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Research Article

Hyers–Ulam Stability of Additive Functional Equation Using Direct and Fixed-Point Methods

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In this present work, we obtain the solution of the generalized additive functional equation and also establish Hyers–Ulam stability results by using alternative fixed point for a generalized additive functional equation $\chi(\sum_{g=1}^{l} v_g) = \sum_{1\leq g<h<i\leq l}\chi(v_g + v_h + v_i) - \sum_{1\leq g<h\leq l}\chi(v_g + v_h) - ((l^2 - 5l + 2)/2)\sum_{g=1}^{l}(\chi(v_g) - \chi(-v_g)/2)$. where l is a nonnegative integer with $\mathbb{N} - \{0, 1, 2, 3, 4\}$ in Banach spaces.

1. Introduction

The problem of Ulam–Hyers stability concerns determining circumstances under which, given an approximate solution of a functional equation, one may locate an exact key that is closer to it in some sense. The investigation of stability problem for functional equations is identified to a question of Ulam [1] about the stability of group homomorphisms and affirmatively answered for Banach space by Hyers [2, 3]. It was further generalized and interesting results were obtained by a number of authors [4–6].

Rassias investigated the Hyers–Ulam stability results for the various functional equations in [7–10] through different spaces. Czerwik [11, 12] examined the stability of the quadratic functional equation involving several variables in the normed spaces. Numerous mathematicians investigated the various stability results in [13–32].

In 2019, Park et al. [33] introduced additive s-functional inequality. Using the fixed-point method and direct method, he established the Hyers–Ulam stability for the above-mentioned one in complex Banach spaces. Also, he examined the Hyers–Ulam stability of homomorphism and derivations in complex Banach algebras. In 2018, Almahalebi [34] investigated the quadratic functional equation in Banach spaces. And, he established the hyperstability outcome of the same equation through the fixed-point approach.

Radu [35] investigated various results about the stability problem by using the fixed-point alternative. He applied the fixed-point method to examine the stability of Cauchy functional equation and Jensen's functional equations. After his work, numerous authors used the fixed-point method to investigate several functional equations [36–41]. The functional equation

$$f(x+y) = f(x) + f(y), \quad (1)$$

is called the Cauchy additive functional equation and it is the most famous functional equation. As $f(x) = cx$ is the solution of (1), every solution of the additive equation is called an additive function.

In this present work, we derive the solution of the generalized additive functional equation along with established Hyers–Ulam stability results by using direct and fixed-point methods for a generalized additive functional equation

$$\chi\left(\sum_{g=1}^{l} v_g\right) = \sum_{1\le g<h<i\le l} \chi\left(v_g + v_h + v_i\right) - \sum_{1\le g<h\le l} \chi\left(v_g + v_h\right) - \left(\frac{l^2-5l+2}{2}\right)\sum_{g=1}^{l} \frac{\chi\left(v_g\right)-\chi\left(-v_g\right)}{2}, \tag{2}$$

where $l \ge 5$ is a nonnegative integer in Banach spaces.

2. General Solution of the Functional Equation (2)

In this section, we derive the general solution of the generalized additive functional equation (2).

Here, we consider Φ and Ω be real vector spaces.

Theorem 1. *If a mapping $\chi\colon \Phi \longrightarrow \Omega$ satisfies the functional equation (2) for all $v_1, v_2, \ldots, v_n \in \Phi$, then the mapping $\chi\colon \Phi \longrightarrow \Omega$ satisfies the functional equation (1) for all $x, y \in \Phi$.*

Proof. Suppose a mapping $\chi\colon \Phi \longrightarrow \Omega$ satisfies the functional equation (2). Replacing $(v_1, v_2, v_3, \ldots, v_l)$ by $(0, 0, 0, \ldots, 0)$ in the functional equation (2), we have $\chi(0) = 0$. Replacing $(v_1, v_2, v_3, \ldots, v_l)$ by $(v, 0, 0, \ldots, 0)$ in equation (2), we get $\chi(-v) = -\chi(v)$ for all $v \in \Phi$. Therefore, the function χ is an odd function. Replacing $(v_1, v_2, v_3, \ldots, v_l)$ by $(v, v, 0, 0, \ldots, 0)$ in equation (2) and using the property of odd function, we have

$$\chi(2v) = 2\chi(v), \tag{3}$$

for all $v \in \Phi$. Replacing v by $2v$ in (3), we obtain

$$\chi\left(2^2 v\right) = 2^2\chi(v), \tag{4}$$

for all $v \in \Phi$. Again, replacing v by $2v$ in (5) and using (3), we have

$$\chi\left(2^3 v\right) = 2^3\chi(v), \tag{5}$$

for all $v \in \Phi$. We can generalize for any nonnegative integer n and we get

$$\chi\left(2^n v\right) = 2^n\chi(v), \tag{6}$$

for all $v \in \Phi$. Now, replacing $(v_1, v_2, v_3, \ldots, v_l)$ by $(x, y, 0, \ldots, 0)$ in (2), we obtain our desired result of equation (1).

Remark 1. Let Ω be a linear space and a function $\chi\colon \Phi \longrightarrow \Omega$ satisfies the functional equation (2). Then, the following claims hold:

(1) $\chi(r^k v) = r^k\chi(v)$ for all $v \in \mathbb{R}$, $r \in \mathbb{Q}$, k integers

(2) $\chi(v) = v\chi(1)$ for all $v \in \mathbb{R}$ if χ is continuous

In Sections 3 and 4, we take Φ be a normed space and Ω be a Banach space. For our convincing effortlessness, we describe a function $\Theta\colon \Phi \longrightarrow \Omega$ as

$$\begin{aligned}\Theta\left(v_1, v_2, v_3, \ldots, v_l\right) = {} & \chi\left(\sum_{g=1}^{l} v_g\right) - \sum_{1\le g<h<i\le l} \chi\left(v_g + v_h + v_i\right) \\ & - \sum_{1\le g<h\le l} \chi\left(v_g + v_h\right) - \left(\frac{l^2-5l+2}{2}\right)\sum_{g=1}^{l} \frac{\chi\left(v_g\right)-\chi\left(-v_g\right)}{2},\end{aligned} \tag{7}$$

for every $v_1, v_2, \ldots, v_l \in \Phi$.

3. Hyers–Ulam Stability of the Functional Equation (2): Direct Method

In this section, we investigated the Hyers–Ulam stability of the generalized additive functional equation (2) in Banach space by using the direct method.

Theorem 2. *Let $\zeta \in \{-1, 1\}$, $\xi\colon \Phi^l \longrightarrow [0, \infty)$ be a mapping such that*

$$\sum_{w=0}^{\infty} \frac{\xi\left(2^{w\zeta} v_1, 2^{w\zeta} v_2, \ldots, 2^{w\zeta} v_l\right)}{2^{w\zeta}}, \tag{8}$$

converges in $\mathbb{R}$ with

$$\lim_{w\longrightarrow\infty} \frac{\xi\left(2^{w\zeta} v_1, 2^{w\zeta} v_2, \ldots, 2^{w\zeta} v_l\right)}{2^{w\zeta}} = 0, \tag{9}$$

for all $v_1, v_2, \ldots, v_l \in \Phi$. If a mapping $\Theta\colon \Phi \longrightarrow \Omega$ satisfies the inequality

$$\left\|\Theta\left(v_1, v_2, \ldots, v_l\right)\right\| \le \xi\left(v_1, v_2, \ldots, v_l\right), \tag{10}$$

for all $v_1, v_2, \ldots, v_l \in \Phi$, then there exists a unique additive mapping $\Psi\colon \Phi \longrightarrow \Omega$ satisfying equation (2) and

$$\|\chi(v) - \Psi(v)\| \le \frac{1}{2(l-4)} \sum_{w=((1-\zeta)/2)}^{\infty} \frac{\xi\left(2^{w\zeta} v, 2^{w\zeta} v, 0, \ldots, 0\right)}{2^{w\zeta}}, \tag{11}$$

for all $v \in \Phi$.

Proof. Assume $\zeta = 1$. Replacing $(v_1, v_2, \ldots, v_l)$ by $(v, v, 0, \ldots, 0)$ in (10), we have

$$\|(l-4)\chi(2v)-2(l-4)\chi(v)\| \leq \xi(v,v,0,\ldots,0), \tag{12}$$

for all $v \in \Phi$. From equality (12), we get

$$\left\|\frac{\chi(2v)}{2}-\chi(v)\right\| \leq \frac{\xi(v,v,0,\ldots,0)}{2(l-4)}, \tag{13}$$

for all $v \in \Phi$. Exchanging v through $2v$ in (13), we obtain

$$\left\|\frac{\chi(2^2 v)}{2}-\chi(2v)\right\| \leq \frac{\xi(2v,2v,0,\ldots,0)}{2(l-4)}, \tag{14}$$

for all $v \in \Phi$. From (14), we achieve

$$\left\|\frac{\chi(2^2 v)}{2^2}-\frac{\chi(2v)}{2}\right\| \leq \frac{1}{2}\,\frac{\xi(2v,2v,0,\ldots,0)}{2(l-4)}, \tag{15}$$

for all $v \in \Phi$. Adding together (13) and (15), we get the following outcome:

$$\left\|\frac{\chi(2^2 v)}{2^2}-\chi(v)\right\| \leq \frac{1}{2(l-4)}\left[\xi(v,v,0,\ldots,0)+\frac{\xi(2v,2v,0,\ldots,0)}{2}\right], \tag{16}$$

for all $v \in \Phi$. It follows from (13), (15), and (16), and we can generalize that as follows:

$$\begin{aligned}
&\left\|\frac{\chi(2^l v)}{2^l}-\chi(v)\right\| \leq \frac{1}{2(l-4)}\sum_{w=0}^{l-1}\frac{\xi(2^w v,2^w v,0,\ldots,0)}{2^w}, \quad \forall v \in \Phi,\\
\Longrightarrow &\left\|\frac{\chi(2^l v)}{2^l}-\chi(v)\right\| \leq \frac{1}{2(l-4)}\sum_{w=0}^{\infty}\frac{\xi(2^w v,2^w v,0,\ldots,0)}{2^w},
\end{aligned} \tag{17}$$

for all $v \in \Phi$. In order to establish the convergence of the sequence $\{\chi(2^w v)/2^w\}$, switch v through $2^s v$ and also divide by 2^s in (17). We conclude that, for some $w, s > 0$,

$$\begin{aligned}
\left\|\frac{\chi(2^{w+s}v)}{2^{(w+s)}}-\frac{\chi(2^s v)}{2^s}\right\| &= \frac{1}{2^s}\left\|\frac{\chi(2^{w+s}v)}{2^w}-\chi(2^s v)\right\|\\
&\leq \frac{1}{2(l-4)}\sum_{w=0}^{l-1}\frac{\xi(2^{w+s}v,2^{w+s}v,0,\ldots,0)}{2^{(w+s)}}\\
&\leq \frac{1}{2(l-4)}\sum_{w=0}^{\infty}\frac{\xi(2^{w+s}v,2^{w+s}v,0,\ldots,0)}{2^{(w+s)}} \longrightarrow 0, \quad \text{as } s \longrightarrow \infty,
\end{aligned} \tag{18}$$

for all $v \in \Phi$. Therefore, the sequence $\{\chi(2^w v)/2^w\}$ is a Cauchy. As Ω is complete, there exists $\Psi\colon \Phi \longrightarrow \Omega$ so that $\Psi(v) = \lim_{w\longrightarrow\infty}(\chi(2^w v)/2^w)$ for all $v \in \Phi$. Taking the limit $w \longrightarrow \infty$ in (17), we obtain that result (11) holds for all $v \in \Phi$. To prove that the function Ψ satisfies equation (2), replacing $(v_1, v_2, \ldots, v_l)$ by $(2^w v_1, 2^w v_2, \ldots, 2^w v_l)$ and also dividing by 2^w in (10), we get

$$\frac{1}{2^w}\left\|\Theta(2^w v_1, 2^w v_2, \ldots, 2^w v_l)\right\| \leq \frac{1}{2^w}\xi(2^w v_1, 2^w v_2, \ldots, 2^w v_l), \tag{19}$$

for all $v_1, v_2, \ldots, v_l \in \Phi$. Taking the limit $w \longrightarrow \infty$ in the above inequality and using the definition of $\Psi(v)$, we have $\Psi(v_1, v_2, \ldots, v_l) = 0$ for all $v_1, v_2, \ldots, v_l \in \Phi$. Thus, the function Ψ satisfies equation (2). To prove that the function

Ψ is unique, let $\varphi\colon \Phi \longrightarrow \Omega$ be another additive mapping satisfying the functional equation (2) and (11). Hence,

$$\begin{aligned}\|\Psi(v)-\varphi(v)\| &\le \frac{1}{2^s}\|\Psi(2^s v)-\chi(2^s v)\| + \|\chi(2^s v)-\varphi(2^s v)\| \\ &\le \frac{1}{2(l-4)}\sum_{w=0}^{\infty}\frac{\xi(2^{w+s}v, 2^{w+s}v, 0, \ldots, 0)}{2^{(w+s)}}, \quad \forall v \in \Phi \\ &\longrightarrow 0 \text{ as } s \longrightarrow \infty.\end{aligned} \tag{20}$$

Hence, Ψ is unique. Now, replacing v through $(v/2)$ in (12), we have

$$\left\|(l-4)\chi(v) - 2(l-4)\chi\left(\frac{v}{2}\right)\right\| \le \xi\left(\frac{v}{2}, \frac{v}{2}, 0, \ldots, 0\right), \tag{21}$$

for all $v \in \Phi$. The rest of the proof is similar to that when $\zeta = 1$. So for $\zeta = -1$, we can prove the results by a similar manner. Hence, the proof is completed. □

Corollary 1. *Let ϕ and ϑ be positive real numbers. If there exists a mapping $\Theta\colon \Phi \longrightarrow \Omega$ satisfying the inequality*

$$\|\Theta(v_1, v_2, \ldots, v_l)\| \le \begin{cases} \phi, \\ \phi\left\{\sum_{i=1}^{l}\|v_i\|^{\vartheta}\right\}, \\ \phi\left\{\prod_{i=1}^{l}\|v_i\|^{\vartheta} + \sum_{i=1}^{l}\|v_i\|^{l\vartheta}\right\}, \end{cases} \tag{22}$$

for all $v_1, v_2, \ldots, v_l \in \Phi$, then there exists a unique additive mapping $\Psi\colon \Phi \longrightarrow \Omega$ such that

$$\|\chi(v) - \Psi(v)\| \le \begin{cases} \dfrac{\phi}{|l-4|}, \\ \dfrac{2\phi\|v\|^{\vartheta}}{(l-4)\left|2-2^{\vartheta}\right|}; & \vartheta \ne 1, \\ \dfrac{2\phi\|v\|^{l\vartheta}}{(l-4)\left|2-2^{l\vartheta}\right|}; & \vartheta \ne \dfrac{1}{l}, \end{cases} \tag{23}$$

for all $v \in \Phi$.

4. Hyers–Ulam Stability of the Functional Equation (2): Fixed-Point Method

In this section, we examined the Hyers–Ulam stability of the generalized additive functional equation (2) in Banach space by using the fixed-point method.

Theorem 3. *Let $\Psi\colon \Phi \longrightarrow \Omega$ be a mapping for which there exists a mapping $\xi\colon \Phi^l \longrightarrow [0, \infty)$ and*

$$\lim_{w\longrightarrow\infty}\frac{\xi\left(\eta_g^w v_1, \eta_g^w v_2, \ldots, \eta_g^w v_l\right)}{\eta_g^w} = 0, \tag{24}$$

where $\eta_g = \begin{cases} 2, & \text{if } g = 0, \\ (1/2), & \text{if } g = 1, \end{cases}$ and such that it satisfies the inequality

$$\|\Psi(v_1, v_2, \ldots, v_l)\| \le \xi(v_1, v_2, \ldots, v_l), \tag{25}$$

for all $v_1, v_2, \ldots, v_l \in \Phi$. If there exists a Lipschitz constant $L = L(g)$ such that

$$v \longrightarrow \lambda(v) = \frac{1}{(l-4)}\xi\left(\frac{v}{2}, \frac{v}{2}, 0, \ldots, 0\right), \tag{26}$$

has the property

$$\frac{\lambda(\eta_g v)}{\eta_g} = L\lambda(v), \tag{27}$$

for all $v \in \Phi$. Then, there exists a unique additive mapping $\Psi\colon \Phi \longrightarrow \Omega$ satisfying equation (2) and

$$\|\chi(v) - \Psi(v)\| \le \frac{L^{1-g}}{1-L}\lambda(v), \tag{28}$$

for all $v \in \Phi$.

Proof. Consider a set $\psi = \{m/m\colon \Phi \longrightarrow \Omega, m(0) = 0\}$ and initiate the generalized metric on ψ, $d(m, n) = \inf\{w \in (0, \infty)\colon \|m(v) - n(v)\| \le w\lambda(v), v \in \Psi\}$. It is easy to view that (ψ, d) is complete. Fix $\nu\colon \psi \longrightarrow \psi$ as $\nu m(v) = (1/\eta_g)m(\eta_g v)$ for all $v \in \Phi$. For $m, n \in \psi$ and $v \in \Phi$, we have

$$\begin{aligned} d(m,n) = w &\Longrightarrow \|m(v) - n(v)\| \le w\lambda(v) \Longrightarrow \left\|\frac{m(\eta_g v)}{\eta_g} - \frac{n(\eta_g v)}{\eta_g}\right\| \le \frac{1}{\eta_g}w\lambda(\eta_g v) \\ &\Longrightarrow \|\nu m(v) - \nu n(v)\| \le \frac{1}{\eta_g}w\lambda(\eta_g v) \Longrightarrow d(\nu m(v), \nu n(v)) \le wL. \end{aligned} \tag{29}$$

That is, $d(\nu m, \nu n) \le L\, d(m, n)$. Accordingly, ν is a strictly contractive mapping on ψ with Lipschitz constant L. From inequality (12), we have

$$\|(l-4)\chi(2v) - 2(l-4)\chi(v)\| \le \xi(v, v, 0, \ldots, 0), \tag{30}$$

for all $v \in \Phi$. It follows from (30) that

$$\left\|\frac{\chi(2\nu)}{2}-\chi(\nu)\right\| \leq \frac{\xi(\nu,\nu,0,\ldots,0)}{2(l-4)}, \tag{31}$$

for all $\nu \in \Phi$. Using inequality (27) when $g=0$, we get

$$\left\|\chi(\nu)-\frac{\chi(2\nu)}{2}\right\| \leq \frac{1}{2}\lambda(\nu) \Longrightarrow \|\chi(\nu)-\nu m(\nu)\| \leq L\lambda(\nu), \tag{32}$$

for all $\nu \in \Phi$. So we obtain

$$d(\nu\chi(\nu),\chi(\nu)) \leq L = L^{1-g} < \infty, \tag{33}$$

for all $\nu \in \Phi$. Replacing ν by $(\nu/2)$ in (31), we have

$$\left\|\chi(\nu)-2\chi\left(\frac{\nu}{2}\right)\right\| \leq \frac{\xi((\nu/2),(\nu/2),0,\ldots,0)}{(l-4)}, \tag{34}$$

for all $\nu \in \Phi$. Using inequality (27) when $g=1$, we have

$$\left\|2\chi\left(\frac{\nu}{2}\right)-\chi(\nu)\right\| \leq \lambda(\nu) \Longrightarrow \|\nu\chi(\nu)-\chi(\nu)\| \leq \lambda(\nu), \tag{35}$$

for all $\nu \in \Phi$. Therefore, we obtain the result that

$$d(\chi(\nu),\nu\chi(\nu)) \leq 1 = L^0 = L^{1-g}, \tag{36}$$

for all $\nu \in \Phi$. From inequalities (33) and (36), we conclude that

$$d(\chi(\nu),\nu\chi(\nu)) \leq L^{1-g} < \infty, \tag{37}$$

for all $\nu \in \Phi$. Next, using fixed-point alternative theorem [35], there exists a fixed point Ψ of ν in ψ such that

$$\Psi(\nu) = \lim_{w\longrightarrow\infty} \frac{\chi\left(\eta_g^w \nu\right)}{\eta_g^w}, \tag{38}$$

for all $\nu \in \Phi$. In order to establish $\Psi\colon \Phi \longrightarrow \Omega$ satisfying equation (2), we use an argument similar to that in the proof of Theorem 2. As Ψ is a unique fixed point of ν in the set $\Delta = \{((\chi \in \psi)/d(\chi,\Psi)) < \infty\}$, Ψ is a unique function such that

$$\begin{aligned} d(\chi,\Psi) &\leq \frac{1}{1-L} d(\chi,\nu\chi) \Longrightarrow d(\chi,\Psi) \leq \frac{L^{1-g}}{1-L} \Longrightarrow \|\chi(\nu)-\Psi(\nu)\| \\ &\leq \frac{L^{1-g}}{1-L}\lambda(\nu), \end{aligned} \tag{39}$$

for all $\nu \in \Phi$. This accomplished the proof. □

Corollary 2. *Let ϕ and ϑ be positive real numbers. If a mapping $\Theta\colon \Phi \longrightarrow \Omega$ satisfies inequality (22) for all $\nu_1,\nu_2,\ldots,\nu_l \in \Phi$, then there exists a unique additive mapping such that (23) for all $\nu \in \Phi$.*

Proof. Consider

$$\xi(\nu_1,\nu_2,\ldots,\nu_l) \leq \begin{cases} \phi, \\ \phi\left\{\sum_{i=1}^{l}\|\nu_i\|^{\vartheta}\right\}, \\ \phi\left\{\prod_{i=1}^{l}\|\nu_i\|^{\vartheta} + \sum_{i=1}^{l}\|\nu_i\|^{l\vartheta}\right\}, \end{cases} \tag{40}$$

for all $\nu_1,\nu_2,\ldots,\nu_l \in \Phi$. Now

$$\frac{\xi\left(\eta_g^w\nu_1,\eta_g^w\nu_2,\ldots,\eta_g^w\nu_l\right)}{\eta_g^w} = \begin{cases} \dfrac{\phi}{\eta_g^w}, \\ \dfrac{\phi}{\eta_g^w}\left\{\sum_{i=1}^{l}\|\eta_g\nu_i\|^{\vartheta}\right\}, \\ \dfrac{\phi}{\eta_g^w}\left\{\prod_{i=1}^{l}\|\eta_g\nu_i\|^{\vartheta} + \sum_{i=1}^{l}\|\eta_g\nu_i\|^{l\vartheta}\right\}, \end{cases} = \begin{cases} \longrightarrow 0, & \text{as } w \longrightarrow \infty, \\ \longrightarrow 0, & \text{as } w \longrightarrow \infty, \\ \longrightarrow 0, & \text{as } w \longrightarrow \infty. \end{cases} \tag{41}$$

That is, (24) holds. As we have

$$\lambda(\nu) = \frac{1}{(l-4)}\xi\left(\frac{\nu}{2},\frac{\nu}{2},0,\ldots,0\right). \tag{42}$$

Also,

$$\lambda(v) = \frac{1}{(l-4)}\xi\left(\frac{v}{2},\frac{v}{2},0,\ldots,0\right) = \begin{cases} \dfrac{\phi}{(l-4)}, \\[2ex] \dfrac{2\phi\|v\|^{\vartheta}}{(l-4)2^{\vartheta}} \\[2ex] \dfrac{2\phi\|v\|^{l\vartheta}}{(l-4)2^{l\vartheta}}. \end{cases} \tag{43}$$

And also,

$$\frac{1}{\eta_g}\lambda(\eta_g v) = \begin{cases} \dfrac{\phi}{(l-4)\eta_g}, \\[2ex] \dfrac{2}{\eta_g}\dfrac{\phi\|v\|^{\vartheta}\eta_g^{\vartheta}}{(l-4)2^{\vartheta}} \\[2ex] \dfrac{2}{\eta_g}\dfrac{\phi\|v\|^{l\vartheta}\eta_g^{l\vartheta}}{(l-4)2^{l\vartheta}}, \end{cases} = \begin{cases} \eta_g^{-1}\lambda(v), \\ \eta_g^{\vartheta-1}\lambda(v), \\ \eta_g^{l\vartheta-1}\lambda(v), \end{cases} \tag{44}$$

for all $v \in \Phi$. Inequity (2) holds for the following cases:

(i) $L = 2^{-1}$ if $g = 0$ and $L = 2^{1}$ if $g = 1$
(ii) $L = 2^{\vartheta-1}$ for $\vartheta < 1$ if $g = 0$ and $L = 2^{1-\vartheta}$ for $\vartheta > 1$ if $g = 1$
(iii) $L = 2^{l\vartheta-1}$ for $\vartheta < (1/l)$ if $g = 0$ and $L = 2^{1-l\vartheta}$ for $\vartheta > (1/l)$ if $g = 1$

Now, from (28), we verify the following cases:

Case 1: $L = 2^{-1}$ if $g = 0$:

$$\|\chi(v) - \Psi(v)\| \leq \frac{L^{1-g}}{1-L}\lambda(v) = \frac{2^{-1}\phi}{(l-4)\left(1-2^{-1}\right)} = \frac{\phi}{(l-4)}. \tag{45}$$

Case 2: $L = 2$ if $g = 1$:

$$\|\chi(v) - \Psi(v)\| \leq \frac{L^{1-g}}{1-L}\lambda(v) = \frac{\phi}{(l-4)(1-2)} = -\frac{\phi}{(l-4)}. \tag{46}$$

Case 3: $L = 2^{\vartheta-1}$ for $\vartheta < 1$ if $g = 0$:

$$\|\chi(v) - \Psi(v)\| \leq \frac{L^{1-g}}{1-L}\lambda(v) = \frac{2^{\vartheta-1}}{1-2^{\vartheta-1}}\frac{2\phi\|v\|^{\vartheta}}{(l-4)2^{\vartheta}} = \frac{2\phi\|v\|^{\vartheta}}{(l-4)\left(2-2^{\vartheta}\right)}. \tag{47}$$

Case 4: $L = 2^{1-\vartheta}$ for $\vartheta > 1$ if $g = 1$:

$$\|\chi(v) - \Psi(v)\| \leq \frac{L^{1-g}}{1-L}\lambda(v) = \frac{1}{1-2^{1-\vartheta}}\frac{2\phi\|v\|^{\vartheta}}{(l-4)2^{\vartheta}} = \frac{2\phi\|v\|^{\vartheta}}{(l-4)\left(2^{\vartheta}-2\right)}. \tag{48}$$

Case 5: $L = 2^{l\vartheta-1}$ for $\vartheta < (1/l)$ if $g = 0$:

$$\|\chi(v) - \Psi(v)\| \leq \frac{L^{1-g}}{1-L}\lambda(v) = \frac{2^{l\vartheta-1}}{1-2^{l\vartheta-1}}\frac{2\phi\|v\|^{l\vartheta}}{(l-4)2^{l\vartheta}} = \frac{2\phi\|v\|^{l\vartheta}}{(l-4)\left(2-2^{l\vartheta}\right)}. \tag{49}$$

Case 6: $L = 2^{1-l\vartheta}$ for $\vartheta > (1/l)$ if $g = 1$:

$$\|\chi(v) - \Psi(v)\| \leq \frac{L^{1-g}}{1-L}\lambda(v) = \frac{1}{1-2^{1-l\vartheta}}\frac{2\phi\|v\|^{l\vartheta}}{(l-4)2^{l\vartheta}} = \frac{2\phi\|v\|^{l\vartheta}}{(l-4)\left(2^{l\vartheta}-2\right)}. \tag{50}$$

Hence, the proof is accomplished. □

5. Counterexample

In this section, we examine the following counterexample changed by the well-known counterexample of Gajda [42] to (2).

Example 1. Let a mapping $\Theta: \Phi \longrightarrow \Omega$ be defined by

$$\Theta(\nu) = \sum_{l=0}^{\infty} \frac{\phi(2^l \nu)}{2^l}, \tag{51}$$

where

$$\phi(\nu) = \begin{cases} \mu\nu, & -1 < \nu < 1, \\ \mu, & \text{otherwise}, \end{cases} \tag{52}$$

where μ is a constant, and then, the mapping $\Theta: \Phi \longrightarrow \Omega$ satisfies the inequality

$$\left|\Theta(\nu_1, \nu_2, \ldots, \nu_l)\right| \leq \left(\frac{l^2 - 5l + 8}{2}\right) 8\mu \left(\sum_{i=1}^{n} |\nu_i|\right), \tag{53}$$

for all $\nu_1, \nu_2, \ldots, \nu_l \in \Phi$, but there exists an additive function $\Psi: \Phi \longrightarrow \Omega$ with a constant α such that

$$|\chi(\nu) - \Psi(\nu)| \leq \alpha|\nu|, \tag{54}$$

for all $\nu \in \Phi$.

Proof. It is easy to show that χ is bounded by 2μ on Φ. If $\sum_{i=1}^{l} |\nu_i| \geq (1/2)$ or 0, then the left side of (53) is less than $((l^2 - 5l + 8)/2)2\mu$, and thus, (53) is true. Assume that $0 < \sum_{i=1}^{l} |\nu_i| \geq (1/2)$. Then there exists an integer m such that

$$\frac{1}{2^{(m+2)}} \leq \sum_{i=1}^{l} |\nu_i| < \frac{1}{2^{(m+1)}}. \tag{55}$$

So that $2^m|\nu_1| < (1/2), 2^m|\nu_2| < (1/2), \ldots, 2^m|\nu_l| < (1/2)$ and $2^l\nu_1, 2^l\nu_2, \ldots, 2^l\nu_l \in (-1, 1)$ for all $l = 0, 1, 2, \ldots, m-1$. So, for $l = 0, 1, \ldots, m-1$

$$\begin{aligned} &\phi\left(\sum_{g=1}^{l} 2^l \nu_g\right) - \sum_{1 \leq g < h < i \leq l} \phi\left(2^l(\nu_g + \nu_h + \nu_i)\right) + \sum_{1 \leq g < h \leq l} \phi\left(2^l(\nu_g + \nu_h)\right) \\ &+ \left(\frac{l^2 - 5l + 2}{2}\right) \sum_{g=1}^{l} \frac{\phi\left(2^l(\nu_g)\right) + \phi\left(2^l(-\nu_g)\right)}{2} = 0. \end{aligned} \tag{56}$$

From the definition of χ, we have

$$\begin{aligned} \left|\Theta(\nu_1, \nu_2, \ldots, \nu_l)\right| &\leq \sum_{i=m}^{\infty} \frac{1}{2^i} \left|\phi\left(2^i\nu_1, 2^i\nu_2, \ldots, 2^i\nu_l\right)\right| \\ &\leq \sum_{i=m}^{\infty} \frac{1}{2^i} \left(\frac{l^2 - 5l + 8}{2}\right)\mu \\ &\leq \left(\frac{l^2 - 5l + 8}{2}\right) 2^{(1-m)}\mu. \end{aligned} \tag{57}$$

It follows from (55) that

$$\left|\Theta(\nu_1, \nu_2, \ldots, \nu_l)\right| \leq \left(\frac{l^2 - 5l + 8}{2}\right) 8\mu \left(\sum_{i=1}^{n} |\nu_i|\right), \tag{58}$$

for all $\nu_1, \nu_2, \ldots, \nu_l \in \Phi$. Thus, χ satisfies (53) for all $\nu_1, \nu_2, \ldots, \nu_l \in \Phi$.

Assume on the contrary that there exists a mapping $\Psi: \Phi \longrightarrow \Omega$ additive with a constant $\alpha > 0$ satisfying (54). As χ is bounded and continuous for all $\nu \in \Phi$, Ψ is bounded on any open interval containing the origin and continuous at the origin.

By Remark 1, Ψ defined by $\Psi(\nu) = a\nu$ for all $\nu \in \Phi$. Hence

$$|\chi(\nu)| \leq (\alpha + |a|)|\nu|, \tag{59}$$

for all $\nu \in \Phi$. However, we can select a nonnegative integer m and $m\mu > \alpha + |a|$. If $\nu \in (0, (1/2^{m-1}))$, then $2^l\nu \in (0, 1)$ for all $l = 0, 1, \ldots, m-1$, and for this ν, we get

$$\chi(\nu) = \sum_{l=0}^{\infty} \frac{\phi(2^l\nu)}{2^l} \geq \sum_{l=0}^{m-1} \frac{\mu(2^l\nu)}{2^l} = m\mu\nu > (\alpha + |a|)\nu, \tag{60}$$

which contradicts. □

6. Conclusion

In this work, we introduced the generalized finite-dimensional additive functional equation (2) and obtain its general solution in Section 2. In Section 3, we investigated the Hyers–Ulam stability results in Banach space by using the direct method, and in Section 4, we examined the Hyers–Ulam stability results in Banach space by using the fixed-point method. In Section 5, we proved the counterexample changed by the well-known counterexample of Gajda [42] to show the nonstability of the generalized additive functional equation (2).

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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