_{t=1}^n h(Y_t)(Y_t - \theta)^2 \right). \quad (5)$$

Then,

$$\hat{\theta}_n(h) = \frac{\sum_{t=1}^n h(Y_t)Y_t}{\sum_{t=1}^n h(Y_t)}. \quad (6)$$

Likewise, the weighted least squares estimators under H_1 after and before the change point, respectively, are

$$\begin{aligned} \hat{\theta}_{k*}(h) &= \frac{\sum_{t=k+1}^n h(Y_t)Y_t}{\sum_{t=k+1}^n h(Y_t)}, \\ \hat{\theta}_k(h) &= \frac{\sum_{t=1}^k h(Y_t)Y_t}{\sum_{t=1}^k h(Y_t)}. \end{aligned} \quad (7)$$

To construct the test statistic, we employ the likelihood ratio test derived as

$$-2 \log \Delta_k = n \log \hat{\sigma}_n^2 - k \log \hat{\sigma}_k^2 - (n - k) \log \hat{\sigma}_{k*}^2. \quad (8)$$

Take $\hat{\sigma}_{k*}^2 = \hat{\sigma}_k^2 + \delta$, and under the null hypothesis consider the case $\delta \rightarrow 0$ as $n \rightarrow \infty$, and we have

$$-\frac{2}{n} \log \Delta_k = \log \left(\frac{\hat{\sigma}_n^2}{\hat{\sigma}_k^2} \right) = \log \left(1 + \frac{\hat{\sigma}_n^2 - \hat{\sigma}_k^2}{\hat{\sigma}_k^2} \right) = \log \left(1 + \frac{H_k}{\hat{\sigma}_k^2} \right), \quad (9)$$

where

$$\begin{aligned} \sigma_n^2(h) &= \sum_{t=1}^n h(Y_t)(Y_t - \hat{\theta}_n(h))^2, \\ \sigma_k^2(h) &= \sum_{t=1}^k h(Y_t)(Y_t - \hat{\theta}_k(h))^2, \\ \sigma_{k*}^2(h) &= \sum_{t=k+1}^n h(Y_t)(Y_t - \hat{\theta}_{k*}(h))^2, \end{aligned} \quad (10)$$

and the variance estimates are now given as

$$\begin{aligned}
H_k = \hat{\sigma}_n^2(h) - \hat{\sigma}_k^2(h) &= \frac{1}{\sum_{t=1}^n h_t} \left(\sum_{t=1}^k h_t (\hat{\theta}_k - \hat{\theta}_n) [(Y_t - \hat{\theta}_k) + (Y_t - \hat{\theta}_n)] \right) \\
&+ \frac{1}{\sum_{t=1}^n h_t} \left(\sum_{t=k+1}^n h_t (\hat{\theta}_{k^*} - \hat{\theta}_n) [(Y_t - \hat{\theta}_{k^*}) + (Y_t - \hat{\theta}_n)] \right).
\end{aligned} \tag{12}$$

Substituting $\hat{\theta}_n$, $\hat{\theta}_k$, and $\hat{\theta}_{k^*}$ into equation (12), we have

$$\begin{aligned}
H_k &= \left(S(k) - \left(\frac{\sum_{t=1}^k h_t}{\sum_{t=1}^n h_t} \right) S(n) \right) \\
&\times \left[\frac{1}{\sum_{t=1}^k h_t \sum_{t=1}^n h_t} \sum_{t=1}^k h_t (Y_t - \hat{\theta}_k) - \frac{1}{\sum_{t=1}^n h_t (\sum_{t=1}^n h_t - \sum_{t=1}^k h_t)} \sum_{t=k+1}^n h_t (Y_t - \hat{\theta}_{k^*}) \right] \\
&+ \left[\frac{1}{\sum_{t=1}^k h_t \sum_{t=1}^n h_t} \sum_{t=1}^k h_t (Y_t - \hat{\theta}_n) - \frac{1}{\sum_{t=1}^n h_t (\sum_{t=1}^n h_t - \sum_{t=1}^k h_t)} \sum_{t=k+1}^n h_t (Y_t - \hat{\theta}_n) \right] := I + J.
\end{aligned} \tag{13}$$

Noting that $\hat{\theta}_{k^*} = \hat{\theta}_k + \delta$, as $\delta \rightarrow 0$, we simplify I and J as follows:

$$\begin{aligned}
I &= \frac{1}{\sum_{t=1}^k h_t \sum_{t=1}^n h_t} \sum_{t=1}^k h_t Y_t - \frac{1}{\sum_{t=1}^n h_t (\sum_{t=1}^n h_t - \sum_{t=1}^k h_t)} \sum_{t=k+1}^n h_t Y_t \\
&= \frac{1}{\sum_{t=1}^k h_t (\sum_{t=1}^n h_t - \sum_{t=1}^k h_t)} \left(S(k) - \left(\frac{\sum_{t=1}^k h_t}{\sum_{t=1}^n h_t} \right) S(n) \right), \\
J &= \frac{1}{\sum_{t=1}^k h_t \sum_{t=1}^n h_t} \sum_{t=1}^k h_t Y_t - \frac{1}{\sum_{t=1}^n h_t (\sum_{t=1}^n h_t - \sum_{t=1}^k h_t)} \sum_{t=k+1}^n h_t Y_t \\
&= \frac{1}{\sum_{t=1}^k h_t (\sum_{t=1}^n h_t - \sum_{t=1}^k h_t)} \left(S(k) - \left(\frac{\sum_{t=1}^k h_t}{\sum_{t=1}^n h_t} \right) S(n) \right).
\end{aligned} \tag{14}$$

Finally, under the likelihood ratio, we have

$$-2 \log \tilde{\Delta}_k = \frac{\sum_{t=1}^n h_t}{\hat{\sigma}_k^2 \sum_{t=1}^k h_t (\sum_{t=1}^n h_t - \sum_{t=1}^k h_t)} \left(S(k) - \left(\frac{\sum_{t=1}^k h_t}{\sum_{t=1}^n h_t} \right) S(n) \right)^2. \tag{15}$$

Hence, the modified weighted form of the test statistic is of the form

$$\tilde{e}_n = \max_{0 < k < n} \left(\frac{(\sum_{t=1}^n h_t)^2}{\sum_{t=1}^k h_t (\sum_{t=1}^n h_t - \sum_{t=1}^k h_t)} \right)^v \frac{1}{\sqrt{n}} \left| S(k) - \left(\frac{\sum_{t=1}^k h_t}{\sum_{t=1}^n h_t} \right) S(n) \right| \quad (16)$$

$$\text{with } v \in \left(0, \frac{1}{2}\right) \text{ where } S(k) = \sum_{t=1}^k h_t Y_t \text{ and } S(n) = \sum_{t=1}^n h_t Y_t.$$

In this paper, we consider observations $Y_1, Y_2, \dots, Y_n$ with α -mixing errors. Define $S_G(k; \theta) = \sum_{t=1}^k h_t G(Y_t; \theta)$ and $S_G(n; \theta) = \sum_{t=1}^n h_t G(Y_t; \theta)$, where $G(Y_t; \theta) =$

$g(\varepsilon_t)$, and in the case of variance change, we take $g(\varepsilon_t) = \varepsilon_t^2 = (Y_t - \theta)^2$. Finally, we have

$$\tilde{e}_n = \max_{1 \leq k < n} \left(\frac{(\sum_{t=1}^n h_t)^2}{\sum_{t=1}^k h_t (\sum_{t=1}^n h_t - \sum_{t=1}^k h_t)} \right)^v \frac{1}{\sqrt{n}} \left| \sum_{t=1}^k h_t \varepsilon_t^2 - \left(\frac{\sum_{t=1}^k h_t}{\sum_{t=1}^n h_t} \right) \sum_{t=1}^n h_t \varepsilon_t^2 \right|, \quad (17)$$

with $v \in (0, 1/2)$, and the estimator is given as $\hat{k} = \operatorname{argmax}_{1 \leq k < n} \tilde{e}_k$.

3. Consistency of the Estimator

In this section, we establish the consistency of the randomised estimator. The general model involves a GARCH(p,q) model with a possible shift at an unknown time, k . We make the assumption below for the weight, h_t .

Assumption 1. The functions $h_t = g(Y_{t-1}, Y_{t-2}, \dots, Y_{t-p}, Y_{t+1}, Y_{t+2}, \dots, Y_{t+p})$ and $g(Y_1, Y_2, \dots, Y_{2p})$ are real and positive functions on $\mathbb{R}^{2p}$ such that $\mathbb{E}\{(1 + \|Y_t\|^{2+\alpha})(h_t + h_t^{2+\alpha})\} < \infty$ for $\alpha > 0$.

We rewrite the test statistic of equation (17) as

$$\begin{aligned} \tilde{e}_n = \max_{1 \leq k < n} & \left(\frac{1}{\left(\sum_{t=1}^k h(Y_t) / \sum_{t=1}^n h(Y_t) \right) \left(\sum_{t=k+1}^n h(Y_t) / \sum_{t=1}^n h(Y_t) \right)} \right)^v \\ & \cdot \frac{1}{\sqrt{n}} \left| \frac{\sum_{t=1}^k h(Y_t) \varepsilon_t^2}{\sum_{t=1}^n h(Y_t)} - \frac{\sum_{t=1}^k h(Y_t)}{\sum_{t=1}^n h(Y_t)} \frac{\sum_{t=1}^n h(Y_t) \varepsilon_t^2}{\sum_{t=1}^n h(Y_t)} \right| \left(\sum_{t=1}^n h(Y_t) \right). \end{aligned} \quad (18)$$

Under the null hypothesis of no variance change, we have

$$\frac{\sum_{t=1}^k h(Y_t) \varepsilon_t^2}{\sum_{t=1}^n h(Y_t)} = \frac{k}{n} \left(\frac{(1/k) \sum_{t=1}^k h(Y_t) \varepsilon_t^2}{(1/n) \sum_{t=1}^n h(Y_t)} \right) = \frac{k}{n} \left(\frac{(\sigma^2/k) \sum_{t=1}^k h(Y_t) z_t^2}{(1/n) \sum_{t=1}^n h(Y_t)} \right) \xrightarrow{P} \tau \left(\frac{\sigma^2 \mathbb{E} h(Y_t) z_t^2}{\mathbb{E} h(Y_t)} \right). \quad (19)$$

Also,

$$\frac{\sum_{t=1}^k h(Y_t)}{\sum_{t=1}^n h(Y_t)} \frac{\sum_{t=1}^n h(Y_t) \varepsilon_t^2}{\sum_{t=1}^n h(Y_t)} = \frac{k}{n} \left(\frac{(1/k) \sum_{t=1}^k h(Y_t)}{(1/n) \sum_{t=1}^n h(Y_t)} \right) \left(\frac{(1/n) \sum_{t=1}^n h(Y_t) \varepsilon_t^2}{(1/n) \sum_{t=1}^n h(Y_t)} \right) \xrightarrow{P} \tau \left(\frac{\sigma^2 \mathbb{E} h(Y_t) z_t^2}{\mathbb{E} h(Y_t)} \right). \quad (20)$$

Subtracting equation (19) from equation (20), the difference $\xrightarrow{P} 0$

Under the alternative hypothesis of a change in variance ($\sigma_1^2 \neq \sigma_2^2$), we have

$$\frac{\sum_{t=1}^k h(Y_t) \varepsilon_t^2}{\sum_{t=1}^n h(Y_t)} = \frac{k}{n} \left(\frac{(1/k) \sum_{t=1}^k h(Y_t) \varepsilon_t^2}{(1/n) \sum_{t=1}^n h(Y_t)} \right) = \frac{k}{n} \left(\frac{(\sigma_1^2/k) \sum_{t=1}^k h(Y_t) z_t^2}{1/n \sum_{t=1}^n h(Y_t)} \right) \xrightarrow{P} \tau \left(\frac{\sigma_1^2 \mathbb{E} h(Y_t) z_t^2}{\mathbb{E} h(Y_t)} \right), \quad (21)$$

$$\begin{aligned} \frac{\sum_{t=1}^k h(Y_t)}{\sum_{t=1}^n h(Y_t)} \frac{\sum_{t=1}^n h(Y_t) \varepsilon_t^2}{\sum_{t=1}^n h(Y_t)} &= \frac{k}{n} \left(\frac{(1/k) \sum_{t=1}^k h(Y_t)}{(1/n) \sum_{t=1}^n h(Y_t)} \right) \left(\frac{(\sigma_1^2/k_0) (k_0/n) \sum_{t=1}^k h(Y_t) z_t^2}{(1/n) \sum_{t=1}^n h(Y_t)} \right) + \left(\frac{(\sigma_2^2/n - k_0) (n - k_0/n) \sum_{t=1}^k h(Y_t) z_t^2}{(1/n) \sum_{t=1}^n h(Y_t)} \right) \\ &= \tau (\tau_0 \sigma_1^2 \lambda) + \tau ((1 - \tau_0) \sigma_2^2 \lambda), \quad \text{where } \lambda = \frac{\sum_{t=1}^k h(Y_t) z_t^2}{\sum_{t=1}^n h(Y_t)}. \end{aligned} \quad (22)$$

Subtracting equation (21) from equation (22), the difference $\xrightarrow{P} (1 - \tau_0) \tau \lambda (\sigma_1^2 - \sigma_2^2)$. Let us also define $\hat{e}_n = (\bar{e}_n / (\sum_{t=1}^n h(Y_t))^{1/2})$

Then, the weight function of equation (18) leads to

$$\max_{1 \leq k \leq n} \left(\frac{1}{\left(\sum_{t=1}^k h(Y_t) / \sum_{t=1}^n h(Y_t) \right) t \left(n \sum_{t=k+1}^n h(Y_t) / \sum_{t=1}^n h(Y_t) \right)} \right)^v \left(\frac{\sum_{t=1}^k h(Y_t)}{n} \right)^{1/2} \xrightarrow{P} \tau^{-v} (1 - \tau)^{-v} (\mathbb{E} (h(Y_t)))^{1/2}. \quad (23)$$

Therefore,

$$\hat{e}_n \xrightarrow{P} \begin{cases} \tau^{1-v} (1 - \tau)^{-v} (\mathbb{E} (h(Y_t)))^{1/2} (1 - \tau_0) \lambda (\sigma_1^2 - \sigma_2^2), & \text{for } \tau \leq \tau_0, \\ \tau^{-v} (1 - \tau)^{1-v} (\mathbb{E} (h(Y_t)))^{1/2} \tau_0 \lambda (\sigma_1^2 - \sigma_2^2), & \text{for } \tau > \tau_0. \end{cases} \quad (24)$$

Take $\bar{e}_k = \lim_{n \rightarrow \infty} \bar{e}_n$, $\lambda_0 = (\mathbb{E} h(Y_t))^{1/2}$, and $\rho = \sigma_1^2 - \sigma_2^2$, and we have

Then, for $\tau = \tau_0$, we obtain

$$\bar{e}_{k_0} = \tau_0^{1-v} (1 - \tau_0)^{1-v} \lambda \lambda_0 \rho. \quad (26)$$

$$\bar{e}_k \xrightarrow{P} \begin{cases} \tau^{1-v} (1 - \tau)^{-v} (1 - \tau_0) \lambda \lambda_0 \rho, & \text{for } \tau \leq \tau_0, \\ \tau^{-v} (1 - \tau)^{1-v} \tau_0 \lambda \lambda_0 \rho, & \text{for } \tau > \tau_0. \end{cases} \quad (25)$$

Also,

$$\begin{aligned} |\bar{e}_{k_0} - \hat{e}_n| &\xrightarrow{P} \begin{cases} (1 - \tau_0)^{1-v} \left[\tau_0^{1-v} - \tau^{(1-v)} \left(\frac{1 - \tau_0}{1 - \tau} \right)^v \right] \lambda_0 \lambda \rho, & \text{for } \tau \leq \tau_0, \\ \tau_0^{1-v} \left[(1 - \tau_0)^{1-v} - (1 - \tau)^{(1-v)} \left(\frac{\tau_0}{\tau} \right)^v \right] \lambda_0 \lambda \rho, & \text{for } \tau > \tau_0, \end{cases} \\ &\geq \begin{cases} (1 - \tau_0)^{1-v} (\tau_0^{1-v} - \tau^{(1-v)}) \lambda_0 \lambda \rho, & \text{for } \tau \leq \tau_0, \\ \tau_0^{1-v} ((1 - \tau_0)^{1-v} - (1 - \tau)^{(1-v)}) \lambda_0 \lambda \rho, & \text{for } \tau > \tau_0. \end{cases} \end{aligned} \quad (27)$$

For $\tau \leq \tau_0$, take $g(\tau) = \tau^\alpha$ and $\alpha = 1 - v$, and applying the mean value theorem, we obtain

Similarly, for $\tau > \tau_0$, take $H(\tau) = (1 - \tau)^{1-v}$ and apply the mean value theorem to obtain

$$\tau_0^{1-v} - \tau^{1-v} \geq (1 - v) \tau_0^{-v} (\tau_0 - \tau). \quad (28)$$

$$(1 - \tau_0)^{1-v} - (1 - \tau)^{1-v} \geq (1 - v) (1 - \tau_0)^{-v} (\tau_0 - \tau). \quad (29)$$

Putting equations (28) and (29) in equation (27), we obtain

$$\begin{aligned} |\bar{e}_{k_0} - \bar{e}_n| &\geq \begin{cases} (1 - \tau_0)^{1-v} (1 - v) \tau_0^{-v} (\tau_0 - \tau) \lambda_0 \lambda \rho, & \text{for } \tau \leq \tau_0, \\ \tau_0^{1-v} (1 - v) (1 - \tau_0)^{-v} (\tau - \tau_0) \lambda_0 \lambda \rho, & \text{for } \tau > \tau_0, \end{cases} \\ &\geq \bar{\tau} |\tau_0 - \tau| \lambda_0 \lambda \rho, \quad \text{where } \bar{\tau} = \begin{cases} (1 - \tau_0)^{1-v} (1 - v) \tau_0^{-v}, & \text{for } \tau \leq \tau_0, \\ \tau_0^{1-v} (1 - v) (1 - \tau_0)^{-v}, & \text{for } \tau > \tau_0. \end{cases} \end{aligned} \quad (30)$$

We note that

$$|\bar{e}_k| - |\bar{e}_{k_0}| \leq 2 \max_{n\delta \leq k \leq n(1-\delta)} |\bar{e}_k - \bar{e}_k| + |\bar{e}_{k_0}| - |\bar{e}_k|. \quad (31)$$

Replacing τ by $\hat{\tau}_n$ and noting that $\hat{e}_{k_0} \leq \hat{e}_{n\tau}$, we have

$$\begin{aligned} \bar{\tau} |\tau_0 - \hat{\tau}_n| &\leq |\bar{e}_{k_0}| - |\bar{e}_k| \leq 2 \max_{n\delta \leq k \leq n(1-\delta)} |\bar{e}_k - \bar{e}_k| + |\bar{e}_k| - |\bar{e}_{k_0}| \\ &\leq 2 \max_{n\delta \leq k \leq n(1-\delta)} |\bar{e}_k - \bar{e}_k|. \end{aligned} \quad (32)$$

Let us consider

$$|\bar{e}_k - \bar{e}_k| \leq \left| M \left(\left(\frac{\sum_{t=1}^k h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} \left(\frac{\sum_{t=k+1}^n h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} - \tau^{-v} (1 - \tau)^{-v} \right) \lambda_0 \right| + |\tau^{-v} (1 - \tau)^{-v} (M \lambda_0 - \tau (1 - \tau_0) \lambda \lambda_0 \rho)| := S_1 + S_2, \quad (33)$$

where $M = ((\sum_{t=1}^k h(Y_t) \varepsilon_t^2 / \sum_{t=1}^n h(Y_t)) - (\sum_{t=1}^k h(Y_t) / \sum_{t=1}^n h(Y_t)) (\sum_{t=1}^n h(Y_t) \varepsilon_t^2 / \sum_{t=1}^n h(Y_t)))$,

$$\begin{aligned} S_1 &= \left| M \left(\left(\frac{\sum_{t=1}^k h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} \left(\frac{\sum_{t=k+1}^n h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} - \tau^{-v} (1 - \tau)^{-v} \right) \lambda_0 \right|, \\ S_2 &= |\tau^{-v} (1 - \tau)^{-v} (M \lambda_0 - \tau (1 - \tau_0) \lambda \lambda_0 \rho)|. \end{aligned} \quad (34)$$

Maximising S_1 , we obtain

$$S_1 \leq \left| M \left(\frac{\sum_{t=k+1}^n h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} \left(\left(\frac{\sum_{t=1}^k h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} - \tau^{-v} \right) \lambda_0 \right| + \left| M \left(\frac{\sum_{t=1}^k h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} \left(\left(\frac{\sum_{t=k+1}^n h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} - (1 - \tau)^{-v} \right) \lambda_0 \right|. \quad (35)$$

Furthermore, we have

$$S_1 \leq \left| \lambda_0 M \left(\frac{\sum_{t=k+1}^n h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} \left[\left(\frac{\sum_{t=1}^k h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} - \tau^{-v} \right] \right| + \left| \lambda_0 M \tau^{-v} \left[\left(\frac{\sum_{t=k+1}^n h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} - (1 - \tau)^{-v} \right] \right|. \quad (36)$$

We apply the mean value theorem to equation (36) by assuming $f(x) = x^{-\tau}$ at $(\sum_{t=1}^k h(Y_t)/\sum_{t=1}^n h(Y_t)) - \tau$ and $(\sum_{t=k+1}^n h(Y_t)/\sum_{t=1}^n h(Y_t))^{-v}$, $\forall x \in (0, 1)$, to obtain

$$S_1 \leq \left| -v\lambda_{k_1}^{-v-1}\lambda_0 M \left(\frac{\sum_{t=k+1}^n h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^{-v} \left(\frac{\sum_{t=1}^k h(Y_t)}{\sum_{t=1}^n h(Y_t)} - \tau \right) \right| + \left| -v\lambda_{k_2}^{-v-1}\lambda_0 M \tau^v \left(\frac{\sum_{t=k+1}^n h(Y_t)}{\sum_{t=1}^n h(Y_t)} - (1-\tau) \right) \right| := K_1 + K_2, \quad (37)$$

where $\lambda_{k_1} \rightarrow \tau$ and $\lambda_{k_2} \rightarrow 1-\tau$, such that

$$\mathbb{P} \left(\max_{n\delta \leq k \leq k_0} |\lambda_{k_1} - \tau| > \varepsilon \right) \leq \mathbb{P} \left(\max_{k > n\delta} |\lambda_{k_1} - \tau| > \varepsilon \right) \xrightarrow{P} 0. \quad (38)$$

Proposition 2. K_1 is given by $|-v\lambda_{k_1}^{-v-1}\lambda_0 M (\sum_{t=k+1}^n h(Y_t)/\sum_{t=1}^n h(Y_t))^{-v} ((\sum_{t=1}^k h(Y_t)/\sum_{t=1}^n h(Y_t)) - \tau)|$ and converges to $(\Psi_a/\sqrt{n})$.

Proof.

$$\begin{aligned} \mathbb{P} \left(\max_{n\delta \leq k \leq k_0} |K_1| > \varepsilon \right) &\leq \varepsilon + \mathbb{P} \left(\max_{n\delta \leq k \leq k_0} \left| \frac{\sum_{t=1}^k h(Y_t)}{\sum_{t=1}^n h(Y_t)} - \tau \right| > \alpha_\varepsilon \right), \\ \alpha_\varepsilon &= \left(\frac{\sum_{t=k+1}^n h(Y_t)}{\sum_{t=1}^n h(Y_t)} \right)^v \varepsilon \lambda_{k_1}^{v+1} \lambda_0^{-1} M^{-1}, \end{aligned} \quad (39)$$

$$\mathbb{P} \left(\max_{n\delta \leq k \leq k_0} |K_1| > \varepsilon \right) \leq \varepsilon + \mathbb{P} \left(\max_{n\delta \leq k \leq k_0} \frac{1}{k} \left| \sum_{t=1}^k (h(Y_t) - \mathbb{E}(h(Y_t))) \right| > \frac{\alpha_\varepsilon}{(1-\delta)\mathbb{E}(h(Y_t))} \right).$$

We apply Theorem 4.1 of [14] to equation (39) as follows:

$$\begin{aligned} \mathbb{P} \left(\max_{n\delta \leq k \leq k_0} \frac{1}{k} \left| \sum_{t=1}^k (h(Y_t) - \mathbb{E}(h(Y_t))) \right| > \frac{\alpha_\varepsilon}{(1-\delta)\mathbb{E}(h(Y_t))} \right) &\leq \frac{1}{n^2 \delta^2} \sum_{i,j=1}^{n\delta} |\text{Cov}_{i,j}| \\ &+ \sum_{k=[n\delta]}^{k_0-1} \left| \frac{1}{(k)^2} - \frac{1}{(k+1)^2} \right| \sum_{i,j=1}^k |\text{Cov}_{i,j}| + 2 \sum_{k=[n\delta]}^{k_0-1} \frac{1}{(k+1)^2} (\text{Cov}_{k+1,k+1})^{1/2} \left(\sum_{i,j=1}^k \text{Cov}_{i,j} \right)^{1/2} \\ &+ \sum_{k=[n\delta]}^{k_0-1} \frac{1}{(k+1)^2} (\text{Cov}_{k+1,k+1}) \leq C \left(\frac{1}{n\delta} + \sum_{k=[n\delta]}^{k_0-1} \frac{1}{k^2} + \sum_{k=[n\delta]}^{k_0-1} \frac{1}{k^2 k^{1/2}} + \sum_{k=[n\delta]}^{k_0-1} \frac{1}{k^2} \right) \leq \frac{C}{\sqrt{\delta} \sqrt{n}} \leq \frac{\Psi_a}{\sqrt{n}}. \end{aligned} \quad (40)$$

Proposition 3. K_2 is given by $|-v\lambda_{k_2}^{-v-1}\lambda_0 M \tau^v ((\sum_{t=k+1}^n h(Y_t)/\sum_{t=1}^n h(Y_t)) - (1-\tau))|$ and converges to $(\Psi_b/\sqrt{n})$.

Proof

□

$$\begin{aligned}
\mathbb{P}\left(\max_{n\delta \leq k \leq k_0} |K_2| > \varepsilon\right) &\leq \varepsilon + \mathbb{P}\left(\max_{n\delta \leq k \leq k_0} \left|\frac{\sum_{t=K+1}^n h(Y_t)}{\sum_{t=1}^n h(Y_t)} - (1 - \tau)\right| > \beta_\varepsilon\right) \beta_\varepsilon = \varepsilon \tau^\nu \lambda_{k_2}^{v+1} \lambda_0^{-1} M^{-1}, \\
\mathbb{P}\left(\max_{n\delta \leq k \leq k_0} |K_2| > \varepsilon\right) &\leq \varepsilon + \mathbb{P}\left(\max_{n\delta \leq k \leq k_0} \frac{1}{n-k} \left|\sum_{t=k+1}^n (h(Y_t) - \mathbb{E}(h(Y_t)))\right| > \frac{\beta_\varepsilon}{n(n-k)\mathbb{E}h(Y_t)}\right) \\
&\leq \mathbb{P}\left(\max_{n\delta \leq k \leq k_0} \frac{1}{n-k} \left|\sum_{t=1}^k (h(Y_t) - \mathbb{E}(h(Y_t)))\right| > \frac{\beta_\varepsilon}{(1-\delta)\mathbb{E}h(Y_t)}\right).
\end{aligned} \tag{41}$$

Next, we apply Theorem 4.1 of [14] to equation (41) as follows:

$$\begin{aligned}
\mathbb{P}\left(\max_{n\delta \leq k \leq k_0} |K_2| > \varepsilon\right) &\leq \frac{1}{n^2 \delta^2} \sum_{i,j=1}^{[n\delta]} |\text{Cov}_{i,j}| + \sum_{k=[n\delta]}^{k_0-1} \left| \frac{1}{(n-(k+1))^2} - \frac{1}{(n-k)^2} \right| \sum_{i,j=1}^k |\text{Cov}_{i,j}| \\
&+ 2 \sum_{k=[n\delta]}^{k_0-1} \frac{1}{(n-(k+1))^2} (\text{Cov}_{k+1,k+1})^{1/2} \left(\sum_{i,j=1}^k \text{Cov}_{i,j} \right)^{1/2} + \sum_{k=[n\delta]}^{k_0-1} \frac{1}{(n-(k+1))^2} (\text{Cov}_{k+1,k+1}) \\
&\leq \frac{\delta C_1}{n(1-\delta)} + C_2 \sum_{k=[n\delta]}^{k_0-1} \frac{1}{(n-k)^2} + C_3 \sum_{k=[n\delta]}^{k_0-1} \frac{1}{(n-k)^{3/2}} + C_4 \sum_{k=[n\delta]}^{k_0-1} \frac{1}{(n-k)^2}.
\end{aligned} \tag{42}$$

Simplifying further, we obtain

$$\mathbb{P}\left(\max_{n\delta \leq k \leq k_0} |K_2| > \varepsilon\right) \leq C \left(\frac{1}{n(1-\delta)^2} + \sum_{k=[n\delta]}^{k_0-1} \frac{1}{(n-k)^2} + \sum_{k=[n\delta]}^{k_0-1} \frac{1}{(n-k)^{3/2}} + \sum_{k=[n\delta]}^{k_0-1} \frac{1}{(n-k)^2} \right) \leq \frac{\Psi_b}{\sqrt{n}} \tag{43}$$

Similarly,

$$\mathbb{P}\left(\max_{n\delta \leq k \leq k_0} |S_2| > \varepsilon\right) \leq \frac{\Psi_c}{\sqrt{n}} \tag{44}$$

Finally, combining S_1 and S_2 , we have

$$\mathbb{P}\left(|\hat{e}_k - \bar{e}_k| > \varepsilon\right) \leq \frac{\Psi_e}{\sqrt{n}} \tag{45}$$

Consequently, from equation (32), we have

$$\mathbb{P}\left(\max_{n\delta \leq k \leq n(1-\delta)} |\tau_0 - \hat{\tau}_n| \right) \leq \frac{\Psi_e}{\sqrt{n}} \xrightarrow{P} 0. \tag{46}$$

This completes the proof of consistency of the randomised estimator. $\square$

4. Simulation Study

We present simulation results to illustrate the validity of the estimator. Data were generated from the model, $\varepsilon_t = \sigma_t z_t$, where $z_t \sim N(0, 1)$, and σ_t^2 is from a standard GARCH(1, 1) model as given in equation (2). The choice of weight $h_t(Y_t, \theta)$ greatly improves the efficiency of the estimator.

We, however, consider the function of equation (47) in this paper:

$$h_t(Y_t, \theta) \longrightarrow \begin{cases} 1, & \text{for } \varepsilon_t \leq B, \\ \frac{B}{|\varepsilon_t|}, & \text{for } \varepsilon_t > B, \end{cases} \tag{47}$$

$B > 0$.

The number of replications is 1000 with sample sizes set at 500 to 4000. We present in Figures 1 and 2 a simulation study of the estimator.

The graphs of Figures 1 and 2 show the mean estimate of test of [20] (Lee), the randomised(Ran) estimator and the True(Tru) value of the estimator. It can be seen that both estimators tend to perform well when the change occurs in the middle of the sample. They, however, tend to underperform when the change occurs at the early (first quarter) or late (third quarter) of the sample, although the randomised estimator outperforms the ordinary CUSUM test of [20]. The estimator is shown to be consistent with increasing sample size, as shown theoretically in Section 3, for

(a)
FIGURE 1: Continued.

(b)

FIGURE 1: Graphs showing changes in volatility: (a) graph showing an early, middle, and late change with a 5% change in volatility of the True(Tru), ordinary CUSUM(Lee) of [20], and the Randomised(Ran) estimator and (b) graph showing an early, middle, and late change with a 200% change in volatility of the True(Tru), ordinary CUSUM(Lee) of [20], and the Randomised(Ran) estimator.

FIGURE 2: Graph showing an early, middle, and late change with a 400% change in volatility of the True(Tru), ordinary CUSUM(Lee) of [20], and Randomised(Ran) estimator.

situations where there is an early, mid, or late change in the variance structure of the data. The efficiency of the estimator greatly improves with larger variance change ratios.

5. Conclusion

In this paper, a randomised change point estimator for GARCH Models is presented. A simulation result of the estimator was carried out and compared to the estimator of [20]. We noticed that depending on the weight function chosen the randomised estimator outperforms the ordinary CUSUM of [20] especially when there is an early or late change in the sample. The consistency of the estimator was also established.

Data Availability

The authors used simulated data from a standard GARCH process which are included within the article.

Conflicts of Interest

The authors declare there are no conflicts of interest.

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