Research Article

On Quantum Differential Subordination Related with Certain Family of Analytic Functions

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1.Introduction

Recently, there is a rapid increase in the area of Quantum calculus (known as $q$-calculus) due to its widespread applications in many areas of study such as geometry functions theory (GFT), combinatorial analysis, Lie theory, mechanical engineering, cosmology, and statistics. The concept of $q$-integral was first introduced and studied by Jackson et al. [6] at the beginning of the twentieth century.

The development of the concept of $q$-calculus in GFT had its history from the work of Ismail et al. [5], where the notion of $q$-starlike functions was extensively studied. As such, many subclasses of univalent functions correlated with $q$-calculus have been on increase (see [1, 14, 17, 20, 23–25, 27, 28]). In recent times, various family of $q$- extension of starlike functions, which are connected to Janowski functions in the open unit disc $U$, were initiated and examined from many different viewpoints and perspective (see [17, 20, 28]).

In an attempt to generalize the notion of uniformly closed-to-convex functions considered by Goodman [3], Kanas and Wisniowska [8–10, 13] and Kanas and Srivastava [12] introduced the conic domain $\Omega_m$ $(m \geq 0)$ and studied the classes $m–UCV$ and $m–UST$ of $m$-uniformly convex and starlike functions. Furthermore, Noor and Malik [22], using the concept of Janowski class, extended the domain $\Omega_m$ to $\Omega_{gm}(y, \lambda)$, $1 \leq \lambda < y \leq 1$. In the latest article by Mahmood et al. [17], the importance of $q$-calculus was used to improve the Noor–Malik conic domains to $\Omega_{gm}(y, \lambda)$. Using this domain, they examined the coefficient inequalities associated with the class $m–UST_q(y, \lambda)$ of $q$-uniformly starlike functions. Afterward, the same coefficient problems were also explored for the classes $m–UCV_q(y, \lambda)$, $m–UKV_q(y, \lambda)$, $m–UC^*V_q(y, \lambda)$ of $m$-uniformly $q$-convex, close-to-convex and quasi-convex functions by Naeem et al. [20].

Motivated by these recent articles [15, 17, 20, 28], our aim is to introduce the novel classes $m–UM_q(\alpha, y, \lambda)$ and $m–UQ_q(\alpha, y, \lambda)$ consisting of $m$-uniformly $q$-$\alpha$-convex and quasi-convex functions. We study the coefficient inequalities associated with these classes and some other related properties. Some relevant consequences of our results which were studied in previous work show the significance of our investigation.

2.Materials and Methods

Now, we give some useful preliminaries which are necessary for our study.
Let $A$ be the class of normalized analytic functions $f(z)$ in $U = \{z: |z| < 1\}$ with

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots. \tag{1}$$

Let $S, CV, ST, QV,$ and $KV$ be the subclasses of $A$ consisting functions that are univalent, convex, starlike, quasi-convex, and close-to-convex functions, respectively. The function $f(z)$ of form (1) is subordinate to the analytic function $g(z)$ (written as $f(z) \prec g(z)$) of the form

$$g(z) = z + b_2z^2 + b_3z^3 + \cdots, \tag{2}$$

if there exists a Schwarz function $w(z)$ in $U$ such that

$$f(z) = g(w(z)), \quad z \in U. \tag{3}$$

Let $-1 \leq \lambda < \gamma \leq 1$. Then, class $P(\gamma, \lambda)$ (see [7]) of function $p(z)$ satisfies the subordination condition

$$p(z) < \frac{1 + \gamma}{1 + \lambda z}, \quad z \in U, \tag{4}$$

or equivalently,

$$p(z) = \frac{(1 + \gamma)h(z) + (1 - \gamma)}{(1 + \lambda)h(z) + (1 - \lambda)}, \quad z \in U, \tag{5}$$

where $h(z)$ is the class of functions with positive real part. For $\gamma = 1 - 2\beta, 0 \leq \beta < 1,$ and $\lambda = -1$, the class $P(\beta)$ reduces to the class $P(\beta)$, the class of functions whose real part is greater than $\beta$.

The conic domains $\Omega_m(\gamma, \lambda)(m \geq 0)$ of Janowski type introduced by Noor and Malik [22] are defined as follows:

$$\Omega_m(\gamma, \lambda) = \left\{u + iv: \left[\left(\lambda^2 - 1\right)\left(u^2 + v^2\right) - 2\left(\gamma\lambda - 1\right)u + \left(\gamma^2 - 1\right)\right]^2 \right. \right.$$

$$\left. > m^2\left[-2\left(\lambda + 1\right)\left(u^2 + v^2\right) + 2\left(\gamma + \lambda + 2\right)u - 2\left(\gamma + 1\right)^2 + 4\left(\gamma - \lambda\right)v^2\right]\right\}. \tag{6}$$

Geometrical interpretation of $\Omega_m(\gamma, \lambda)$ and its effect on $\Omega_m$ were also demonstrated in [22]. The class $m = P(\gamma, \lambda)$ represents the class of all functions that maps $U$ onto $\Omega_m(\gamma, \lambda)$. Equivalently, a function $p(z)$ belongs to $m = P(\gamma, \lambda)$ if and only if

$$p(z) < \frac{(\gamma + 1)p_m(z) - (\gamma - 1)}{(\lambda + 1)p_m(z) - (\lambda - 1)}, \quad m \geq 0, -1 \leq \lambda < \gamma \leq 1, \tag{7}$$

where $p_m(z)$ has its definition in [10, 11] and given by

$$p_m(z) = \begin{cases} 1 + \frac{z}{1-z}, & m = 0, \\ 1 + \frac{2}{n\lambda}\left(\log \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}}\right)^2, & m = 1, \\ 1 + \frac{2}{1 - m}\sinh^2\left[\frac{\sqrt{2}}{\pi} \arccos m \right] \arctan \sqrt{\lambda}, & 0 < m < 1, \\ 1 + \frac{1}{1 - m}\sin\left[\frac{\pi}{2R(t)} \arcsin \left(\frac{1}{\sqrt{1 - x^2}} \sqrt{1 - (tx)^2}\right)\right] + \frac{1}{\sqrt{1 - x^2}}, & m > 1, \end{cases} \tag{8}$$

where $u(t) = ((\sqrt{1 - x^2})/(1 - \sqrt{\lambda})), t \in (0, 1), z \in U$ and $t$ is chosen such that $m = \cosh((\pi R'(t))/(4R(t)))$, $R(t)$ is Legendre’s complete elliptic integral of the first kind, and $R'(t)$ is the complementary integral of $R(t)$.

**Definition 1** (see [2]). Let $q \in (0, 1)$. Then, the $q$-number $[n]_q$ is given as

$$[n]_q = \begin{cases} 1 - q^n \quad \text{if } n \in \mathbb{C}, \\ \sum_{i=0}^{n-1} q^i = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbb{N}, \text{ as } q \rightarrow 1^{-}, \end{cases} \tag{9}$$
and the $q$-derivative of a complex valued function $f(\zeta)$ in $U$ is given by
\[
D_q f(\zeta) = \begin{cases} 
\frac{f(q\zeta) - f(\zeta)}{(q-1)\zeta}, & \zeta \neq 0, \\
f'(0), & \zeta = 0, \\
f'(\zeta), & \text{as } q \to 1^-.
\end{cases}
\] (10)

From the above explanation, it is easy to see that, for $f(\zeta)$ given by (1),
\[
D_q f(\zeta) = 1 + \sum_{n=2}^{\infty} [l]_n a_n \zeta^n.
\] (11)

**Definition 2** (see [28]). An analytic function $p(\zeta)$ in $U$ belongs to $P_q(\gamma, \lambda)$ if and only if the condition
\[
p(\zeta) = \frac{(O_1 y + O_3) h(\zeta) - (y - 1) O_1}{(O_1 \lambda + O_3) h(\zeta) - (\lambda - 1) O_1},
\] (12)
is satisfied, where $O_1 = 1 + q, O_3 = 3 - q$ and $h \in P$.

**Definition 3** (see [17]). An analytic function $p(\zeta)$ in $U$ belongs to $m - P_q(\gamma, \lambda)$ if and only if the subordination condition
\[
p(\zeta) < \frac{(O_1 y + O_3) p_m(\zeta) - (y - 1) O_1}{(O_1 \lambda + O_3) p_m(\zeta) - (\lambda - 1) O_1},
\] (13)
is satisfied, where $p_m(\zeta)$ is given by (8). Equivalently, $p(\zeta) \in m - P_q(\gamma, \lambda)$ if and only if $p(\zeta)$ conformally maps $U$ onto the domain $\Omega_{q,m}(\gamma, \lambda)$ defined by
\[
\Omega_{q,m}(\gamma, \lambda) = \{u + iv : \Re \Phi > m |\Phi - 1|, \Phi = \frac{(\lambda - 1) O_1, p(\zeta) - (y - 1) O_1}{(\lambda O_1 + O_3) p(\zeta) - (\lambda - 1) O_1}\},
\] (15)
where
\[
\Phi = \frac{(\lambda - 1) O_1, p(\zeta) - (y - 1) O_1}{(\lambda O_1 + O_3) p(\zeta) - (\lambda - 1) O_1}.
\] (16)

We note that
(i) $m - P_q(\gamma, \lambda) \subset P(\beta)$, where $\beta$ is given by
\[
\beta = \frac{4m + (1 - y) O_1}{4m + (1 - \lambda) O_1}.
\] (17)

(ii) As $\lambda \to 1^-$, the class $m - P_q(\gamma, \lambda)$ becomes the class $m - P(\gamma, \lambda)$ and $\Omega_{q,m}(\gamma, \lambda) = \Omega_m(\gamma, \lambda)$ [22].

(iii) When $\lambda \to 1^-$ and $y = 1, \lambda = -1$, the class $m - P_q(\gamma, \lambda)$ reduces to the class $P(p_m)$ and $\Omega_{q,m}(\gamma, \lambda) = \Omega_m$ [10].

**Definition 4** (see [20]). Let $g(\zeta)$ of form (2) be in $\mathcal{A}$ and $\alpha \geq 0, m \in [0, \infty)$. Then, $g(\zeta) \in m - UCV_q(\gamma, \lambda)$ if and only if
\[
D_q \left( cD_q g(\zeta) \right) \frac{(O_1 y + O_3) p_m(\zeta) - (y - 1) O_1}{(O_1 \lambda + O_3) p_m(\zeta) - (\lambda - 1) O_1},
\] (18)
where $p_m(\zeta)$ is given by (8).

Inspired by the above recent mentioned work, we announce the following novel classes of analytic functions.

**Definition 5.** Let $f(\zeta) \in \mathcal{A}, \alpha \geq 0, m \in [0, \infty)$. Then, $f(\zeta) \in m - U_{M_q}(\alpha, \gamma, \lambda)$ if and only if
\[
\Re \left( \frac{(\lambda - 1) O_1 J_q(\alpha, f; \zeta) - (y - 1) O_1}{(\lambda O_1 + O_3) J_q(\alpha, f; \zeta) - (\gamma O_1 + O_3)} \right) > m \frac{(\lambda - 1) O_1 J_q(\alpha, f; \zeta) - (y - 1) O_1}{(\lambda O_1 + O_3) J_q(\alpha, f; \zeta) - (\gamma O_1 + O_3)} - 1.
\] (19)

Equivalently, $f(\zeta) \in m - U_{M_q}(\alpha, \gamma, \lambda)$ if and only if
\[
J_q(\alpha, f; \zeta) = (1 - \alpha) \frac{cD_q f(\zeta)}{f(\zeta)} + \alpha \frac{D_q \left( cD_q f(\zeta) \right)}{D_q f(\zeta)} \frac{(O_1 y + O_3) p_m(\zeta) - (y - 1) O_1}{(O_1 \lambda + O_3) p_m(\zeta) - (\lambda - 1) O_1}.
\] (20)
**Definition 6.** Let \( f(z) \in \mathcal{A}, \alpha \geq 0, m \in [0, \infty) \). Then, \( f \in m - \text{UQ}_q(\alpha, \gamma, \lambda) \) if and only if there exists an analytic function \( g(z) \in m - \text{UCV}_q(\gamma, \lambda) \) such that

\[
\Re \left( \frac{(\lambda - 1)O_1J_q(\alpha, f; \cdot) - (\gamma - 1)O_1}{(\lambda O_1 + O_3)J_q(\alpha, f; \cdot) - (\gamma O_1 + O_3)} \right) > m \left| \frac{(\lambda - 1)O_1J_q(\alpha, f; \cdot) - (\gamma - 1)O_1}{(\lambda O_1 + O_3)J_q(\alpha, f; \cdot) - (\gamma O_1 + O_3)} - 1 \right| \tag{21}
\]

Equivalently, \( f \in m - \text{UQ}_q(\alpha, \gamma, \lambda) \) if and only if

\[
J_q(\alpha, f; \cdot) = (1 - \alpha) \frac{D_qf(c)}{D_qg(c)} + \alpha \frac{D_q(cD_qf(c))}{D_qg(c)} \frac{(O_1y + O_3)p_m(c) - (\gamma - 1)O_1}{(O_1\lambda + O_3)p_m(c) - (\lambda - 1)O_1} \tag{22}
\]

We note the following special cases:

(i) When \( q \to 1^- \), the classes \( m - \text{UM}_q(\alpha, \gamma, \lambda) \) and \( m - \text{UQ}_q(\alpha, \gamma, \lambda) \) reduce to the classes \( m - \text{UM}(\alpha, \gamma, \lambda) \) [21] and \( m - \text{UQ}(\alpha, \gamma, \lambda) \) [18], respectively.

(ii) When \( q \to 1^- \) and \( \alpha = 0 \), the classes \( m - \text{UM}_q(\alpha, \gamma, \lambda) \) and \( m - \text{UQ}_q(\alpha, \gamma, \lambda) \) cut down to the classes \( m - \text{UST}(\gamma, \lambda) \) [22] and \( m - \text{UKV}(\gamma, \lambda) \) [16].

(iii) When \( q \to 1^- \) and \( \alpha = 1 \), the classes \( m - \text{UM}_q(\alpha, \gamma, \lambda) \) and \( m - \text{UQ}_q(\alpha, \gamma, \lambda) \) scale down to the classes \( m - \text{UCV}(\gamma, \lambda) \) [22] and \( m - \text{UCV}(\gamma, \lambda) \) [16].

(iv) When \( \alpha = 0 \) and \( \alpha = 1 \), the classes \( m - \text{UM}_q(\alpha, \gamma, \lambda) \) and \( m - \text{UQ}_q(\alpha, \gamma, \lambda) \) diminish, respectively, to those classes of functions considered in [17, 20].

(v) When \( 0 = \alpha \) in Definition 2, the class \( m - \text{UM}_q(\alpha, \gamma, \lambda) \) becomes the class of \( q \)-starlike functions \( \text{ST}_q(\gamma, \lambda) \) of Janowski type recently explored by Srivastava et al. [28].

To effectively establish our findings, the following set of lemmas is required.

### 3. A Set of Lemmas

**Lemma 1** (see [4]). Let \( m \geq 0 \) and \( p_m(c) \) given by (8) be of the form \( p_m(c) = 1 + \sum_{j=1}^{\infty} c_j^2 + \cdots \). Then,

\[
|c_2 - \delta c_1^2| \leq \begin{cases} -4\delta + 2, & \delta \leq 0, \\ 2, & 0 \leq \delta \leq 1, \\ 4\delta + 2, & \delta \geq 1, \end{cases} \tag{25}
\]
when $\delta < 0$ or $\delta > 1$, the equality holds if and only if $p(\zeta) = (1 + \zeta)/(1 - \zeta)$ or one of its rotations. If $0 < \delta < 1$, then the equality holds if and only if $p(\zeta) = (1 + \zeta)/(1 - \zeta^2)$ or one of its rotations. If $\delta = 0$, then the equality holds if and only if

$$p(\zeta) = \left(\frac{1 + \gamma}{2}\right) \frac{1 + \zeta}{1 - \zeta} + \left(\frac{1 - \gamma}{2}\right) \frac{1 - \zeta}{1 + \zeta}, \quad 0 \leq \gamma \leq 1,$$

or one of its rotations. If $\delta = 1$, then the equality holds if and only if $\zeta$ is the reciprocal of one of the functions such that $\zeta = \gamma > 0$ or one of its rotations. If $\delta > 0$, then the equality holds if and only if $p(\zeta) = (1 + \zeta)/(1 - \zeta)$ or one of its rotations.

Lemma 3 (see [20]). Let $g(\zeta)$ be of form (2) and $g \in m - \text{UCV}_q(y, \lambda)$. Then,

$$|c_2 - \delta c_1| + |c_1|^2 \leq 2, \quad \left(0 < \delta \leq \frac{1}{2}\right),$$

or

$$|c_2 - \delta c_1| + (1 - \delta)|c_1|^2 \leq 2, \quad \left(\frac{1}{2} < \delta \leq 1\right).$$

4. Results and Discussion

We now turn our attention to the main results of this article.

4.1. Sufficient Conditions

Theorem 1. A function $f(\zeta)$ of form (1) belongs to the class $m - \text{UM}_q(\alpha, \gamma, \lambda)$ if it satisfies the condition

$$\phi_n(\alpha, m, q, \gamma, \lambda)<|\lambda - \gamma|O_1,$$

where

$$|b_n| \leq \frac{1}{m} \prod_{i=0}^{n-2} \left(\left|\bar{Q}_i \Omega_1 (\gamma - \lambda) - 4q[i]_q\right|\right),$$

where $\bar{Q}_i$ is defined by (23).

Proof. Suppose condition (29) holds. Then, we need to prove that

$$m^2 \left|\frac{(\lambda - 1)O_1 f(q, \alpha; \gamma) - (\gamma - 1)O_1}{(\lambda O_1 + O_3)f(q, \alpha; \gamma) - (\gamma O_1 + O_3)} - 1\right| - \Re \left(\frac{(\lambda - 1)O_1 f(q, \alpha; \gamma) - (\gamma - 1)O_1}{(\lambda O_1 + O_3)f(q, \alpha; \gamma) - (\gamma O_1 + O_3)} - 1\right) < 1.$$

Therefore,

$$m^2 \left|\frac{(\lambda - 1)O_1 f(q, \alpha; \gamma) - (\gamma - 1)O_1}{(\lambda O_1 + O_3)f(q, \alpha; \gamma) - (\gamma O_1 + O_3)} - 1\right| - \Re \left(\frac{(\lambda - 1)O_1 f(q, \alpha; \gamma) - (\gamma - 1)O_1}{(\lambda O_1 + O_3)f(q, \alpha; \gamma) - (\gamma O_1 + O_3)} - 1\right)$$

$$\leq (m + 1)^2 \left|\frac{(\lambda - 1)O_1 f(q, \alpha; \gamma) - (\gamma - 1)O_1}{(\lambda O_1 + O_3)f(q, \alpha; \gamma) - (\gamma O_1 + O_3)} - 1\right|$$

$$\leq 4(m + 1)^2 \left|\frac{(\lambda - 1)O_1 f(q, \alpha; \gamma) - (\gamma - 1)O_1}{(\lambda O_1 + O_3)f(q, \alpha; \gamma) - (\gamma O_1 + O_3)} - 1\right|$$

$$= 4(m + 1)^2 \left|\frac{(1 - \alpha)\Omega Df(q, \zeta)Df(q, \zeta) + \alpha f(q)Df(q, \zeta) - f(q)Df(q, \zeta)}{(1 - \alpha)(\lambda O_1 + O_3)f(q, \alpha; \gamma) + \alpha(\lambda O_1 + O_3)f(q)Df(q, \zeta) - (\gamma O_1 + O_3)f(q)Df(q, \zeta)}\right|. $$
We have
\[
cD_q f(c)D_q f(c) = \left( \sum_{n=0}^{\infty} [n]_q a_n c^n \right) \left( \sum_{n=0}^{\infty} [n]_q a_n c^{n-1} \right), \quad a_0 = 0, a_1 = [0]_q = [1]_q = 1
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} [i]_q [n-i]_q a_i a_{n-i} \right) c^{n-1}
\]
\[
= \zeta + \sum_{n=2}^{\infty} \left( \sum_{i=1}^{n-1} [i]_q [n+1-i]_q a_i a_{n-i} \right) c^n
\]
\[
= \zeta + \sum_{n=2}^{\infty} \left( 2[n]_q a_n + \sum_{i=2}^{n-1} [i]_q [n+1-i]_q a_i a_{n-i} \right) c^n.
\]

Similarly,
\[
D_q cD_q f(c) = \zeta + \sum_{n=2}^{\infty} \left( 2[n]_q a_n + \sum_{i=2}^{n-1} [i]_q [n+1-i]_q a_i a_{n-i} \right) c^n,
\]
\[
f(c)D_q f(c) = \zeta + \sum_{n=2}^{\infty} \left( 2[n]_q a_n + \sum_{i=2}^{n-1} [i]_q [n+1-i]_q a_i a_{n-i} \right) c^n.
\]

From inequality (32) and equations (33) and (34), we arrive at
\[
\left| \frac{(\lambda - 1)O_{1}f_q(\alpha, f; c) - (\gamma - 1)O_{3}}{(\lambda O_{1} + O_{3})f_q(\alpha, f; c) - (\gamma O_{1} + O_{3})} - 1 \right| \leq 4(m + 1) \left| \sum_{n=2}^{\infty} \left( [n]_q - 1 \right) \left( 1 + a([n]_q - 1) \right) a_n + \sum_{i=2}^{n-1} \left( [i]_q - 1 \right) + a([n+1-i]_q - [i]_q) \right|
\]
\[
\cdot \left( \lambda - \gamma \right)O_{1} + \sum_{n=2}^{\infty} \left( \lambda O_{1} + O_{3} \right) \left( 2[n]_q + a([n]_q - 1)^2 \right)
\]
\[
- (\gamma O_{1} + O_{3}) \left( [n]_q + 1 \right) a_n + \sum_{i=2}^{n-1} \left( \lambda O_{1} + O_{3} \right) \left( [i]_q + a([n+1-i]_q - [i]_q) \right)
\]
\[
- (\gamma O_{1} + O_{3}) \left( [n+1-i]_q \right) a_{n-1-i} \right|^{-1}
\]
\[
\leq 4(m + 1) \left| \sum_{n=2}^{\infty} \left( [n]_q - 1 \right) \left( 1 + a([n]_q - 1) \right) a_n + \sum_{i=2}^{n-1} \left( [i]_q - 1 \right) + a([n+1-i]_q - [i]_q) \right|
\]
\[
\cdot \left( \lambda - \gamma \right)O_{1} - \sum_{n=2}^{\infty} \left( \lambda O_{1} + O_{3} \right) \left( 2[n]_q + a([n]_q - 1)^2 \right)
\]
\[
- (\gamma O_{1} + O_{3}) \left( [n]_q + 1 \right) a_n - \sum_{i=2}^{n-1} \left( \lambda O_{1} + O_{3} \right) \left( [i]_q + a([n+1-i]_q - [i]_q) \right) - (\gamma O_{1} + O_{3}) \left( [n+1-i]_q \right) a_{n-1-i} \right|^{-1}
\]
\[
(36)
\]
The last inequality is bounded by 1 if (29) is satisfied. This completes the proof.

As \( q \rightarrow 1^{-} \) in Theorem 1, we are led to Theorem 1 in [21].

\[
\phi_n(\alpha, m, y, \lambda) = \sum_{n=2}^{\infty} [2(m + 1)(n - 1)(1 + \alpha(n - 1)) + |(\lambda + 1)(2n + \alpha(n - 1)^2)|
- (y + 1)(n + 1)||a_n| + \sum_{n=2}^{\infty} \sum_{i=n-2}^{\infty} 2(m + 1)(i - 1 + \alpha(n + 1 - 2i)) - (y + 1)](n + 1 - i)|a_n|a_{n+1-i}|
\]  

Corollary 1. A function \( f(\zeta) \) of type (1) is in \( m - UST(\alpha, y, \lambda) \) if it satisfies the condition

\[
\phi_n(\alpha, m, y, \lambda) < |\lambda - y|,
\]  

where

\[ \phi_n(\alpha, m, y, \lambda) = \sum_{n=2}^{\infty} [2(m + 1)(n - 1)(1 + \alpha(n - 1)) + |(\lambda + 1)(2n + \alpha(n - 1)^2)|
- (y + 1)(n + 1)||a_n| + \sum_{n=2}^{\infty} \sum_{i=n-2}^{\infty} 2(m + 1)(i - 1 + \alpha(n + 1 - 2i)) - (y + 1)](n + 1 - i)|a_n|a_{n+1-i}|
\]

Corollary 2. If \( f(\zeta) \) having representation (1) satisfies the condition

\[
\sum_{n=2}^{\infty} \left[ 4q(m + 1)(n - 1)|a_n|\bigg((\lambda O_1 + O_3)[n]_q - (\gamma O_1 + O_3)\bigg)|a_n| < |\lambda - y|O_1,
\]

Then \( f \in m - UST_q(y, \lambda) \).

If we choose \( \alpha = 0 \) and allow \( q \rightarrow 1^{-} \) in the above theorem, our investigation comes down to Theorem 1 in [22].

Corollary 4 (see [11]). If \( f(\zeta) \in \mathcal{A} \) satisfies the condition

\[
\sum_{n=2}^{\infty} (m(n - 1) + n)|a_n| < 1,
\]

then \( f \in m - UST_{q-1}(0, 1, -1) = m - UST \).

Theorem 2. Let \( f \in \mathcal{A} \). Then, \( f \in m - UQ_2(\alpha, y, \lambda) \) if inequality (42) is satisfied,

\[
\sum_{n=2}^{\infty} \prod_{i=0}^{\infty} \frac{(4(m + 1) + (\gamma O_1 + O_3))|\zeta_1 O_1 (y - \lambda) - 4q|1|_q^q|}{4q[1 + 1]|_q^q}
+ \sum_{n=2}^{\infty} (4(m + 1) + (\lambda O_1 + O_3))(1 + aq[n - 1]|_q)|a_n| < |\lambda - y|O_1
\]

where \( \zeta_1 \) is given by (23).

Proof. Assuming (42) holds, then it suffices to establish that

\[
m \left( \frac{(\lambda - 1)O_1 f_q(\alpha, f; \zeta) - (y - 1)O_1}{(\lambda O_1 + O_3)f_q(\alpha, f; \zeta) - (y O_1 + O_3)} - 1 \right) - \text{Re} \left( \frac{(\lambda - 1)O_1 f_q(\alpha, f; \zeta) - (y - 1)O_1}{(\lambda O_1 + O_3)f_q(\alpha, f; \zeta) - (y O_1 + O_3)} - 1 \right)
\]

< 1.
Following the same process in the proof of Theorem 1, we have

\[
\begin{align*}
&\left| m \left( (\lambda - 1)O_1 \mathcal{J}_q (\alpha, f; \zeta) - (y - 1)O_1 \right) \right| - \Re \left( \left( (\lambda - 1)O_1 \mathcal{J}_q (\alpha, f; \zeta) - (y - 1)O_1 \right) \right) \\
&\leq 4(m + 1) \left| \frac{(1 - a)D_q f (\zeta) + aD_q (cD_q f (\zeta)) - D_q g (\zeta)}{(\lambda O_1 + O_3)(1 - a)D_q f (\zeta) + aD_q (cD_q f (\zeta))} - (\gamma O_1 + O_3) \right|
\end{align*}
\]

(44)

(\text{where } g \in m - \text{UCV}_q (\gamma, \lambda))

where we have used Lemma 3. Thus, the last inequality is bounded by 1 if (42) is satisfied. Hence, we complete the proof.

As \( q \rightarrow 1^- \) in Theorem 2, we obtain the similar result proved in [18].

**Corollary 5.** Let \( f \in \mathcal{A} \). Then, \( f \in m - \text{UQ} (\alpha, \gamma, \lambda) \) if

\[
\sum_{n=2}^{\infty} \prod_{i=0}^{n-2} \left( 2(m + 1) + (\gamma + 1) \right) |\partial_1 O_1 (y - \lambda) - 2\lambda | \\
\left( 2(i + 1) \right)
\]

\[
+ \sum_{n=2}^{\infty} (2(m + 1) + (\lambda + 1) + (1 + a(n - 1))n|a_n| < |\lambda - \gamma|,
\]

(45)

holds.

For \( \alpha = 0 \) in Theorem 2, we get the result established by Naeem et al. [20].

**Corollary 6.** Let \( f \in \mathcal{A} \). Then, \( f \in m - \text{UKV}_q (\gamma, \lambda) \) if inequality (46) is satisfied,

\[
\sum_{n=2}^{\infty} \prod_{i=0}^{n-2} \left( 2(m + 1) + (\gamma + 1) \right) |\partial_1 O_1 (y - \lambda) - 2\lambda | \\
\left( 2(i + 1) \right)
\]

\[
+ \sum_{n=2}^{\infty} (2(m + 1) + (\lambda + 1) + (1 + a(n - 1))n|a_n| < |\lambda - \gamma|
\]

(46)

For \( \alpha = 0 \) and \( \alpha = 1 \) as \( q \rightarrow 1^- \) in Theorem 2, we obtain the following results established in [16].

**Corollary 7.** Let \( f \in \mathcal{A} \). Then, \( f \in m - \text{UKV} (\gamma, \lambda) \) if

\[
\sum_{n=2}^{\infty} \prod_{i=0}^{n-2} \left( 2(m + 1) + (\gamma + 1) \right) |\partial_1 O_1 (y - \lambda) - 2\lambda | \\
\left( 2(i + 1) \right)
\]

\[
+ \sum_{n=2}^{\infty} (2(m + 1) + (\lambda + 1) + (1 + a(n - 1))n|a_n| < |\lambda - \gamma|
\]

is satisfied.

**Corollary 8.** Let \( f \in \mathcal{A} \). Then, \( f \in m - \text{UKV} (\gamma, \lambda) \) if
exists a Schwarz function

\[ \sum_{n=2}^{\infty} \prod_{i=0}^{n-1} \frac{(2(m+1)+(n+1))|C_i O_1(y - \lambda) - 2\lambda|}{2(i+1)} \]

+ \sum_{n=2}^{\infty} (2(m+1)+(\lambda+1))n^2|a_n|<|\lambda - \gamma|,

is satisfied.

4.2. Fekete Szegö Inequality

Theorem 3. Let \( f \in m - UM_q(\alpha, \gamma, \lambda) \). Then, for any real number \( \delta \), we have

\[ |a_3 - \delta a_2^2| \leq \frac{(\gamma - \lambda)q}{8q(1 + aq [2]^q)} \times \begin{cases} \Lambda_1 (\alpha, m, q, \gamma, \lambda), & \delta < \rho_1, \\ 2, & \rho_1 \leq \delta \leq \rho_2, \\ \Lambda_2 (\alpha, m, q, \gamma, \lambda), & \delta > \rho_2, \end{cases} \]

where

\[ \rho_1 = \frac{1}{[2]^q \left( 1 + aq [2]^q \right)} \left[ \frac{8q(1 + aq)^2}{(\gamma - \lambda)O_1 \cdot \frac{1}{2} \left( \frac{\lambda O_1 + O_3}{O_1} \cdot \frac{1}{2} \right)} \right] \left( 1 + aq \left[ 2^2 \right] \right) \]

\[ \rho_2 = \frac{1}{[2]^q \left( 1 + aq [2]^q \right)} \left[ \frac{8q(1 + aq)^2}{(\gamma - \lambda)O_1 \cdot \frac{1}{2} \left( \frac{\lambda O_1 + O_3}{O_1} \cdot \frac{1}{2} \right)} \right] \left( 1 + aq \left[ 2^2 \right] \right) \]

\[ \Lambda_1 (\alpha, m, q, \gamma, \lambda) = \frac{4O_2 - (\lambda O_1 + O_3) \cdot \frac{1}{2} \left( \frac{\lambda O_1 + O_3}{O_1} \cdot \frac{1}{2} \right)}{2q(1 + aq)^2} \]

\[ \Lambda_2 (\alpha, m, q, \gamma, \lambda) = \frac{(\lambda O_1 + O_3) \cdot \frac{1}{2} \left( \frac{\lambda O_1 + O_3}{O_1} \cdot \frac{1}{2} \right)}{2q(1 + aq)^2} \]

and \( \varrho_1 \) and \( \varrho_2 \) are given by (23) and (24). This result cannot be improved. Then, \( f (\zeta) \in m - UM_q(\alpha, \gamma, \lambda) \) implies there exists a Schwarz function \( w(\zeta) \) such that

\[ J_q (\alpha, f; \zeta) = \frac{(O_1 + O_3) p_m (w(\zeta)) - (y - 1)O_1}{(\lambda O_1 + O_3) p_m (w(\zeta)) - (\lambda - 1)O_1} \]

Using the representation for \( p_m (\zeta) \) in Lemma 3 and the relationship between \( w(\zeta) \) and \( p (\zeta) = 1 + c_1 \zeta + c_2 \zeta^2 + \cdots \in P \), we can write

\[ J_q (\alpha, f; \zeta) = \frac{4 + ((y O_1 + O_3) c_1 \varrho_1)/2) \zeta + ((y O_1 + O_3)/4) \left( 2c_2 - c_1^2 \right) \varrho_1 + c_1^2 \varrho_2 \zeta^2 + \cdots}{4 + ((\lambda O_1 + O_3) c_1 \varrho_1)/2) \zeta + ((\lambda O_1 + O_3)/4) \left( 2c_2 - c_1^2 \right) \varrho_1 + c_1^2 \varrho_2 \zeta^2 + \cdots} \]

\[ = 1 + \frac{(y - \lambda) \zeta}{8} + \left( \frac{y - \lambda}{16} \right) O_1 ((2c_2 - c_1^2) \varrho_1 + c_1^2 \varrho_2) \]

\[ \frac{(y - \lambda) (\lambda O_1 + O_3) c_1^2 O_1 \varrho_1^2}{64} \zeta^2 + \cdots. \]
However,

\[
J_q(\xi, f; q) = (1 - a) \frac{\xi D_q f (q) - f (q)}{f (q)} + a \frac{D_q (\xi D_q f (q))}{D_q f (q)}
\]

\[
= 1 + q(1 + aq)\xi q + q[ [2]_q (1 + aq[2]_q) a_3 - (1 + aq[2]_q + 1)] a_3^2] \xi^2 + \cdots .
\]  

On comparing the coefficients of $\xi$ and $\xi^2$ of (53) and (54), we obtain

\[
a_2 = \frac{(y - \lambda) \xi_1 O_l \xi_1}{8q(1 + aq)}
\]

\[
a_3 = \frac{1}{q[2]_q(1 + aq[2]_q)} \left[ \left( y - \lambda \right) \xi_1 \left( (2c_2 - c_1^2) \xi_2 + c_1^2 \xi_2 \right) - \frac{(y - \lambda)(\lambda O_l + O_2) c_2^2 O_l \xi_1^2}{64} \right] + \frac{(1 + aq[2]_q + 1) a_2^2}{q[2]_q(1 + aq[2]_q)}
\]

Now, for a real number $\delta$, we have

\[
|a_3 - \delta a_3^2| = \frac{1}{q[2]_q(1 + aq[2]_q)} \left[ \left( y - \lambda \right) \xi_1 \left( (2c_2 - c_1^2) \xi_2 + c_1^2 \xi_2 \right) - \frac{(y - \lambda)(\lambda O_l + O_2) c_2^2 O_l \xi_1^2}{64} \right]
\]

\[
+ (y - \lambda) O_l \xi_1 c_2^2 \left( y - \frac{B}{8} \right) O_l \left[ (1 + aq[2]_q + 1) - \delta [2]_q (1 + aq[2]_q) \right] \]

\[
= \frac{(y - \lambda) O_l \xi_1}{8q[2]_q(1 + aq[2]_q)} \left[ c_2 - \frac{1}{2 \xi_1} - \frac{(\lambda O_l + O_2) \xi_1}{8} \right]
\]

\[
\left[ (1 + aq[2]_q + 1) - \delta [2]_q (1 + aq[2]_q) \right] \left( y - \lambda \right) O_l \xi_1 \]

\[
= \frac{(y - \lambda) O_l \xi_1}{8q[2]_q(1 + aq[2]_q)} \left| c_2 - \beta c_2^2 \right|
\]

where

\[
\beta = \frac{c_1 - c_2}{2 \xi_1} + \frac{(\lambda O_l + O_2) c_1}{8} - \frac{1}{8q(1 + aq)} \left( [2]_q (1 + aq[2]_q + 1) - \delta [2]_q (1 + aq[2]_q) \right) \left( y - \lambda \right) O_l \xi_1 .
\]

Hence, the result follows from Lemma 2.

If $a = 0$ and $0 < m < 1$, then Theorem 3 reduces to Theorem 10 in [17].

**Corollary 9.** Let $0 < m < 1$ and $f \in m - UST_q (y, \lambda)$ be of form (1). Then, for any real number $\delta$, we have
\[|a_3 - \delta a_2^3| \leq \frac{(y - \lambda)^2}{4q(1 - m^3)} \times \begin{cases} 
\Lambda_1(m, q, y, \lambda), & \delta < \nu_1, \\
\Lambda_2(m, q, y, \lambda), & \delta \geq \nu_2,
\end{cases}\] where

\[
\nu_1 = \frac{1}{1 + q} - \frac{q\pi^2}{(y - \lambda)(1 + q)^2} \left( \frac{1}{6} - \frac{5}{6} \frac{(\lambda + 1 + q)}{\pi^2} \right),
\]

and

\[
\nu_2 = \frac{1}{1 + q} - \frac{q\pi^2}{(y - \lambda)(1 + q)^2} \left( \frac{1}{6} - \frac{5}{6} \frac{(\lambda + 1 + q - \delta)}{\pi^2} \right) + \frac{5}{6} \frac{(\lambda + 1 + q)}{\pi^2} + \frac{1}{(1 - \delta)(1 + q + \delta)(y - \lambda)}.\]

The result is sharp.

Proof. The proof is straightforward from Theorem 3 and Lemma 1.

If \(\alpha = 0\) in Theorem 3, the case \(m = 1\) is contained in the following corollary.

**Corollary 10.** Let \(m = 1\), and \(f(x)\) of form (1) belongs to \(m - \text{UST}(y, \lambda)\). Then, for a real number \(\delta\), we have the following sharp inequality:

\[|a_3 - \delta a_2^3| \leq \frac{(y - \lambda)}{q\pi^2} \times \begin{cases} 
\Lambda_1(q, y, \lambda), & \delta < \nu_1, \\
\Lambda_2(q, y, \lambda), & \delta > \nu_2,
\end{cases}\] where

\[
\nu_1 = \frac{1}{1 + q} - \frac{q\pi^2}{(y - \lambda)(1 + q)^2} \left( \frac{1}{6} + \frac{(\lambda + 1 + q)}{\pi^2} \right),
\]

and

\[
\nu_2 = \frac{1}{1 + q} - \frac{q\pi^2}{(y - \lambda)(1 + q)^2} \left( \frac{1}{6} + \frac{(\lambda + 1 + q - \delta)}{\pi^2} \right) + \frac{5}{6} \frac{(\lambda + 1 + q)}{\pi^2} + \frac{1}{(1 - \delta)(1 + q + \delta)(y - \lambda)}.\]
The result is sharp.
Theorem 3 becomes Theorem 1 in [28] when \( \alpha = m = 0 \).

**Theorem 4.** The range of every univalent functions \( f \in m - UM_q(\alpha, \gamma, \lambda) \) contains the disc:

\[
|\zeta| < \frac{2q(1 + aq)}{8q(1 + aq) + (\gamma - \lambda)O_1|\bar{f}_1|},
\]

(66)

**Proof.** From the proof of Theorem 3, we can see that

\[
|a_2| \leq \frac{(\gamma - \lambda)O_1|\bar{f}_1|}{4q(1 + aq)},
\]

(67)

Since the Koebe one-quarter theorem asserted that each omitted value \( w \) of the univalent function \( f(\zeta) \) of form (1) satisfies

\[
|w| > \frac{1 + |a_2|}{2 + |a_2|}
\]

\[
\geq \frac{4q(1 + aq)}{8q(1 + aq) + (\gamma - \lambda)O_1|\bar{f}_1|}.
\]

(68)

\[
(1 - \alpha)D_qf(\zeta) + aD_q(cD_qf(\zeta)) = \left(\frac{O_1(\gamma + O_2)P_m(w(\zeta)) - (\gamma - 1)O_1}{O_1(\lambda + O_2)P_m(w(\zeta)) - (\lambda - 1)O_1}\right)D_q\zeta,
\]

(72)

which in turn implies

\[
1 + [2]_q(1 + aq)a_2\zeta + \cdots = 1 + (y_1 + [2]_q b_2)\zeta \cdots.
\]

(73)

It is easy to see that

\[
b_2 = \frac{y_1}{q[2]_q},
\]

(74)

such that

\[
a_2 = \frac{y_1}{q(1 + aq)},
\]

(75)

**Corollary 11.** Let \( f(\zeta) \) of the series representation (1) be in \( ST_q(\gamma, \lambda) \). Then, we have the sharp inequality:

\[
|\zeta| < \frac{2q}{4q + (\gamma - \lambda)O_1},
\]

(69)

**Theorem 5.** The range of every univalent function \( f \in m - UQ_q(\alpha, \gamma, \lambda) \) contains the same disc given by (66).

**Proof.** Let \( w(\zeta) \) be a Schur function. We note first in Theorem 3 that

\[
\frac{(O_1(\gamma + O_2)P_m(w(\zeta)) - (\gamma - 1)O_1}{O_1(\lambda + O_2)P_m(w(\zeta)) - (\lambda - 1)O_1} = 1 + y_1(\zeta + y_2\zeta^2 + \cdots,
\]

(70)

where

\[
y_1 = \frac{(y - \lambda)c_1O_1}{8},
\]

\[
y_2 = \left(\frac{y - \lambda}{16}\right)O_1\left(2c_2 - c_1^2\right)\bar{c}_1 + c_1^2\bar{c}_2 - \frac{(y - \lambda)(\lambda O_1 + O_2)c_1^2\bar{c}_1}{64},
\]

(71)

Since \( f \in m - UQ_q(\alpha, \gamma, \lambda) \), then for some \( g(\zeta) = \zeta + b_2\zeta^2 + b_3\zeta^3 + \cdots \in m - UCV_q(\gamma, \lambda) \), we have

\[
\]

Therefore, comparing the coefficients of \( \zeta \) of (73) and applying (74), we obtain

\[
\]

such that

\[
\]
\[ |a_2| \leq \frac{(y - \lambda)\Omega_1|G_1|}{4q(1 + aq)} \]  

(76)

Now, proceeding the same way as in the proof of Theorem 4, we have the required result.

5. Conclusion

Using the concept of $q$-calculus, we have introduced some new subclasses of analytic functions in the unit disc related to Janowski class of functions. In addition, sufficient conditions, Fekete–Szegő inequality as well as covering results for functions belonging to these new classes were established. Consequently, many remarkable special cases of our findings which were studied in the previous work were obtained [19, 26].

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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