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Research Article

Iterative Methods of Weak and Strong Convergence Theorems for the Split Common Solution of the Feasibility Problems, Generalized Equilibrium Problems, and Fixed Point Problems

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The purpose of this paper is to introduce the extragradient methods for solving split feasibility problems, generalized equilibrium problems, and fixed point problems involved in nonexpansive mappings and pseudocontractive mappings. We establish the results of weak and strong convergence under appropriate conditions. As applications of our three main theorems, when the mappings and their domains take different types of cases, we can obtain nine iterative approximation theorems and corollas on fixed points, variational inequality solutions, and equilibrium points.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces, and let C and Q be two nonempty closed and convex subsets of H_1 and H_2, respectively. Let A: $H_1 \longrightarrow H_2$ be a bounded linear operator with its adjoint A^*. The split feasibility problem (SFP) is to find a point x such that

$$x \in C, \quad Ax \in Q. \tag{1}$$

We denote the solution set of the split feasibility problem (SFP) by

$$\Omega = \{x \in C : Ax \in Q\} = C \cap A^{-1}Q. \tag{2}$$

Problem (1) was first introduced by Censor and Elfving [1] in the finite-dimensional spaces and further has been studied by many researchers (see, for example, [2–6]) and the references therein. To solve the SFP, Byrne [2, 7] first introduced the so-called CQ algorithm as follows:

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = P_C\left(I - \lambda A^*\left(I - P_Q\right)A\right)x_n, \quad \forall n \geq 0, \end{cases} \tag{3}$$

where $0 < \lambda < 2/\rho(A^*A)$, P_C denotes the projection onto C, and $\rho(A^*A)$ is the spectral radius of the self-adjoint operator A^*A. Many authors continue to study the CQ algorithm in its various forms (see, for example, [8–14]). The CQ algorithm can be viewed from two different but equivalent ways: optimization and fixed point [6]. From the view of optimization point, $x^* \in \Omega$ in (2) if and only if x^* is a solution of the following minimization problem with zero optimal value $\min_{x\in C} f(x) := (1/2)\|Ax - P_Q Ax\|^2$, where f is a differentiable convex function and has a Lipschitz gradient given by $\nabla f(x) = A^*(I - P_Q)A$, with Lipschitz constant $L = \rho(A^*A)$. Thus, x^* solves the (SFP) if and only if x^* solves the variational inequality problem of finding $x^* \in C$ such that $\langle \nabla f(x^*), y - x^* \rangle \geq 0$ for all $y \in C$.

Xu [6] considered the following Tikhonov regularized problem:

$$\min_{x\in C} f_\alpha(x) := \frac{1}{2}\left\|Ax - P_Q Ax\right\|^2 + \frac{1}{2}\alpha\|x\|^2, \quad (4)$$

where $\alpha > 0$ is the regularization parameter. We observe that the gradient

$$\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^*\left(I - P_Q\right)A + \alpha I, \quad (5)$$

is $(\alpha + \|A\|^2)$-Lipschitz continuous and α-strongly monotone. The fixed point approach method to solve the SFP is based on the following observations. Let $\lambda > 0$, and assume that $x^* \in \Omega$. Then, $Ax^* \in Q$, which implies that $(I - P_Q)Ax^* = 0$, and thus, $\lambda A^*(I - P_Q)Ax^* = 0$. Hence, we have the fixed point equation $(I - \lambda A^*(I - P_Q)A)x^* = x^*$. Requiring that $x^* \in C$, we consider the fixed point equation

$$P_C(I - \lambda\nabla f)x^* = P_C\left(I - \lambda A^*\left(I - P_Q\right)A\right)x^* = x^*. \quad (6)$$

In [6], it is proved that the solutions of fixed point equation (6) are precisely the solutions of the *SFP*.

Let $A: C \longrightarrow H$ be a nonlinear mapping and F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The generalized equilibrium problem is to find $x^* \in C$ such that $F(x^*, y) + \langle Ax^*, y - x^*\rangle \geq 0$, $\forall y \in C$. The set of solutions is denoted by $\mathrm{GEP}(F, A)$. If $A = 0$, then $\mathrm{GEP}(F, A)$ is denoted by $\mathrm{EP}(F)$. If $F(x, y) = 0$ for all $x, y \in C$, then $\mathrm{GEP}(F, A)$ is denoted by $\mathrm{VI}(C, A) = \left\{x^* \in C\text{: } \langle Ax^*, y - x^*\rangle \geq 0, \forall y \in C\right\}$. This is the set of solutions of the variational inequality for A (see, for example, [15–21]). If $C = H$, then $\mathrm{VI}(H, A) = A^{-1}(0)$ where $A^{-1}(0) = \{x \in H\text{: } Ax = 0\}$.

In 2008, Takahashi and Takahashi [15] have suggested the following iterative method. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \langle Ax_n, y - y_n\rangle + \dfrac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = a_n x_n + (1 - a_n)T[\beta_n u + (1 - \beta_n)y_n], \quad \forall n \geq 1. \end{cases} \quad (7)$$

Under some appropriate conditions, they proved that the sequence $\{x_n\}$ converges strongly to a point $P_{F(T)\cap \mathrm{GEP}(F,A)}u$.

Motivated and inspired by the above works, we will investigate the weak and strong convergence methods for solving the split feasibility problems, generalized equilibrium problems, and fixed point problems involved in nonexpansive mappings and pseudocontractive mappings. As applications of our three main theorems, when the mappings and their domains take different types of cases, we can obtain nine iterative approximation theorems and corollaries on fixed points, variational inequality solutions, and equilibrium points. So, our results in this paper generalize and improve upon the corresponding modern results of many other authors.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and C be a nonempty, closed, and convex subset of H. Recall that a mapping A: $C \longrightarrow H$ is said to be monotone if $\langle Au - Av, u - v\rangle \geq 0$ for all $u, v \in C$ [18, 19]. A mapping A is said to be α-strongly monotone whenever there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|u - v\|^2$ for all $u, v \in C$. A mapping A is said to be α-inverse strongly monotone if there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$ for all $u, v \in C$. Recall that the classical variational inequality problem, which we denote by $\mathrm{VI}(C, A)$, is to find $x \in C$ such that $\langle Ax, y - x\rangle \geq 0$, for all $y \in C$ [16, 17]. It is well known that, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| = \inf_{y\in C}\|x - y\| =: d(x, C)$. It is well known that P_C is a nonexpansive and monotone mapping from H onto C and satisfy the following:

(1) $\langle x - P_C x, z - P_C x\rangle \leq 0$ for all $x \in H, z \in C$

(2) $\|x - z\|^2 \geq \|x - P_C x\|^2 + \|z - P_C x\|^2$ for all $x \in H, z \in C$

(3) The relation $\langle P_C x - P_C z, x - z\rangle \geq \|P_C x - P_C z\|^2$ holds for all $z, x \in H$

Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see from (2) that

$$p \in \mathrm{VI}(C, A) \Leftrightarrow p = P_C(p - \lambda Ap), \quad \forall\lambda > 0. \quad (8)$$

For solving the equilibrium problem, we assume that F satisfies the following conditions:

(i) $(A_1)F(x, x) = 0$ for all $x \in C$

(ii) $(A_2)F$ is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$

(iii) (A_3) for each $x, y, z \in C$, $\lim_{t\longrightarrow 0}F(tz + (1 - t)x, y) \leq F(x, y)$

(iv) (A_4) for each $x \in C$, the function $y \longrightarrow F(x, y)$ is convex and lower semicontinuous

If $F(x, y) = \langle Ax, y - x\rangle$ for every $x, y \in C$, we see that the equilibrium problem is reduced to the variational inequality problem.

Lemma 1 (see [22]). *Let C be a nonempty, closed, and convex subset of H, and let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1) - (A_4)$. For $r > 0$ and $x \in H$, consider the mapping T_r: $H \longrightarrow C$ defined by*

$$T_r(x) = \left\{z \in C: F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}. \quad (9)$$

Then, $T_r(x) \neq \varnothing$ for all $x \in H$, T_r is single-valued, $\mathrm{EP}(F)$ is closed and convex, $F(T_r) = \mathrm{EP}(F)$, and T_r is firmly nonexpansive, that is, $\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle$ for all $x, y \in H$.

Lemma 2 (see [23]). *Let C be a nonempty, closed, and convex subset of H, F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1 - A_4)$, ad A_F be a multivalue mapping from H into itself defined by $A_F x = \{z \in C\colon F(z, y) \leq \langle y - x, z\rangle, \forall y \in C\}$ whenever $x \in C$ and $A_F x = \varnothing$ otherwise. Then, A_F is a maximal monotone operator with the domain $T_r(x) = (I + rA_F)^{-1}x$, for all $x \in H$ and $r > 0$.*

Definition 1. Let $T\colon H \longrightarrow H$ be a nonlinear operator.

(1) T is said to be L-Lipschitz whenever there exists $L \geq 0$ such that $\|Tu - Tv\| \leq L\|u - v\|$, $\forall u, v \in H$. If $L = 1$, we call T is nonexpansive, and T is said to be a contraction if $L < 1$.

(2) T is said to be firmly nonexpansive if $2T - I$ is nonexpansive and I is the identity mapping, or equivalently, $\langle Tu - Tv, u - v\rangle \geq \|Tu - Tv\|^2$, $\forall u, v \in H$. Alternatively, T is firmly nonexpansive if and only if T can be expressed as $T = (1/2)(I + S)$, where $S\colon H \longrightarrow H$ is nonexpansive.

(3) T is said to be α-averaged nonexpansive mapping, if there exists a nonexpansive mapping S, such that $T = (1 - \alpha)I + \alpha S$, where $\alpha \in (0, 1)$. Thus, firmly nonexpansive mappings are $(1/2)$-averaged mapping.

(4) T is said to be pseudocontractive if and only if $\|Tu - Tv\|^2 \leq \|u - v\|^2 + \|(I - T)u - (I - T)v\|^2$, $\forall u, v \in H$.

(5) T is said to be k-strictly pseudocontractive if and only if there exists $0 \leq k < 1$, such that

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 + k\|(I - T)u - (I - T)v\|^2, \quad \forall u, v \in H. \tag{10}$$

Remark 1 (see [2]). Let $T\colon C \longrightarrow C$ be a given mapping:

(i) T is nonexpansive if and only if the complement $I - T$ is $(1/2)$-inverse strongly monotone.

(ii) If T is α-inverse strongly monotone, then for $\gamma > 0, \gamma T$ is (α/γ)-inverse strongly monotone.

(iii) T is averaged if and only if the complement $I - T$ is α-inverse strongly monotone for some $\alpha > 1/2$. Indeed, for $\alpha \in (0, 1), T$ is α-averaged if and only if $I - T$ is $(1/2\alpha)$-inverse strongly monotone.

We denote by $F(T)$ the set of fixed points of T. Note that every α-inverse strongly monotone mapping T is Lipschitz and $\|Tu - Tv\| \leq (1/\alpha)\|u - v\|$. Every nonexpansive mapping is a k-strictly pseudocontractive mapping and every k-strictly pseudocontractive mapping is pseudocontractive. Assume that $T\colon C \longrightarrow C$ is a strictly pseudocontractive. If $A = I - T$, we easily find that A is $(1 - k/2)$-inverse strongly monotone and $F(T) = \mathrm{VI}(C, A)$. Note that T is pseudocontractive if and only if $A = I - T$ is monotone, and $F(T) = A^{-1}(0) = \{x \in H\colon Ax = 0\}$. There are a lot works associated with the fixed point algorithms for nonexpansive mappings and pseudocontractive mappings (see, for example, [24–28]).

A set-valued mapping $T\colon H \longrightarrow 2^H$ is called monotone if for all $x, y \in H, f \in Tx$, and $h \in Ty$ imply $\langle x - y, f - h\rangle \geq 0$. A monotone mapping $T\colon H \longrightarrow 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mappings. Also, a monotone mapping $T\colon H \longrightarrow 2^H$ is maximal if and only if, for $(x, f) \in H \times H, \langle x - y, f - h\rangle \geq 0$ for every $(y, h) \in G(T)$ implies $f \in Tx$. Let $A\colon C \longrightarrow H$ be an inverse strongly monotone mapping and let $N_C u$ be the normal cone to C at $u \in C$, i.e., $N_C u = \{v \in H\colon \langle u - w, v\rangle \geq 0, \forall w \in C\}$. Define

$$Tu := \begin{cases} Au + N_C u, & u \in C, \\ \varnothing, & u \notin C. \end{cases} \tag{11}$$

It is known that T is maximal monotone and $0 \in Tu$ if and only if $u \in \mathrm{VI}(C, A)$ [29, 30].

Lemma 3 (see [8]). *Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A\colon H_1 \longrightarrow H_2$ be a bounded linear operator and $f\colon H_1 \longrightarrow \mathbb{R}$ be a continuous differentiable function. If $\alpha > 0$ and $\lambda \in (0, (1/\|A\|^2))$, then*

(1) $\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^*(I - P_Q)A + \alpha I$ is $(1/\alpha + \|A\|^2)$-inverse strongly monotone mapping

(2) $I - \lambda\nabla f_\alpha$ is $(\lambda(\alpha + \|A\|^2)/2)$-averaged

(3) $P_C(I - \lambda\nabla f_\alpha)$ is ζ-averaged, with $\zeta = (2 + \lambda(\alpha + \|A\|^2)/4)$

(4) $P_C(I - \lambda\nabla f_\alpha)$ is nonexpansive

Lemma 4 (see [31]). *Let H be a real Hilbert space, C be a closed convex subset of H, and $T\colon C \longrightarrow C$ be a continuous pseudocontractive mapping. Then,*

(i) $F(T)$ is a closed convex subset of C

(ii) $(I - T)$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \longrightarrow x$ and $Tx_n - x_n \longrightarrow 0$; as $n \longrightarrow \infty$, then $x = T(x)$.

Lemma 5 (see [32]). *Let H be a real Hilbert space. Then, for all $x_j \in H$ and $a_j \in [0, 1]$, for $j = 1, 2, 3$ such that $a_1 + a_2 + a_3 = 1$, the following equality holds:*

$$\left\|a_1x_1 + a_2x_2 + a_3x_3\right\|^2 = a_1\left\|x_1\right\|^2 + a_2\left\|x_2\right\|^2 + a_3\left\|x_3\right\|^2 - \sum_{1\leq i,j\leq 3} a_ia_j\left\|x_i - x_j\right\|^2. \tag{12}$$

Lemma 6 (see [33]). *Let C be a nonempty closed and convex subset of a real Hilbert space H and $T\colon C \longrightarrow C$ be a nonexpansive mapping. Then, $I - T$ is demiclosed at zero.*

Lemma 7 (see [34]). *Let $\{x_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative real numbers satisfying $x_{n+1} \leq x_n + \gamma_n$. If $\sum_{n=0}^{\infty} \gamma_n$ converges, then $\lim_{n\longrightarrow\infty} x_n$ exists.*

Lemma 8 (see [35]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T: C \longrightarrow C$ be a k-strictly pseudocontraction with a fixed point. Define $S: C \longrightarrow C$ by $Sx = ax + (1-a)Tx$ for each $x \in C$. Then, as $a \in [k, 1)$, S is nonexpansive such that $F(S) = F(T)$.*

Lemma 9 (see [36]). *Let $\{x_n\}$ be a sequence of nonnegative real numbers satisfying $x_{n+1} \leq (1-\beta_n)x_n + \beta_n\gamma_n + \alpha_n$, where $\{\beta_n\} \subset (0,1)$ and $\{\gamma_n\}$ is a sequence such that $\sum_{n=0}^{\infty} \beta_n = \infty$, $\limsup_{n\longrightarrow\infty} \gamma_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n\beta_n| < \infty$, and $\sum_{n=0}^{\infty} \alpha_n < \infty$ where $\alpha_n \geq 0$. Then, $\lim_{n\longrightarrow\infty} x_n = 0$.*

Lemma 10 (see [37]). *Let $\{x_n\}$, $\{\varepsilon_n\}$, and $\{\alpha_n\}$ be the sequences in $[0,\infty)$ such that*

$$x_{n+1} \leq x_n + \varepsilon_n (x_n - x_{n-1}) + \alpha_n, \quad \forall n \geq 0, \tag{13}$$

$\sum_{n=0}^{\infty} \alpha_n < \infty$, and there exists a real number ε with $0 \leq \varepsilon_n \leq \varepsilon < 1$ for all $n \geq 0$. Then, the following holds:

(i) $\sum_{n=0}^{\infty} [x_n - x_{n-1}]_+ < \infty$, where $[t]_+ = \max\{t, 0\}$

(ii) There exists $x^* \in [0,\infty)$ such that $\lim_{n\longrightarrow\infty} x_n = x^*$

Lemma 11 (see [31]). *Let H be a real Hilbert space. Then, for any given $x, y \in H$, the following inequality holds: $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$.*

3. Weak and Strong Convergence Results

Now, we are ready to state and prove some of our main results in this section.

Theorem 1. *Assume that C and Q are 2 nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A: H_1 \longrightarrow H_2$ be a bounded linear operator, $f: H_1 \longrightarrow \mathbb{R}$ be a continuous differentiable function, F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1)-(A_4)$, M be an α-inverse strongly monotone mapping from C into H_1, $S: C \longrightarrow C$ be a nonexpansive mapping, and $T: C \longrightarrow C$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap \Omega \cap GEP(F, M) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{v_n\}$ be sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ F(v_n, y) + \langle Mx_n, y - v_n\rangle + \dfrac{1}{r_n}\langle y - v_n, v_n - x_n\rangle \geq 0, \\ \quad \forall y \in C, \\ z_n = P_C\left(I - \lambda_n \nabla f_{\alpha_n}\right)v_n, \\ y_n = P_C\left(v_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \\ x_{n+1} = a_n x_0 + b_n x_n + c_n\left((1-\beta_n)Sy_n + \beta_n T_n z_n\right), \quad \forall n \geq 0, \end{cases} \tag{14}$$

where $T_n = (1-\gamma_n)I + \gamma_n T$ and $\gamma_n \in (k, 1)$. Suppose the following conditions are satisfied:

(a) $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\lim_{n\longrightarrow\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$
(b) $\{\beta_n\} \subset [\beta_1^*, \beta_2^*]$ for some $\beta_1^*, \beta_2^* \in (0,1)$
(c) $\{\lambda_n\} \subset [e, d]$ for some $e, d \in (0, (1/\|A\|^2))$
(d) $0 < a_n \leq a' < 1, 0 < b \leq b_n \leq b' < 1, 0 < c \leq c_n \leq c' < 1$ and $a_n + b_n + c_n = 1$,
(e) $0 < q_1 \leq r_n \leq q_2 < 2\alpha$

Then, $\{x_n\}$ converges strongly to the point $u = P_\Gamma(x_0)$ provided $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$.

Proof. For any fixed $u \in \Gamma$, we find that $u = P_C(I - \lambda\nabla f)u$ for $\lambda \in (0, (1/\|A\|^2))$ and $Su = u$. We see from Lemma 8 that T_n is nonexpansive and $F(T_n) = F(T)$. It is observed that v_n can be rewritten as $v_n = T_{r_n}(x_n - r_n Mx_n)$, $n \geq 0$. From condition (e) and Lemma 1, we have

$$\begin{aligned} \|v_n - u\|^2 &= \left\|T_{r_n}(x_n - r_n Mx_n) - u\right\|^2 \\ &= \left\|T_{r_n}(x_n - r_n Mx_n) - T_{r_n}(u - r_n Mu)\right\|^2 \\ &\leq \left\|(x_n - r_n Mx_n) - (u - r_n Mu)\right\|^2 \\ &= \left\|x_n - u\right\|^2 + r_n(r_n - 2\alpha)\left\|Mx_n - Mu\right\|^2 \\ &\leq \left\|x_n - u\right\|^2. \end{aligned} \tag{15}$$

From (14), (15), and Lemma 3, it follows that

$$\begin{aligned} \|z_n - u\| &= \left\|P_C\left(I - \lambda_n\nabla f_{\alpha_n}\right)v_n - P_C(I - \lambda_n\nabla f)u\right\| \\ &\leq \left\|P_C\left(I - \lambda_n\nabla f_{\alpha_n}\right)v_n - P_C\left(I - \lambda_n\nabla f_{\alpha_n}\right)u\right\| \\ &\quad + \left\|P_C\left(I - \lambda_n\nabla f_{\alpha_n}\right)u - P_C(I - \lambda_n\nabla f)u\right\| \\ &\leq \|v_n - u\| + \left\|\left(I - \lambda_n\nabla f_{\alpha_n}\right)u - (I - \lambda_n\nabla f)u\right\| \\ &\leq \|v_n - u\| + \lambda_n\alpha_n\|u\| \\ &\leq \|x_n - u\| + \lambda_n\alpha_n\|u\|. \end{aligned} \tag{16}$$

By the property of metric projection, we have

$$\begin{aligned}
\|y_n - u\|^2 &\le \left\|v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - u\right\|^2 - \left\|v_n - \lambda_n \nabla f_{\alpha_n} z_n - y_n\right\|^2 \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), u - y_n\rangle \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n) \\
&\quad - \nabla f_{\alpha_n}(u), u - z_n\rangle \\
&\quad + 2\lambda_n \big(\langle \nabla f_{\alpha_n}(u), u - z_n\rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle \big) \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(u), u - z_n\rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle \\
&= \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle (\alpha_n I + \nabla f) u, u - z_n\rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \alpha_n \langle u, u - z_n\rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle \\
&= \|v_n - u\|^2 - \|v_n - z_n\|^2 - 2\langle v_n - z_n, z_n - y_n\rangle \\
&\quad - \|z_n - y_n\|^2 \\
&\quad + 2\lambda_n \left[\alpha_n \langle u, u - z_n\rangle + \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle\right] \\
&= \|v_n - u\|^2 - \|v_n - z_n\|^2 \\
&\quad + 2\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n\rangle \\
&\quad + 2\lambda_n \alpha_n \langle u, u - z_n\rangle - \|z_n - y_n\|^2.
\end{aligned} \tag{17}$$

Furthermore, by the property of metric projection, we have

$$\begin{aligned}
&\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n\rangle \\
&\quad = \langle v_n - \lambda_n \nabla f_{\alpha_n}(v_n) - z_n, y_n - z_n\rangle \\
&\qquad + \langle \lambda_n \nabla f_{\alpha_n}(v_n) - \lambda_n \nabla f_{\alpha_n}(z_n), y_n - z_n\rangle \\
&\quad \le \langle \lambda_n \nabla f_{\alpha_n}(v_n) - \lambda_n \nabla f_{\alpha_n}(z_n), y_n - z_n\rangle \\
&\quad \le \lambda_n \left\|\nabla f_{\alpha_n}(v_n) - \nabla f_{\alpha_n}(z_n)\right\| \|y_n - z_n\| \\
&\quad \le \lambda_n \left(\alpha_n + \|A\|^2\right) \|v_n - z_n\| \|y_n - z_n\|.
\end{aligned} \tag{18}$$

Hence, we have

$$\begin{aligned}
\|y_n - u\|^2 &\le \|v_n - u\|^2 - \|v_n - z_n\|^2 \\
&\quad + 2\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n\rangle \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n \alpha_n \langle u, u - z_n\rangle \\
&\le \|v_n - u\|^2 - \|v_n - z_n\|^2 + 2\lambda_n \left(\alpha_n + \|A\|^2\right) \\
&\quad \cdot \|v_n - z_n\| \|y_n - z_n\| \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n \alpha_n \|u\| \|u - z_n\| \\
&\le \|v_n - u\|^2 - \|v_n - z_n\|^2 + \lambda_n^2 \left(\alpha_n + \|A\|^2\right)^2 \\
&\quad \cdot \|v_n - z_n\|^2 + \|y_n - z_n\|^2 \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n \alpha_n \|u\| \|u - z_n\| \\
&= \|v_n - u\|^2 + \left(\lambda_n^2 \left(\alpha_n + \|A\|^2\right)^2 - 1\right) \|v_n - z_n\|^2 \\
&\quad + 2\lambda_n \alpha_n \|u\| \|u - z_n\| \\
&\le \|v_n - u\|^2 + 2\lambda_n \alpha_n \|u\| \|u - z_n\| \\
&\le \|v_n - u\|^2 + 2\lambda_n \alpha_n \|u\| \left(\|v_n - u\| + \lambda_n \alpha_n \|u\|\right) \\
&\le \|v_n - u\|^2 + 4\lambda_n \alpha_n \|u\| \|v_n - u\| + 4\lambda_n^2 \alpha_n^2 \|u\|^2 \\
&= \left(\|v_n - u\| + 2\lambda_n \alpha_n \|u\|\right)^2.
\end{aligned} \tag{19}$$

So, from (15), we obtain

$$\|y_n - u\|^2 \le \left(\|x_n - u\| + 2\lambda_n \alpha_n \|u\|\right)^2. \tag{20}$$

We find from (14) and (16) and the last inequality that

$$\begin{aligned}
\|x_{n+1} - u\| &= \left\|a_n x_0 + b_n x_n + c_n \left((1 - \beta_n) S y_n + \beta_n T_n z_n\right) - u\right\| \\
&\le a_n \|x_0 - u\| + b_n \|x_n - u\| \\
&\quad + c_n \left[(1 - \beta_n) \|S y_n - u\| + \beta_n \|T_n z_n - u\|\right] \\
&\le a_n \|x_0 - u\| + b_n \|x_n - u\| \\
&\quad + c_n \left[(1 - \beta_n) \|y_n - u\| + \beta_n \|z_n - u\|\right] \\
&\le a_n \|x_0 - u\| + b_n \|x_n - u\| \\
&\quad + c_n (1 - \beta_n) \left(\|x_n - u\| + 2\lambda_n \alpha_n \|u\|\right) \\
&\quad + c_n \beta_n \left(\|x_n - u\| + \lambda_n \alpha_n \|u\|\right) \\
&\le a_n \|x_0 - u\| + (1 - a_n) \|x_n - u\| + 2\lambda_n \alpha_n \|u\| \\
&\le \max\left\{\|x_0 - u\|, \|x_n - u\|\right\} + 2\lambda_n \alpha_n \|u\| \\
&\quad \vdots \\
&\le \|x_0 - u\| + 2\, d \|u\| \sum_{i=0}^{\infty} \alpha_i.
\end{aligned} \tag{21}$$

Consequently, from condition (a), we deduce that $\{x_n\}$ is bounded and so there exist the sequences $\{z_n\}$, $\{v_n\}$, and $\{y_n\}$. Put $t_n = (1-\beta_n)Sy_n + \beta_n T_n z_n$ for all $n \geq 0$. We find from (15), (16), (19), and Lemma 5 that

$$\begin{aligned}
\|t_n - u\|^2 &= \|(1-\beta_n)Sy_n + \beta_n T_n z_n - u\|^2 \\
&\leq (1-\beta_n)\|Sy_n - u\|^2 + \beta_n\|T_n z_n - u\|^2 - \beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2 \\
&\leq (1-\beta_n)\|y_n - u\|^2 + \beta_n\|z_n - u\|^2 - \beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2 \\
&\leq (1-\beta_n)\left[\|x_n - u\|^2 + \left(\lambda_n^2(\alpha_n + \|A\|^2)^2 - 1\right)\|v_n - z_n\|^2 + r_n(r_n - 2\alpha)\|Mx_n - Mu\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n - u\|\right] \\
&\quad + \beta_n\left(\|x_n - u\| + \lambda_n\alpha_n\|u\|\right)^2 - \beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2 \\
&\leq \|x_n - u\|^2 + \left(\lambda_n^2(\alpha_n + \|A\|^2)^2 - 1\right)\|v_n - z_n\|^2 \\
&\quad + r_n(r_n - 2\alpha)\|Mx_n - Mu\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n - u\| \\
&\quad + 2\lambda_n\alpha_n\|u\|\|x_n - u\| + \lambda_n^2\alpha_n^2\|u\|^2 - \beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2.
\end{aligned} \tag{22}$$

From (14) and the last inequality, we conclude that

$$\begin{aligned}
\|x_{n+1} - u\|^2 &= \|a_n x_0 + b_n x_n + c_n t_n - u\|^2 \\
&\leq a_n\|x_0 - u\|^2 + b_n\|x_n - u\|^2 + c_n\|t_n - u\|^2 - b_n c_n\|x_n - t_n\|^2 \\
&\leq a_n\|x_0 - u\|^2 + b_n\|x_n - u\|^2 - b_n c_n\|x_n - t_n\|^2 \\
&\quad + c_n\Big[\|x_n - u\|^2 + \left(\lambda_n^2(\alpha_n + \|A\|^2)^2 - 1\right)\|v_n - z_n\|^2 + r_n(r_n - 2\alpha)\|Mx_n - Mu\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n - u\| \\
&\quad + 2\lambda_n\alpha_n\|u\|\|x_n - u\| + \lambda_n^2\alpha_n^2\|u\|^2 - \beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2\Big] \\
&\leq a_n\|x_0 - u\|^2 + (1-a_n)\|x_n - u\|^2 - b_n c_n\|x_n - t_n\|^2 \\
&\quad + c_n\left(\lambda_n^2(\alpha_n + \|A\|^2)^2 - 1\right)\|v_n - z_n\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n - u\| \\
&\quad + 2\lambda_n\alpha_n\|u\|\|x_n - u\| + r_n(r_n - 2\alpha)\|Mx_n - Mu\|^2 \\
&\quad + \lambda_n^2\alpha_n^2\|u\|^2 - c_n\beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2 \\
&\leq a_n\|x_0 - u\|^2 + \|x_n - u\|^2 - b_n c_n\|x_n - t_n\|^2 \\
&\quad + c_n\left(\lambda_n^2(\alpha_n + \|A\|^2)^2 - 1\right)\|v_n - z_n\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n - u\| \\
&\quad + 2\lambda_n\alpha_n\|u\|\|x_n - u\| + r_n(r_n - 2\alpha)\|Mx_n - Mu\|^2 \\
&\quad + \lambda_n^2\alpha_n^2\|u\|^2 - c_n\beta_n(1-\beta_n)\|Sy_n - T_n z_n\|^2.
\end{aligned} \tag{23}$$

This yields that

$$\begin{aligned}
&c\left(1-d^2\left(\alpha_n+\|A\|^2\right)^2\right)\|v_n-z_n\|^2+cb\|x_n-t_n\|^2\\
&\quad+r_n(2\alpha-r_n)\|Mx_n-Mu\|^2+\beta_1(1-\beta_2)c\|T_nz_n-Sy_n\|^2\\
&\le c_n\left(1-\lambda_n^2\left(\alpha_n+\|A\|^2\right)^2\right)\|v_n-z_n\|^2+c_nb_n\|x_n-t_n\|^2\\
&\quad+r_n(2\alpha-r_n)\|Mx_n-Mu\|^2+c_n\beta_n(1-\beta_n)\|T_nz_n-Sy_n\|^2\\
&\le a_n\|x_0-u\|^2+\|x_n-u\|^2-\|x_{n+1}-u\|^2\\
&+2\lambda_n\alpha_n\|u\|\left(\|z_n-u\|+\|x_n-u\|+\lambda_n\alpha_n\|u\|\right).
\end{aligned} \tag{24}$$

Since $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=0$, we have

$$\begin{aligned}
&\|x_n-u\|^2-\|x_{n+1}-u\|^2\\
&\quad\le\left(\|x_n-u\|-\|x_{n+1}-u\|\right)\\
&\quad\cdot\left(\|x_n-u\|+\|x_{n+1}-u\|\right)\\
&\quad\le\|x_{n+1}-x_n\|\left(\|x_n-u\|+\|x_{n+1}-u\|\right)\longrightarrow 0,\\
&\qquad\qquad\text{as } n\longrightarrow\infty.
\end{aligned} \tag{25}$$

From (97) and the condition (a)–(d), we also obtain

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\|T_nz_n-Sy_n\|&=\lim_{n\longrightarrow\infty}\|Mx_n-Mu\|=\lim_{n\longrightarrow\infty}\|x_n-t_n\|\\
&=\lim_{n\longrightarrow\infty}\|v_n-z_n\|=0.
\end{aligned} \tag{26}$$

It is observe that

$$\begin{aligned}
\|y_n-z_n\|&=\left\|P_C\left(v_n-\lambda_n\nabla f_{\alpha_n}(z_n)\right)-P_C\left(v_n-\lambda_n\nabla f_{\alpha_n}(v_n)\right)\right\|\\
&\le\left\|v_n-\lambda_n\nabla f_{\alpha_n}(z_n)-\left(v_n-\lambda_n\nabla f_{\alpha_n}(v_n)\right)\right\|\\
&=\lambda_n\left\|\nabla f_{\alpha_n}(z_n)-\nabla f_{\alpha_n}(v_n)\right\|\\
&\le\lambda_n\left(\alpha_n+\|A\|^2\right)\|z_n-v_n\|\longrightarrow 0,\quad\text{as } n\longrightarrow\infty.
\end{aligned} \tag{27}$$

Using Lemma 1 and (14), we have

$$\begin{aligned}
\|v_n-u\|^2&=\left\|T_{r_n}(x_n-r_nMx_n)-T_{r_n}(u-r_nMu)\right\|^2\\
&\le\langle(x_n-r_nMx_n)-(u-r_nMu),v_n-u\rangle\\
&=\frac{1}{2}\|(x_n-r_nMx_n)-(u-r_nMu)\|^2+\frac{1}{2}\|v_n-u\|^2\\
&\quad-\frac{1}{2}\|(x_n-r_nMx_n)-(u-r_nMu)-(v_n-u)\|^2\\
&\le\frac{1}{2}\left[\|x_n-u\|^2+\|v_n-u\|^2-\|(x_n-v_n)-2r_n(Mx_n-Mu)\|^2\right]\\
&=\frac{1}{2}\left[\|x_n-u\|^2+\|v_n-u\|^2-\|x_n-v_n\|^2+2r_n\langle x_n-v_n,Mx_n-Mu\rangle\right.\\
&\quad\left.-r_n^2\|Mx_n-Mu\|^2\right].
\end{aligned} \tag{28}$$

It follows that

$$\|v_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - v_n\|^2 + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle. \tag{29}$$

From (19) and (29), we find that

$$\begin{aligned}\|t_n - u\|^2 &= \|(1-\beta_n)Sy_n + \beta_n T_n z_n - u\|^2 \\ &\le (1-\beta_n)\|Sy_n - u\|^2 + \beta_n\|T_n z_n - u\|^2 \\ &\le (1-\beta_n)\|y_n - u\|^2 + \beta_n\|z_n - u\|^2 \\ &\le (1-\beta_n)\left[\|x_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\|\right] \\ &\quad + \beta_n\left(\|x_n - u\|^2 - \|x_n - v_n\|^2 + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle\right) \\ &\le \|x_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| \\ &\quad + \beta_n\left(-\|x_n - v_n\|^2 + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle\right).\end{aligned} \tag{30}$$

From (14) and the last inequality, we conclude that

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|a_n x_0 + b_n x_n + c_n t_n - u\|^2 \\ &\le a_n\|x_0 - u\|^2 + b_n\|x_n - u\|^2 + c_n\|t_n - u\|^2 \\ &\le a_n\|x_0 - u\|^2 + b_n\|x_n - u\|^2 \\ &\quad + c_n\left[\|x_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| + \beta_n\left(-\|x_n - v_n\|^2 + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle\right)\right] \\ &\le a_n\|x_0 - u\|^2 + (1-a_n)\|x_n - u\|^2 - c_n\beta_n\|x_n - v_n\|^2 \\ &\quad + c_n\left[2\lambda_n\alpha_n\|u\|\|u - z_n\| + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle\right).\end{aligned} \tag{31}$$

This yields that

$$\begin{aligned}c_n\beta_n\|x_n - v_n\|^2 &\le a_n\|x_0 - u\|^2 + (1-a_n)\|x_n - u\|^2 - \|x_{n+1} - u\|^2 \\ &\quad + c_n\left[2\lambda_n\alpha_n\|u\|\|u - z_n\| + 2r_n\langle x_n - v_n, Mx_n - Mu\rangle\right].\end{aligned} \tag{32}$$

It follows from condition (a) and $\lim_{n\longrightarrow\infty}\|x_n - x_{n+1}\| = \lim_{n\longrightarrow\infty}\|Mx_n - Mu\| = 0$ that

$$\lim_{n\longrightarrow\infty}\|x_n - v_n\| = 0. \tag{33}$$

Since $\|x_n - z_n\| \le \|x_n - v_n\| + \|v_n - z_n\|$, $\|T_n z_n - z_n\| \le \|T_n z_n - x_n\| + \|x_n - z_n\|$, $\|T_n z_n - x_n\| \le \|T_n z_n - t_n\| + \|t_n - x_n\|$, $\|T_n z_n - t_n\| = (1-\beta_n)\|T_n z_n - Sy_n\|$, we obtain $\|T_n z_n - t_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Note that $1-\beta_n > 0$. This implies that

$$\lim_{n\longrightarrow\infty}\|Tz_n - z_n\| = 0. \tag{34}$$

Also, from $\|y_n - x_n\| \le \|y_n - z_n\| + \|z_n - x_n\|$, $\|Sy_n - x_n\| \le \|Sy_n - t_n\| + \|t_n - x_n\|$, $\|Sy_n - t_n\| = \beta_n\|Sy_n - T_n z_n\|$, and $\|Sy_n - y_n\| \le \|Sy_n - x_n\| + \|x_n - y_n\|$, we get

$$\lim_{n\longrightarrow\infty}\|Sy_n - y_n\| = 0. \tag{35}$$

Since $\nabla f = A^*(I - P_Q)A$ is Lipschitz continuous, we obtain $\lim_{n\longrightarrow\infty}\|\nabla f(z_n) - \nabla f(y_n)\| = 0$.

Next, we show that

$$\limsup_{n\longrightarrow\infty}\langle x_0 - u, x_n - u\rangle \le 0, \tag{36}$$

where $u = P_\Gamma(x_0)$. To show it, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle x_0 - u, x_n - u\rangle = \lim_{k\longrightarrow\infty}\langle x_0 - u, x_{n_k} - u\rangle. \tag{37}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, converges weakly to x^*. Without loss of generality, we assume that $x_{n_k} \longrightarrow x^*$. Since $\|x_n - v_n\| \longrightarrow 0$, $\|x_n - y_n\| \longrightarrow 0$, $\|x_n - z_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain that $y_{n_k} \rightharpoonup x^*$, $v_{n_k} \rightharpoonup x^*$, $z_{n_k} \rightharpoonup x^*$. Since $\{y_{n_k}\} \subset C$ and C is closed and convex, we obtain $x^* \in C$. First, we show that $x^* \in F(T) \cap F(S)$. Then, from (34), (35), Lemma 6, and

Lemma 4, we have that $x^* \in F(T) \cap F(S)$. We now show that $x^* \in \text{GEP}(F, M)$. By $v_n = T_{r_n}(x_n - r_n M x_n)$, we know that

$$F(v_n, y) + \langle M x_n, y - v_n \rangle + \frac{1}{r_n} \langle y - v_n, v_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{38}$$

It follows from (A_2) that

$$\langle M x_n, y - v_n \rangle + \frac{1}{r_n} (y - v_n, v_n - x_n) \geq F(y, v_n), \quad \forall y \in C. \tag{39}$$

Hence,

$$\langle M x_{n_j}, y - v_{n_j} \rangle + \langle y - v_{n_j}, \frac{v_{n_j} - x_{n_j}}{r_{n_j}} \rangle \geq F\left(y, v_{n_j}\right), \quad \forall y \in C. \tag{40}$$

For t with $0 < t \leq 1$ and $y \in C$, let $v_t = ty + (1-t)x^*$. Since $y \in C$ and $x^* \in C$, we obtain $v_t \in C$. So, from (74), we have

$$\begin{aligned}\langle v_t - v_{n_j}, M v_t \rangle &\geq \left(v_t - v_{n_j}, M v_t\right) - \langle v_t - v_{n_j}, M x_{n_j} \rangle \\ &\quad - \left(v_t - v_{n_j}, \frac{v_{n_j} - x_{n_j}}{r_{n_j}}\right) + F\left(v_t, v_{n_j}\right) \\ &= \left(v_t - v_{n_j}, M v_t - M v_{n_j}\right) \\ &\quad + \left(v_t - v_{n_j}, M v_{n_j} - M x_{n_j}\right) \\ &\quad - \left(v_t - v_{n_j}, \frac{v_{n_j} - x_{n_j}}{r_{n_j}}\right) + F\left(v_t, v_{n_j}\right).\end{aligned} \tag{41}$$

Since $\|v_{n_j} - x_{n_j}\| \longrightarrow 0$, we have $\|M v_{n_j} - M x_{n_j}\| \longrightarrow 0$. Furthermore, from the inverse strongly monotonicity of M, we have $\langle v_t - v_{n_j}, M v_t - M v_{n_j} \rangle \geq 0$. It follows from condition (A_4) and $(v_{n_j} - x_{n_j}/r_{n_j}) \longrightarrow 0$ and $v_{n_j} \rightharpoonup x^*$, we have

$$\langle v_t - x^*, M v_t \rangle \geq F(v_t, x^*), \tag{42}$$

as $j \longrightarrow \infty$. From (A_1) and (A_4), we have

$$\begin{aligned}0 &= F(v_t, v_t) \\ &\leq t F(v_t, y) + (1-t) F(v_t, x^*) \\ &\leq t F(v_t, y) + (1-t)(v_t - x^*, M v_t) \\ &= t F(v_t, y) + (1-t) t (y - x^*, M v_t),\end{aligned} \tag{43}$$

and hence,

$$0 \leq F(v_t, y) + (1-t) \langle y - x^*, M v_t \rangle. \tag{44}$$

Letting $t \longrightarrow 0$, we have, for each $y \in C$,

$$F(x^*, y) + \langle y - x^*, M x^* \rangle \geq 0. \tag{45}$$

This implies that $x^* \in \text{GEP}(F, M)$. Next, we show that $x^* \in \Omega$ (1). Let

$$T' p := \begin{cases} \nabla f(p) + N_C p, & p \in C, \\ \varnothing, & p \notin C. \end{cases} \tag{46}$$

Then, T' is maximal monotone and $0 \in T' p$ if and only if $p \in \text{VI}(C, \nabla f)$ [29]. Let $G(T')$ be the graph of T', let $(p, v) \in G(T')$. Then, we have $v \in T'(p) = \nabla f(p) + N_C p$ and hence $v - \nabla f(p) \in N_C p$. Therefore, we have $\langle p - w, v - \nabla f(p) \rangle \geq 0$ for all $w \in C$. By the property of metric projection, from $y_n = P_C(v_n - \lambda_n \nabla f_{\alpha_n} z_n)$ and $p \in C$, we have $\langle p - y_n, y_n - (v_n - \lambda_n \nabla f_{\alpha_n} z_n) \rangle \geq 0$, and hence,

$$\langle p - y_n, \frac{y_n - v_n}{\lambda_n} + \nabla f_{\alpha_n} z_n \rangle \geq 0. \tag{47}$$

From $\langle p - w, v - \nabla f(p) \rangle \geq 0$ for all $w \in C$ and $y_{n_k} \in C$, we have

$$\begin{aligned}\langle p - y_{n_k}, v \rangle &\geq \langle p - y_{n_k}, \nabla f(p) \rangle \\ &\geq \langle p - y_{n_k}, \nabla f(p) \rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} + \nabla f_{\alpha_n} z_{n_k} \rangle \\ &\geq \langle p - y_{n_k}, \nabla f(p) \rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} + \nabla f\left(z_{n_k}\right) \rangle - \alpha_{n_k} \langle p - y_{n_k}, z_{n_k} \rangle \\ &= \langle p - y_{n_k}, \nabla f(p) - \nabla f\left(y_{n_k}\right) \rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} \rangle - \alpha_{n_k} \langle p - y_{n_k}, z_{n_k} \rangle \\ &\quad + \langle p - y_{n_k}, \nabla f\left(y_{n_k}\right) - \nabla f\left(z_{n_k}\right) \rangle \\ &\geq \langle p - y_{n_k}, \nabla f\left(y_{n_k}\right) - \nabla f\left(z_{n_k}\right) \rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} \rangle - \alpha_{n_k} \langle p - y_{n_k}, z_{n_k} \rangle.\end{aligned} \tag{48}$$

Thus, we obtain $\langle p - x^*, v\rangle \geq 0$ as $k \longrightarrow \infty$. Since T' is maximal monotone, we have $x^* \in T'^{-1}0$, and hence, $x^* \in VI(C, \nabla f)$. This implies $x^* \in \Omega$. This implies that $x^* \in \Gamma$. Thanks to (37), we arrive at

$$\limsup_{n\longrightarrow\infty} x_0 - u, x_n - u\rangle = \lim_{k\longrightarrow\infty} \langle x_0 - u, x_{n_k} - u\rangle = \langle x_0 - u, x^* - u\rangle \leq 0. \tag{49}$$

Next, we show that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. Observe that

$$\begin{aligned}\|t_n - u\| &= \|(1-\beta_n)(Sy_n - u) + \beta_n(T_n z_n - u)\| \\ &\leq (1-\beta_n)\|Sy_n - u\| + \beta_n\|T_n z_n - u\| \\ &\leq (1-\beta_n)\|y_n - u\| + \beta_n\|z_n - u\| \\ &\leq (1-\beta_n)\big(\|x_n - u\| + 2\lambda_n\alpha_n\|u\|\big) \\ &\quad + \beta_n\big(\|x_n - u\| + \lambda_n\alpha_n\|u\|\big) \\ &\leq \|x_n - u\| + 2\lambda_n\alpha_n\|u\|.\end{aligned} \tag{50}$$

With the help of (14), we obtain

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \langle a_n x_0 + b_n x_n + c_n t_n - u, x_{n+1} - u\rangle \\ &\leq a_n\langle x_0 - u, x_{n+1} - u\rangle + \langle b_n(x_n - u) + c_n(t_n - u), x_{n+1} - u\rangle \\ &\leq a_n\langle x_0 - u, x_{n+1} - u\rangle + \big(b_n\|x_n - u\| + c_n\|t_n - u\|\big)\|x_{n+1} - u\| \\ &\leq a_n\langle x_0 - u, x_{n+1} - u\rangle + \big(b_n\|x_n - u\| + c_n\|x_n - u\| + 2\lambda_n\alpha_n\|u\|\big)\|x_{n+1} - u\| \\ &\leq a_n\langle x_0 - u, x_{n+1} - u\rangle + \big((1-a_n)\|x_n - u\| + 2\lambda_n\alpha_n\|u\|\big)\|x_{n+1} - u\| \\ &\leq a_n\langle x_0 - u, x_{n+1} - u\rangle + 2\lambda_n\alpha_n\|u\|\|x_{n+1} - u\| \\ &\quad + \frac{(1-a_n)}{2}\left(\|x_n - u\|^2 + \|x_{n+1} - u\|^2\right),\end{aligned} \tag{51}$$

which implies that

$$\begin{aligned}\|x_{n+1} - u\|^2 &\leq (1-a_n)\|x_n - u\|^2 + 2a_n\langle x_0 - u, x_{n+1} - u\rangle \\ &\quad + 4\lambda_n\alpha_n\|u\|\|x_{n+1} - u\|.\end{aligned} \tag{52}$$

It follows from condition (a) and Lemma 9 that

$$\lim_{n\longrightarrow\infty}\|x_n - u\| = 0. \tag{53}$$

Therefore, from $\|x_n - z_n\| \longrightarrow 0$, $\|x_n - y_n\| \longrightarrow 0$, we can conclude that $\{x_n\}$, $\{z_n\}$, $\{v_n\}$, and $\{y_n\}$ converge strongly to the same point $u = P_\Gamma(x_0)$. The proof is complete. □

In the following, we will discuss the weak convergence of the sequence of the new iteration.

Theorem 2. *Assume that C and Q are 2 nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let A: $H_1 \longrightarrow H_2$ be a bounded linear operator and f: $H_1 \longrightarrow \mathbb{R}$ be a continuous differentiable function. Assume that C and Q are 2 nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let A_i: $H_1 \longrightarrow H_2$ bounded linear operators, f_i: $H_1 \longrightarrow \mathbb{R}$ be a continuous differentiable function, $i = 1, 2$, and F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1) - (A_4)$, M be an α-inverse strongly monotone mapping from C into H_1, S: $C \longrightarrow C$ be a nonexpansive mapping, and T: $C \longrightarrow C$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap GEP(F, M) \cap (\cap_{i=1}^2 \Omega_i) \neq \varnothing$. Suppose $\{x_n\}$ and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = \gamma_n P_C\left(x_n - \lambda_n \nabla f_{1\alpha_n} x_n\right) + (1-\gamma_n) P_C\left(x_n - \lambda_n \nabla f_{2s_n} x_n\right), \\ F(v_n, y) + \langle M z_n t, nyq - hv_n\rangle + \dfrac{1}{r_n}\langle y - tv_n n, qv_n h - z_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = a_n x_n + b_n\left((1-\beta_n) z_n + \beta_n T z_n\right) + c_n\left((1-\delta_n) v_n + \delta_n S v_n\right), \quad \forall n \geq 0. \end{cases} \tag{54}$$

Suppose the following conditions are satisfied:

(a) $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\sum_{n=0}^{\infty} s_n < \infty$

(b) $\{\beta_n\} \subset [k, r]$ *for some* $r, k \in (0,1)$, $\{\lambda_n\} \subset [e, d]$ *for some* $e, d \in (0, (1/\|A\|^2))$

(c) $\{\gamma_n\} \subset [t, m]$ *for some* $t, m \in (0,1)$, $\{\delta_n\} \subset [\delta_1^*, \delta_2^*]$ *for some* $\delta_1^*, \delta_2^* \in (0,1)$

(d) $0 < a \leq a_n \leq a' < 1$, $0 < b \leq b_n \leq b' < 1$ *and* $0 < c \leq c_n \leq c' < 1$ *and* $a_n + b_n + c_n = 1$

(e) $0 < q_1 \leq r_n \leq q_2 < 2\alpha$

Then, $\{x_n\}$ *converges weakly to an element* $u \in \Gamma$.

Proof. For any fixed $u \in \Gamma$, we find that $u = P_C(I - \lambda \nabla f)u$ for $\lambda \in (0, (1/\|A\|^2))$ and $Su = u$. Let $y_n = P_C(I - \lambda_n \nabla f_{1\alpha_n})x_n$, $t_n = P_C(I - \lambda_n \nabla f_{2s_n})x_n$, and $T_n = (1-\beta_n)I + \beta_n T$. We see from Lemma 8 that T_n is nonexpansive and $F(T_n) = F(T)$. From (54) and Lemma 3, it follows that

$$\begin{aligned} \|y_n - u\| &\leq \left\|P_C\left(I - \lambda_n \nabla f_{1\alpha_n}\right)x_n - P_C\left(I - \lambda_n \nabla f_{1\alpha_n}\right)u\right\| \\ &\quad + \left\|P_C\left(I - \lambda_n \nabla f_{1\alpha_n}\right)u - P_C\left(I - \lambda_n \nabla f_1\right)u\right\| \\ &\leq \|x_n - u\| + \left\|\left(I - \lambda_n \nabla f_{1\alpha_n}\right)u - \left(I - \lambda_n \nabla f_1\right)u\right\| \\ &\leq \|x_n - u\| + \lambda_n \alpha_n \|u\|. \end{aligned} \tag{55}$$

In a similar way, we have

$$\begin{aligned} \|t_n - u\| &\leq \left\|P_C\left(I - \lambda_n \nabla f_{2s_n}\right)x_n - P_C\left(I - \lambda_n \nabla f_{2s_n}\right)u\right\| \\ &\quad + \left\|P_C\left(I - \lambda_n \nabla f_{2s_n}\right)u - P_C\left(I - \lambda_n \nabla f_2\right)u\right\| \\ &\leq \|x_n - u\| + \left\|\left(I - \lambda_n \nabla f_{2s_n}\right)u - \left(I - \lambda_n \nabla f_2\right)u\right\| \\ &\leq \|x_n - u\| + \lambda_n s_n \|u\|. \end{aligned} \tag{56}$$

This implies that

$$\begin{aligned} \|z_n - u\| &\leq \gamma_n \|y_n - u\| + (1-\gamma_n)\|t_n - u\| \\ &\leq \gamma_n\left(\|x_n - u\| + \lambda_n \alpha_n \|u\|\right) + (1-\gamma_n) \\ &\quad \cdot \left(\|x_n - u\| + \lambda_n s_n \|u\|\right) \\ &\leq \|x_n - u\| + \lambda_n \|u\| (\alpha_n + s_n). \end{aligned} \tag{57}$$

Observe that v_n can be rewritten as $v_n = T_{r_n}(z_n - r_n M z_n)$, $n \geq 0$. From (e) and Lemma 1, we have

$$\begin{aligned} \|v_n - u\|^2 &= \left\|T_{r_n}(z_n - r_n M z_n) - u\right\|^2 \\ &= \left(T_{r_n}(z_n - r_n M z_n) - T_{r_n}(u - r_n M u)\right)^2 \\ &\leq \left\|(z_n - r_n M z_n) - (u - r_n M u)\right\|^2 \\ &= \|z_n - u\|^2 + r_n(r_n - 2\alpha)\|M z_n - M u\|^2 \\ &\leq \|z_n - u\|^2 \\ &\leq \|x_n - u\| + \lambda_n \|u\| (\alpha_n + s_n). \end{aligned} \tag{58}$$

We find from (54) and the last inequality that

$$\begin{aligned} \|x_{n+1} - u\| &\leq a_n \|x_n - u\| + b_n \|T_n z_n - u\| \\ &\quad + c_n\left((1-\delta_n)\|v_n - u\| + \delta_n \|S v_n - u\|\right) \\ &\leq a_n \|x_n - u\| + (1-a_n)\|z_n - u\| \\ &\leq a_n \|x_n - u\| + (1-a_n)\left(\|x_n - u\| + \lambda_n \|u\| (\alpha_n + s_n)\right) \\ &\leq \|x_n - u\| + \lambda_n \|u\| (\alpha_n + s_n). \end{aligned} \tag{59}$$

Consequently, from condition (a) and Lemma 7, we deduce that, for every $u \in \Gamma$, $\lim_{n \longrightarrow \infty} \|x_n - u\|$ exists and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded. It follows from (55), (56), and Lemma 5 that

$$\begin{aligned}\|z_n - u\|^2 &\le \gamma_n\|y_n - u\|^2 + (1-\gamma_n)\|t_n - u\|^2 - \gamma_n(1-\gamma_n)\|y_n - t_n\|^2\\ &\le \gamma_n\left(\|x_n - u\| + \lambda_n\alpha_n\|u\|\right)^2 + (1-\gamma_n)\left(\|x_n - u\| + \lambda_n s_n\|u\|\right)^2\\ &\quad - \gamma_n(1-\gamma_n)\|y_n - t_n\|^2\\ &\le \gamma_n\left(2\|x_n - u\|^2 + 2\lambda_n^2\alpha_n^2\|u\|^2\right) + (1-\gamma_n)\left(2\|x_n - u\|^2 + 2\lambda_n^2 s_n^2\|u\|^2\right)\\ &\quad - \gamma_n(1-\gamma_n)\|y_n - t_n\|^2\\ &\le \|x_n - u\|^2 + 2\lambda_n^2\|u\|^2\left(\alpha_n^2 + s_n^2\right) - \gamma_n(1-\gamma_n)\|y_n - t_n\|^2.\end{aligned} \tag{60}$$

Let $S_n v_n = (1-\delta_n)v_n + \delta_n S v_n$. We find from (54), (58), and Lemma 5 and the last inequality that

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|a_n x_n + b_n T_n z_n + c_n((1-\delta_n)v_n + \delta_n S v_n) - u\|^2\\ &\le a_n\|x_n - u\|^2 + b_n\|T_n z_n - u\|^2 + c_n\Big[(1-\delta_n)\|v_n - u\|^2 + \delta_n\|S v_n - u\|^2\\ &\quad - (1-\delta_n)\delta_n\|v_n - S v_n\|^2\Big] - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S_n v_n\|^2\\ &\le a_n\|x_n - u\|^2 + (1-a_n)\|z_n - u\|^2 + r_n(r_n - 2\alpha)\|M z_n - M u\|^2\\ &\quad - c_n(1-\delta_n)\delta_n\|v_n - S v_n\|^2 - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S_n v_n\|^2\\ &\le a_n\|x_n - u\|^2 + (1-a_n)\Big[\|x_n - u\|^2 + 2\lambda_n^2\|u\|^2\left(\alpha_n^2 + s_n^2\right)\\ &\quad - \gamma_n(1-\gamma_n)\|y_n - t_n\|^2\Big] - c_n(1-\delta_n)\delta_n\|v_n - S v_n\|^2\\ &\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S_n v_n\|^2 + r_n(r_n - 2\alpha)\|M z_n - M u\|^2\\ &\le \|x_n - u\|^2 + 2\lambda_n^2\|u\|^2\left(\alpha_n^2 + s_n^2\right) - (1-a_n)\gamma_n(1-\gamma_n)\|y_n - t_n\|^2\\ &\quad - c_n(1-\delta_n)\delta_n\|v_n - S v_n\|^2 - a_n b_n\|x_n - T_n z_n\|^2\\ &\quad - a_n c_n\|x_n - S_n v_n\|^2 + r_n(r_n - 2\alpha)\|M z_n - M u\|^2.\end{aligned} \tag{61}$$

From conditions (b)–(e) and (61), we also obtain

$$
\begin{aligned}
&(1-a')t(1-m)\|y_n-t_n\|^2+ab\|x_n-T_nz_n\|^2+c(1-\delta_2)\delta_1\|v_n-Sv_n\|^2\\
&\quad+ac\|x_n-S_nv_n\|^2+r_n(2\alpha-r_n)\|Mz_n-Mu\|^2\\
&\quad\le(1-a_n)\gamma_n(1-\gamma_n)\|y_n-t_n\|^2+c_n(1-\delta_n)\delta_n\|v_n-Sv_n\|^2\\
&\quad+a_nb_n\|x_n-T_nz_n\|^2+a_nc_n\|x_n-S_nv_n\|^2+r_n(2\alpha-r_n)\|Mz_n-Mu\|^2\\
&\quad\le\|x_n-u\|^2-\|x_{n+1}-u\|^2+2\lambda_n^2\|u\|^2\left(\alpha_n^2+s_n^2\right).
\end{aligned}\tag{62}
$$

Since $\lim_{n\longrightarrow\infty}\|x_n-u\|$ exists and $\sum_{n=0}^{\infty}(\alpha_n+s_n)<\infty$, we see that

$$
\begin{aligned}
\lim_{n\longrightarrow\infty}\|v_n-Sv_n\|&=\lim_{n\longrightarrow\infty}\|x_n-T_nz_n\|=\lim_{n\longrightarrow\infty}\|y_n-t_n\|\\
&=\lim_{n\longrightarrow\infty}\|x_n-S_nv_n\|=\lim_{n\longrightarrow\infty}\|Mz_n-Mu\|=0.
\end{aligned}\tag{63}
$$

Since $\|x_{n+1}-x_n\|\le b_n\|x_n-T_nz_n\|+c_n\|x_n-S_nv_n\|$ and $\|z_n-y_n\|\le\|y_n-t_n\|$, $\|z_n-t_n\|\le\|y_n-t_n\|$, it follows that

$$
\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=\lim_{n\longrightarrow\infty}\|y_n-z_n\|=\lim_{n\longrightarrow\infty}\|t_n-z_n\|=0.\tag{64}
$$

Using Lemma 1 and (58), we have

$$
\begin{aligned}
\|v_n-u\|^2&=\left\|T_{r_n}(z_n-r_nMz_n)-T_{r_n}(u-r_nMu)\right\|^2\\
&\le\langle(z_n-r_nMz_n)-(u-r_nMu),v_n-u\rangle\\
&=\frac{1}{2}\|(z_n-r_nMz_n)-(u-r_nMu)\|^2+\frac{1}{2}\|v_n-u\|^2\\
&\quad-\frac{1}{2}\|(z_n-r_nMz_n)-(u-r_nMu)-(v_n-u)\|^2\\
&\le\frac{1}{2}\left[\|z_n-u\|^2+\|v_n-u\|^2-\|(z_n-v_n)-2r_n(Mz_n-Mu)\|^2\right]\\
&=\frac{1}{2}\left[\|z_n-u\|^2+\|v_n-u\|^2-\|z_n-v_n\|^2+2r_n\langle z_n-v_n,Mz_n-Mu\rangle-r_n^2\|Mx_n-Mu\|^2\right].
\end{aligned}\tag{65}
$$

It follows that

$$
\|v_n-u\|^2\le\|z_n-u\|^2-\|z_n-v_n\|^2+2r_n\langle z_n-v_n,Mz_n-Mu\rangle.\tag{66}
$$

We find from (54) and (66) that

$$
\begin{aligned}
\|x_{n+1}-u\|^2&=\|a_nx_n+b_nT_nz_n+c_n((1-\delta_n)v_n+\delta_nSv_n)-u\|^2\\
&\le a_n\|x_n-u\|^2+b_n\|T_nz_n-u\|^2+c_n\left[(1-\delta_n)\|v_n-u\|^2+\delta_n\|Sv_n-u\|^2\right]\\
&\le a_n\|x_n-u\|^2+(1-a_n)\|z_n-u\|^2-c_n\|z_n-v_n\|^2+2r_n\langle z_n-v_n,Mz_n-Mu\rangle\\
&\le a_n\|x_n-u\|^2+(1-a_n)\left[\|x_n-u\|^2+2\lambda_n^2\|u\|^2\left(\alpha_n^2+s_n^2\right)\right]\\
&\quad-c_n\|z_n-v_n\|^2+2r_n\langle z_n-v_n,Mz_n-Mu\rangle\\
&\le\|x_n-u\|^2+(1-a_n)2\lambda_n^2\|u\|^2\left(\alpha_n^2+s_n^2\right)\\
&\quad-c_n\|z_n-v_n\|^2+2r_n\langle z_n-v_n,Mz_n-Mu\rangle.
\end{aligned}\tag{67}
$$

This yields that

$$\|z_n - v_n\|^2 \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + 2\lambda_n^2\|u\|^2\left(\alpha_n^2 + s_n^2\right) + 2r_n\langle z_n - v_n, Mz_n - Mu\rangle. \tag{68}$$

It follows from condition (a), $\lim_{n\longrightarrow\infty}\|Mz_n - Mu\| = 0$, and $\lim_{n\longrightarrow\infty}\|x_n - u\|$ exists that

$$\lim_{n\longrightarrow\infty}\|z_n - v_n\| = 0. \tag{69}$$

Also, from $\|T_n z_n - z_n\| \le \|T_n z_n - x_n\| + \|x_n - z_n\|$, $\|v_n - x_n\| \le \|v_n - S_n v_n\| + \|x_n - S_n v_n\|$, and $\|v_n - S_n v_n\| \le \|v_n - Sv_n\|$, we get

$$\lim_{n\longrightarrow\infty}\|T_n z_n - z_n\| = \lim_{n\longrightarrow\infty}\|v_n - x_n\| = 0. \tag{70}$$

Note that $\|x_n - y_n\| \le \|x_n - z_n\| + \|z_n - y_n\|$, $\|x_n - t_n\| \le \|x_n - z_n\| + \|z_n - t_n\|$, $\beta_n\|Tz_n - z_n\| = \|T_n z_n - z_n\|$. This implies that

$$\lim_{n\longrightarrow\infty}\|x_n - y_n\| = \lim_{n\longrightarrow\infty}\|x_n - t_n\| = \lim_{n\longrightarrow\infty}\|Tz_n - z_n\| = 0. \tag{71}$$

Since $\nabla f = A^*(I - P_Q)A$ is Lipschitz continuous, we obtain $\lim_{n\longrightarrow\infty}\|\nabla f(x_n) - \nabla f(y_n)\| = 0$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that it converges weakly to some x^*. Since $\|x_n - y_n\| \longrightarrow 0$, $\|x_n - z_n\| \longrightarrow 0$, and $\|x_n - v_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain that $y_{n_k} \rightharpoonup x^*$, $z_{n_k} \rightharpoonup x^*$, and $v_{n_k} \rightharpoonup x^*$. Since $\{y_{n_k}\} \subset C$ and C is closed and convex, we obtain $x^* \in C$. First, we show that $x^* \in F(T) \cap F(S)$. Then, from (63), (71), Lemma 6, and Lemma 4, we have that $x^* \in F(T) \cap F(S)$. We now show $x^* \in \mathrm{GEP}(F, M)$. By $v_n = T_{r_n}(z_n - r_n Mz_n)$, we know that

$$F(v_n, y) + \langle Mz_n, y - v_n\rangle + \frac{1}{r_n}\langle y - v_n, v_n - z_n\rangle \ge 0, \quad \forall y \in C. \tag{72}$$

It follows from (A_2) that

$$\langle Mz_n, y - v_n\rangle + \frac{1}{r_n}\langle y - v_n, v_n - x_n\rangle \ge F(y, v_n), \quad \forall y \in C. \tag{73}$$

Hence,

$$\langle Mz_{n_j}, y - v_{n_j}\rangle + \langle y - v_{n_j}, \frac{v_{n_j} - z_{n_j}}{r_{n_j}}\rangle \ge F\left(y, v_{n_j}\right), \quad \forall y \in C. \tag{74}$$

For t with $0 < t \le 1$ and $y \in C$, let $v_t = ty + (1 - t)x^*$. Since $y \in C$ and $x^* \in C$, we obtain $v_t \in C$. So, from (74), we have

$$\begin{aligned}\langle v_t - v_{n_j}, Mv_t\rangle &\ge \langle v_t - v_{n_j}, Mv_t\rangle - \langle v_t - v_{n_j}, Mz_{n_j}\rangle \\ &\quad - \langle v_t - v_{n_j}, \frac{v_{n_j} - z_{n_j}}{r_{n_j}}\rangle + F\left(v_t, v_{n_j}\right) \\ &= \langle v_t - v_{n_j}, Mv_t - Mv_{n_j}\rangle \\ &\quad + \langle v_t - v_{n_j}, Mv_{n_j} - Mz_{n_j}\rangle \\ &\quad - \langle v_t - v_{n_j}, \frac{v_{n_j} - z_{n_j}}{r_{n_j}}\rangle + F\left(v_t, v_{n_j}\right).\end{aligned} \tag{75}$$

Since $\|v_{n_j} - z_{n_j}\| \longrightarrow 0$, we have $\|Mv_{n_j} - Mz_{n_j}\| \longrightarrow 0$. Furthermore, from the inverse strongly monotonicity of M, we have $\langle v_t - v_{n_j}, Mv_t - Mv_{n_j}\rangle \ge 0$. It follows from A_4 and $(v_{n_j} - z_{n_j}/r_{n_j}) \longrightarrow 0$ and $v_{n_j} \rightharpoonup x^*$, and we have

$$\langle v_t - v, Mv_t\rangle \ge F(v_t, x^*), \tag{76}$$

as $j \longrightarrow \infty$. From (A_1) and (A_4), we have

$$\begin{aligned} 0 &= F(v_t, v_t) \\ &\le tF(v_t, y) + (1 - t)F(v_t, x^*) \\ &\le tF(v_t, y) + (1 - t)\langle v_t - x^*, Mv_t\rangle \\ &= tF(v_t, y) + (1 - t)t\langle y - x^*, Mv_t\rangle, \end{aligned} \tag{77}$$

and hence,

$$0 \le F(v_t, y) + (1 - t)\langle y - x^*, Mv_t\rangle. \tag{78}$$

Letting $t \longrightarrow 0$, we have, for each $y \in C$,

$$F(x^*, y) + \langle y - x^*, Mx^*\rangle \ge 0. \tag{79}$$

This implies that $x^* \in \mathrm{GEP}(F, M)$. Next, we show that $x^* \in \cap_{i=1}^2 \Omega_i$ (1). For $i = 1, 2$, let

$$T_i' p := \begin{cases} \nabla f_i(p) + N_C p, & p \in C, \\ \varnothing, & p \notin C. \end{cases} \tag{80}$$

Then, T_i' is maximal monotone and $0 \in T_i' p$ if and only if $p \in \mathrm{VI}(C, \nabla f_i)$ [29]. Let $G(T_i')$ be the graph of T_i', and $(p, v) \in G(T_i')$. Then, we have $v \in T_i'(p) = \nabla f_i(p) + N_C p$, and hence, $v - \nabla f_i(p) \in N_C p$. Therefore, we have $\langle p - w, v - \nabla f_i(p)\rangle \ge 0$ for all $w \in C$. By the property of metric projection, from $y_n = P_C(x_n - \lambda_n \nabla f_{1\alpha_n} x_n)$ and $p \in C$, we have $\langle p - y_n, y_n - (x_n - \lambda_n \nabla f_{1\alpha_n} x_n)\rangle \ge 0$, and hence,

$$\langle p - y_n, \frac{y_n - x_n}{\lambda_n} + \nabla f_{1\alpha_n} x_n\rangle \ge 0. \tag{81}$$

From $\langle p - w, v - \nabla f_1(p)\rangle \ge 0$ for all $w \in C$ and $y_{n_k} \in C$, we have

$$\begin{aligned}\langle p - y_{n_k}, v\rangle &\geq \langle p - y_{n_k}, \nabla f_1(p)\rangle \\ &\geq \langle p - y_{n_k}, \nabla f_1(p)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_{n_k}} + \nabla f_{1\alpha_n} x_{n_k}\rangle \\ &\geq \langle p - y_{n_k}, \nabla f_1(p)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_{n_k}} + \nabla f_1(x_{n_k})\rangle - \alpha_{n_k}\langle p - y_{n_k}, x_{n_k}\rangle \\ &= \langle p - y_{n_k}, \nabla f_1(p) - \nabla f_1(y_{n_k})\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_{n_k}}\rangle - \alpha_{n_k}\langle p - y_{n_k}, x_{n_k}\rangle \\ &\quad + \langle p - y_{n_k}, \nabla f_1(y_{n_k}) - \nabla f_1(x_{n_k})\rangle \\ &\geq \langle p - y_{n_k}, \nabla f_1(y_{n_k}) - \nabla f_1(x_{n_k})\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_{n_k}}\rangle - \alpha_{n_k}\langle p - y_{n_k}, x_{n_k}\rangle.\end{aligned} \tag{82}$$

Thus, we obtain $\langle p - x^*, v\rangle \geq 0$ as $k \longrightarrow \infty$. Since T_1' is maximal monotone, we have $x^* \in T_1'^{-1}0$, and hence, $x^* \in \mathrm{VI}(C, \nabla f_1)$. Similarly, we have $x^* \in \mathrm{VI}(C, \nabla f_2)$. This implies $x^* \in \Omega_i$ for $i = 1, 2$. This implies that $x^* \in \Gamma$. Therefore, from $\|x_n - z_n\| \longrightarrow 0$, we can conclude that $\{x_n\}$, $\{z_n\}$, and $\{v_n\}$ converge weakly to a point $u \in \Gamma$. The proof is complete. □

Theorem 3. *Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let A: $H_1 \longrightarrow H_2$ be a bounded linear operator, S: $C \longrightarrow C$ be a nonexpansive map, and T: $C \longrightarrow C$ be a strictly pseudo-contractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap \Omega \neq \emptyset$. Suppose $\{x_n\}$, $\{v_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ v_n = x_n + \varepsilon_n(x_n - x_{n-1}), \\ z_n = P_C(I - \lambda_n \nabla f_{\alpha_n})v_n, \\ x_{n+1} = a_n x_n + b_n((1 - \beta_n)z_n + \beta_n T z_n) \\ + c_n \mathrm{S}P_C(v_n - \lambda_n \nabla f_{\alpha_n} z_n), \quad \forall n \geq 0. \end{cases} \tag{83}$$

Suppose the following conditions are satisfied:

(a) $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\sum_{n=0}^{\infty} b_n < \infty$

(b) $\{\beta_n\} \subset [k, r]$ for some $r, k \in (0, 1)$

(c) $\{\lambda_n\} \subset [e, d]$ for some $e, d \in (0, (1/\|A\|^2))$

(d) $0 < a \leq a_n \leq a' < 1, 0 < b_n \leq b' < 1, 0 < c \leq c_n \leq c' < 1$ and $a_n + b_n + c_n = 1$

(e) $\{\varepsilon_n\} \subset [0, \varepsilon]$ and $\varepsilon \in [0, 1)$, $\sum_{n=0}^{\infty} \varepsilon_n \|x_n - x_{n-1}\| < \infty$

Then, $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Proof. For any fixed $u \in \Gamma$, we find that $u = P_C(I - \lambda \nabla f)u$ for $\lambda \in (0, (1/\|A\|^2))$ and $Su = u$. Putting $T_n = (1 - \beta_n)I + \beta_n T$, we see from Lemma 8 that T_n is nonexpansive and $F(T_n) = F(T)$. We observe that

$$\begin{aligned}\|v_n - u\| &= \|x_n + \varepsilon_n(x_n - x_{n-1}) - u\| \\ &\leq \|x_n - u\| + \varepsilon_n \|x_n - x_{n-1}\|.\end{aligned} \tag{84}$$

From (83) and Lemma 3, it follows that

$$\begin{aligned}\|z_n - u\| &= \|P_C(I - \lambda_n \nabla f_{\alpha_n})v_n - P_C(I - \lambda_n \nabla f)u\| \\ &\leq \|P_C(I - \lambda_n \nabla f_{\alpha_n})v_n - P_C(I - \lambda_n \nabla f_{\alpha_n})u\| \\ &\quad + \|P_C(I - \lambda_n \nabla f_{\alpha_n})u - P_C(I - \lambda_n \nabla f)u\| \\ &\leq \|v_n - u\| + \|(I - \lambda_n \nabla f_{\alpha_n})u - (I - \lambda_n \nabla f)u\| \\ &\leq \|v_n - u\| + \lambda_n \alpha_n \|u\|.\end{aligned} \tag{85}$$

Put $y_n = P_C(v_n - \lambda_n \nabla f_{\alpha_n}(z_n))$ for all $n \geq 0$. Then, by property of metric projection, we have

$$\begin{aligned}
\|y_n - u\|^2 &\le \left\|v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - u\right\|^2 - \left\|v_n - \lambda_n \nabla f_{\alpha_n} z_n - y_n\right\|^2 \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), u - y_n\rangle \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n) - \nabla f_{\alpha_n}(u), u - z_n\rangle \\
&\quad + 2\lambda_n \left(\langle \nabla f_{\alpha_n}(u), u - z_n\rangle + \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle\right) \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(u), u - z_n\rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \langle (\alpha_n I + \nabla f) u, u - z_n\rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle \\
&\le \|v_n - u\|^2 - \|v_n - y_n\|^2 + 2\lambda_n \alpha_n \langle u, u - z_n\rangle \\
&\quad + 2\lambda_n \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle \\
&= \|v_n - u\|^2 - \|v_n - z_n\|^2 - 2\langle v_n - z_n, z_n - y_n\rangle - \|z_n - y_n\|^2 \\
&\quad + 2\lambda_n \left[\alpha_n \langle u, u - z_n\rangle + \langle \nabla f_{\alpha_n}(z_n), z_n - y_n\rangle\right] \\
&= \|v_n - u\|^2 - \|v_n - z_n\|^2 + 2\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n\rangle \\
&\quad + 2\lambda_n \alpha_n \langle u, u - z_n\rangle - \|z_n - y_n\|^2.
\end{aligned} \tag{86}$$

Furthermore, by property of metric projection, we have

$$\begin{aligned}
&\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n\rangle \\
&\quad = \langle v_n - \lambda_n \nabla f_{\alpha_n}(v_n) - z_n, y_n - z_n\rangle + \langle \lambda_n \nabla f_{\alpha_n}(v_n) - \lambda_n \nabla f_{\alpha_n}(z_n), y_n - z_n\rangle \\
&\quad \le \langle \lambda_n \nabla f_{\alpha_n}(v_n) - \lambda_n \nabla f_{\alpha_n}(z_n), y_n - z_n\rangle \\
&\quad \le \lambda_n \left\|\nabla f_{\alpha_n}(v_n) - \nabla f_{\alpha_n}(z_n)\right\| \|y_n - z_n\| \\
&\quad \le \lambda_n \left(\alpha_n + \|A\|^2\right) \|v_n - z_n\| \|y_n - z_n\|.
\end{aligned} \tag{87}$$

Hence, we have

$$\begin{aligned}
\|y_n - u\|^2 &\le \|v_n - u\|^2 - \|v_n - z_n\|^2 + 2\langle v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - z_n, y_n - z_n\rangle \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n\alpha_n\langle u, u - z_n\rangle \\
&\le \|v_n - u\|^2 - \|v_n - z_n\|^2 + 2\lambda_n\left(\alpha_n + \|A\|^2\right)\|v_n - z_n\|\|y_n - z_n\| \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| \\
&\le \|v_n - u\|^2 - \|v_n - z_n\|^2 + \lambda_n^2\left(\alpha_n + \|A\|^2\right)^2\|v_n - z_n\|^2 + \|y_n - z_n\|^2 \\
&\quad - \|z_n - y_n\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| \\
&= \|v_n - u\|^2 + \left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| \\
&\le \|v_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\|u - z_n\| \\
&\le \|v_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\left(\|v_n - u\| + \lambda_n\alpha_n\|u\|\right) \\
&\le \|v_n - u\|^2 + 4\lambda_n\alpha_n\|u\|\|v_n - u\| + 4\lambda_n^2\alpha_n^2\|u\|^2 \\
&= \left(\|v_n - u\| + 2\lambda_n\alpha_n\|u\|\right)^2.
\end{aligned} \tag{88}$$

We find from (83), (84), and (85) and the last inequality that

$$\begin{aligned}
\|x_{n+1} - u\| &= \left\|a_n x_n + b_n T_n z_n + c_n S P_C\left(v_n - \lambda_n \nabla f_{\alpha_n}(z_n)\right) - u\right\| \\
&\le a_n\|x_n - u\| + b_n\|T_n z_n - u\| + c_n\|S y_n - u\| \\
&\le a_n\|x_n - u\| + b_n\|z_n - u\| + c_n\|y_n - u\| \\
&\le a_n\|x_n - u\| + b_n\left(\|v_n - u\| + \lambda_n\alpha_n\|u\|\right) + c_n\left(\|v_n - u\| + 2\lambda_n\alpha_n\|u\|\right) \\
&\le a_n\|x_n - u\| + (1 - a_n)\left(\|v_n - u\| + 2\lambda_n\alpha_n\|u\|\right) \\
&\le \|x_n - u\| + \varepsilon_n\|x_n - x_{n-1}\| + 2\lambda_n\alpha_n\|u\|.
\end{aligned} \tag{89}$$

Consequently, from conditions (a) and (e) and Lemma 10, we deduce that, for every $u \in \Gamma$, $\lim_{n\longrightarrow\infty}\|x_n - u\|$ exists and the sequences $\{x_n\}$, $\{z_n\}$, and $\{y_n\}$ are bounded. We find from (83), (84), (85), (88), Lemma 5, and Lemma 11 that

$$
\begin{aligned}
\|x_{n+1}-u\|^2 &= \left\|a_n x_n + b_n T_n z_n + c_n S P_C\left(v_n - \lambda_n \nabla f_{\alpha_n}(z_n)\right) - u\right\|^2 \\
&\le a_n\|x_n-u\|^2 + b_n\|T_n z_n - u\|^2 + c_n\|S y_n - u\|^2 \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le a_n\|x_n-u\|^2 + b_n\|z_n - u\|^2 + c_n\|y_n - u\|^2 \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le a_n\|x_n-u\|^2 + b_n\left(\|v_n-u\| + \lambda_n\alpha_n\|u\|\right)^2 \\
&\quad + c_n\left[\|v_n-u\|^2 + \left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2\right] \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 + 2c_n\lambda_n\alpha_n\|u\|\|z_n - u\| \\
&\le a_n\|x_n-u\|^2 + b_n\left(2\|v_n-u\|^2 + 2\lambda_n^2\alpha_n^2\|u\|^2\right) \\
&\quad + c_n\left[\|v_n-u\|^2 + \left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2 + 2\lambda_n\alpha_n\|u\|\|z_n-u\|\right] \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le (1+b_n)\|x_n-u\|^2 + 2b_n\lambda_n^2\alpha_n^2\|u\|^2 + 2(2b_n + c_n)\varepsilon_n\langle x_n - x_{n-1}, v_n - u\rangle \\
&\quad + c_n\left[\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2 + \alpha_n\left(\lambda_n^2\|u\|^2 + \|z_n-u\|^2\right)\right] \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le (1+b_n)\|x_n-u\|^2 + 2b_n\lambda_n^2\alpha_n^2\|u\|^2 + 2(2b_n + c_n)\varepsilon_n\|x_n - x_{n-1}\|\|v_n - u\| \\
&\quad + c_n\left[\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2 + \alpha_n\left(\lambda_n^2\right)\|u\|^2 + \left(\|v_n-u\| + \lambda_n\alpha_n\|u\|\right)^2\right] \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le (1+b_n)\|x_n-u\|^2 + 2b_n\lambda_n^2\alpha_n^2\|u\|^2 + 2(2b_n + c_n)\varepsilon_n\|x_n - x_{n-1}\|\|v_n - u\| \\
&\quad + c_n\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|v_n - z_n\|^2 + \alpha_n\left(\lambda_n^2\|u\|^2 + 2\|v_n-u\|^2 + 2\lambda_n^2\alpha_n^2\|u\|^2\right) \\
&\quad - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\le (1+b_n+2\alpha_n)\|x_n-u\|^2 + c_n\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|x_n - z_n\|^2 \\
&\quad + \alpha_n\lambda_n^2\|u\|^2\left(1 + 2b_n\alpha_n + 2\alpha_n^2\right) - a_n b_n\|x_n - T_n z_n\|^2 - a_n c_n\|x_n - S y_n\|^2 \\
&\quad + 2(2b_n + c_n + 2\alpha_n)\varepsilon_n\|x_n - x_{n-1}\|\|v_n - u\|.
\end{aligned} \tag{90}
$$

From conditions (b) and (d), we obtain

$$
\begin{aligned}
&c\left(1 - d^2\left(\alpha_n + \|A\|^2\right)^2\right)\|v_n - z_n\|^2 + ab_n\|x_n - T_n z_n\|^2 + ac\|x_n - S y_n\|^2 \\
&\le c_n\left(1 - \lambda_n^2\left(\alpha_n + \|A\|^2\right)^2\right)\|v_n - z_n\|^2 + a_n b_n\|x_n - T_n z_n\|^2 + a_n c_n\|x_n - S y_n\|^2 \\
&\le (1+b_n+2\alpha_n)\|x_n-u\|^2 - \|x_{n+1}-u\|^2 + \alpha_n\lambda_n^2\|u\|^2\left(1 + 2\alpha_n^2\right) \\
&\quad + 2(2b_n + c_n + 2\alpha_n)\varepsilon_n\|x_n - x_{n-1}\|\|v_n - u\|.
\end{aligned} \tag{91}
$$

From conditions (a) and (e), we also obtain

$$\lim_{n\longrightarrow\infty}\left\|x_n - Sy_n\right\| = \lim_{n\longrightarrow\infty}\left\|x_n - T_n z_n\right\| = \lim_{n\longrightarrow\infty}\left\|v_n - z_n\right\| = 0. \tag{92}$$

By the definition of $\{v_n\}$ and (e), we have

$$\lim_{n\longrightarrow\infty}\left\|v_n - x_n\right\| = \lim_{n\longrightarrow\infty}\varepsilon_n\left\|x_n - x_{n-1}\right\| = 0. \tag{93}$$

This implies that

$$\left\|z_n - x_n\right\| \leq \left\|z_n - v_n\right\| + \left\|v_n - x_n\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{94}$$

It is observe that

$$\begin{aligned}\left\|y_n - z_n\right\| &= \left\|P_C\left(v_n - \lambda_n \nabla f_{\alpha_n}(z_n)\right) - P_C\left(v_n - \lambda_n \nabla f_{\alpha_n}(v_n)\right)\right\| \\ &\leq \left\|v_n - \lambda_n \nabla f_{\alpha_n}(z_n) - \left(v_n - \lambda_n \nabla f_{\alpha_n}(v_n)\right)\right\| \\ &= \lambda_n \left\|\nabla f_{\alpha_n}(z_n) - \nabla f_{\alpha_n}(v_n)\right\| \\ &\leq \lambda_n\left(\alpha_n + \|A\|^2\right)\left\|z_n - v_n\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{95}$$

Also, from $\|T_n z_n - z_n\| \leq \|T_n z_n - x_n\| + \|x_n - z_n\|$, $\|y_n - x_n\| \leq \|y_n - z_n\| + \|z_n - x_n\|$, and $\|y_n - v_n\| \leq \|y_n - z_n\| + \|z_n - v_n\|$, we get

$$\lim_{n\longrightarrow\infty}\left\|T_n z_n - z_n\right\| = \lim_{n\longrightarrow\infty}\left\|y_n - x_n\right\| = \lim_{n\longrightarrow\infty}\left\|y_n - v_n\right\| = 0. \tag{96}$$

Note that $\|Sy_n - y_n\| \leq \|Sy_n - x_n\| + \|x_n - y_n\|$, $\beta_n\|Tz_n - z_n\| = \|T_n z_n - z_n\|$. This implies that

$$\lim_{n\longrightarrow\infty}\left\|Sy_n - y_n\right\| = \lim_{n\longrightarrow\infty}\left\|Tz_n - z_n\right\| = 0. \tag{97}$$

Since $\nabla f = A^*(I - P_Q)A$ is Lipschitz continuous, we obtain $\lim_{n\longrightarrow\infty}\|\nabla f(z_n) - \nabla f(y_n)\| = 0$.

Since, $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that it converges weakly to some x^*. Since $\|x_n - y_n\| \longrightarrow 0$, $\|x_n - z_n\| \longrightarrow 0$, and $\|x_n - v_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain that $y_{n_k} \rightharpoonup x^*, z_{n_k} \rightharpoonup x^*$, and $v_{n_k} \rightharpoonup x^*$. Since $\{y_{n_k}\} \subset C$ and C is closed and convex, we obtain $x^* \in C$. First, we show that $x^* \in F(T) \cap F(S)$. Then, from (97), Lemma 6, and Lemma 4, we have that $x^* \in F(T) \cap F(S)$. We now show $x^* \in \Omega$ (1). Let

$$T' p := \begin{cases} \nabla f(p) + N_C p, & p \in C, \\ \varnothing, & p \notin C. \end{cases} \tag{98}$$

Then, T' is maximal monotone and $0 \in T' p$ if and only if $p \in \mathrm{VI}(C, \nabla f)$ [29]. Let $G(T')$ be the graph of T', and let $(p, v) \in G(T')$. Then, we have $v \in T'(p) = \nabla f(p) + N_C p$, and hence, $v - \nabla f(p) \in N_C p$. Therefore, we have $\langle p - w, v - \nabla f(p)\rangle \geq 0$ for all $w \in C$. By property of metric projection, from $y_n = P_C(v_n - \lambda_n \nabla f_{\alpha_n} z_n)$ and $p \in C$, we have $\langle p - y_n, y_n - (v_n - \lambda_n \nabla f_{\alpha_n} z_n)\rangle \geq 0$, and hence,

$$\langle p - y_n, \frac{y_n - v_n}{\lambda_n} + \nabla f_{\alpha_n} z_n\rangle \geq 0. \tag{99}$$

From $\langle p - w, v - \nabla f(p)\rangle \geq 0$ for all $w \in C$ and $y_{n_k} \in C$, we have

$$\begin{aligned}\langle p - y_{n_k}, v\rangle &\geq \langle p - y_{n_k}, \nabla f(p)\rangle \\ &\geq \langle p - y_{n_k}, \nabla f(p)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} + \nabla f_{\alpha_{n_k}} z_{n_k}\rangle \\ &\geq \langle p - y_{n_k}, \nabla f(p)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}} + \nabla f\left(z_{n_k}\right)\rangle - \alpha_{n_k}\langle p - y_{n_k}, z_{n_k}\rangle \\ &= \langle p - y_{n_k}, \nabla f(p) - \nabla f\left(y_{n_k}\right)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}}\rangle - \alpha_{n_k}\langle p - y_{n_k}, z_{n_k}\rangle \\ &\quad + \langle p - y_{n_k}, \nabla f\left(y_{n_k}\right) - \nabla f\left(z_{n_k}\right)\rangle \\ &\geq \langle p - y_{n_k}, \nabla f\left(y_{n_k}\right) - \nabla f\left(z_{n_k}\right)\rangle - \langle p - y_{n_k}, \frac{y_{n_k} - v_{n_k}}{\lambda_{n_k}}\rangle - \alpha_{n_k}\langle p - y_{n_k}, z_{n_k}\rangle.\end{aligned} \tag{100}$$

Thus, we obtain $\langle p - x^*, v\rangle \geq 0$ as $k \longrightarrow \infty$. Since T' is maximal monotone, we have $x^* \in T'^{-1}0$, and hence, $x^* \in \mathrm{VI}(C, \nabla f)$. This implies that $x^* \in \Omega$. This implies that $x^* \in \Gamma$. Therefore, from $\|x_n - z_n\| \longrightarrow 0$ and $\|x_n - v_n\| \longrightarrow 0$, we can conclude that $\{x_n\}$, $\{z_n\}$, and $\{v_n\}$ converge weakly to a point $u \in \Gamma$. The proof is complete. □

4. Applications

If, in Theorem 3 and Theorem 1, we assume that $C = H_1$, then we can get the following theorems.

Theorem 4. *Let H_1 and H_2 be real Hilbert spaces, $A_i\colon H_1 \longrightarrow H_2$ be a bounded linear operator, for $i = 1, 2$, $S\colon H_1 \longrightarrow H_1$ be a nonexpansive mapping, and $T\colon H_1 \longrightarrow H_1$ a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap_{i=1}^{2} (\nabla f_i)^{-1} 0 \neq \varnothing$. Suppose $\{x_n\}$ and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = \gamma_n\left(x_n - \lambda_n \nabla f_{1\alpha_n} x_n\right) + (1-\gamma_n)\left(x_n - \lambda_n \nabla f_{2s_n} x_n\right), \\ x_{n+1} = a_n x_n + b_n\left((1-\beta_n) z_n + \beta_n T z_n\right) \\ + c_n\left((1-\delta_n) z_n + \delta_n S z_n\right), \quad \forall n \geq 0. \end{cases} \tag{101}$$

If conditions $(a)-(d)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Proof. We have $(\nabla f_i)^{-1} 0 = \mathrm{VI}(H_1, \nabla f_i)$ for $i = 1, 2$ and $P_{H_1} = I$; by Theorem 3, we obtain the desired result. □

Theorem 5. *Let H_1 and H_2 be real Hilbert spaces, $A\colon H_1 \longrightarrow H_2$ be a bounded linear operator, $S\colon H_1 \longrightarrow H_1$ be a nonexpansive map, and $T\colon H_1 \longrightarrow H_1$ be a strictly pseudocontractive mapping with constants k such that $\Gamma = F(T) \cap F(S) \cap (\nabla f)^{-1} 0 \neq \varnothing$. Suppose $\{x_n\}$, $\{v_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ v_n = x_n + \varepsilon_n (x_n - x_{n-1}), \\ z_n = \left(I - \lambda_n \nabla f_{\alpha_n}\right) v_n, \\ x_{n+1} = a_n x_n + b_n\left((1-\beta_n) z_n + \beta_n T z_n\right) \\ + c_n S\left(v_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \quad \forall n \geq 0. \end{cases} \tag{102}$$

If conditions $(a)-(e)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Proof. We have $(\nabla f)^{-1} 0 = \mathrm{VI}(H_1, \nabla f)$ and $P_{H_1} = I$; by Theorem 3, we obtain the desired result. □

Theorem 6. *Let H_1 and H_2 be real Hilbert spaces, $A\colon H_1 \longrightarrow H_2$ be a bounded linear operator, $S\colon H_1 \longrightarrow H_1$ be a nonexpansive map, and $T\colon H_1 \longrightarrow H_1$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap (\nabla f)^{-1} 0 \neq \varnothing$. Suppose $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = \left(I - \lambda_n \nabla f_{\alpha_n}\right) x_n, \\ y_n = x_n - \lambda_n \nabla f_{\alpha_n} z_n, \\ x_{n+1} = a_n x_0 + b_n x_n + c_n\left((1-\beta_n) S y_n + \beta_n T_n z_n\right), \quad \forall n \geq 0, \end{cases} \tag{103}$$

where $T_n = (1-\gamma_n) I + \gamma_n T$ and $\gamma_n \subset (k, 1)$. If conditions $(a)-(d)$ are satisfied, then $\{x_n\}$ converges strongly to the point $u = P_\Gamma(x_0)$.

Proof. We have $(\nabla f)^{-1} 0 = \mathrm{VI}(H_1, \nabla f)$ and $P_{H_1} = I$; by Theorem 1, we obtain the desired result. □

Let $B\colon H \longrightarrow 2^H$ be a maximal monotone mapping. Then, for any $x \in H$ and $r > 0$, consider $J_r^B x = \{y \in H\colon x = y + rBy\}$. Likewise, a J_r^B is called the resolvent of B and is denoted by $J_r^B = (I + rB)^{-1}$.

Theorem 7. *Let H_1 and H_2 be real Hilbert spaces, $B_i\colon H_1 \longrightarrow 2^{H_1}$ be maximal monotone mappings, for $i = 1, 2$, $A_i\colon H_1 \longrightarrow H_2$ be bounded linear operators, for $i = 1, 2$, $J_r^{B_i}$ be the resolvents of B_i for each $r > 0$, and $T\colon H_1 \longrightarrow H_1$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap F(S) \cap B_i^{-1} 0 \cap (\nabla f_i)^{-1} 0 \neq \varnothing$. Suppose $\{x_n\}$ and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = \gamma_n J_r^{B_1}\left(x_n - \lambda_n \nabla f_{1\alpha_n} x_n\right) + (1-\gamma_n) J_r^{B_2}\left(x_n - \lambda_n \nabla f_{2s_n} x_n\right), \\ x_{n+1} = a_n x_n + b_n\left((1-\beta_n) z_n + \beta_n T z_n\right) + c_n z_n, \quad \forall n \geq 0. \end{cases} \tag{104}$$

If conditions $(a)-(d)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Proof. We have $F(J_r^{B_i}) = B_i^{-1} 0$, $(\nabla f_i)^{-1} 0 = \mathrm{VI}(H_1, \nabla f_i)$ for $i = 1, 2$ and $P_{H_1} = I$; by Theorem 3, we obtain the desired result. □

Theorem 8. *Let H_1 and H_2 be real Hilbert spaces, $A\colon H_1 \longrightarrow H_2$ be a bounded linear operator, $B\colon H_1 \longrightarrow 2^{H_1}$ be a maximal monotone mapping, J_r^B be the resolvent of B for each $r > 0$, and $T\colon H_1 \longrightarrow H_1$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap B^{-1} 0 \cap (\nabla f)^{-1} 0 \neq \varnothing$. Suppose $\{x_n\}$, $\{v_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ v_n = x_n + \varepsilon_n (x_n - x_{n-1}), \\ z_n = \left(I - \lambda_n \nabla f_{\alpha_n}\right) v_n, \\ x_{n+1} = a_n x_n + b_n\left((1-\beta_n) z_n + \beta_n T z_n\right) \\ + c_n J_r^B\left(v_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \quad \forall n \geq 0. \end{cases} \tag{105}$$

If conditions $(a)-(e)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Proof. We have $F(J_r^B) = B^{-1}0$, $(\nabla f)^{-1}0 = \mathrm{VI}(H_1, \nabla f)$, and $P_{H_1} = I$; by Theorem 3, we obtain the desired result. □

Theorem 9. *Let H_1 and H_2 be real Hilbert spaces, $A\colon H_1 \longrightarrow H_2$ be a bounded linear operator, $B\colon H_1 \longrightarrow 2^{H_1}$ be a maximal monotone mapping, J_r^B be the resolvent of B for each $r > 0$, and $T\colon H_1 \longrightarrow H_1$ be a strictly pseudocontractive mapping with constant k such that $\Gamma = F(T) \cap B^{-1}0 \cap (\nabla f)^{-1}0 \neq \varnothing$. Suppose $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = \left(I - \lambda_n \nabla f_{\alpha_n}\right) x_n, \\ y_n = J_r^B\left(x_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \\ x_{n+1} = a_n x_0 + b_n x_n + c_n\left((1-\beta_n) y_n + \beta_n T_n z_n\right), \quad \forall n \geq 0, \end{cases} \tag{106}$$

where $T_n = (1-\gamma_n)I + \gamma_n T$ and $\gamma_n \subset (k, 1)$. If conditions $(a)-(d)$ are satisfied, then $\{x_n\}$ converges strongly to the point $u = P_\Gamma(x_0)$.

Proof. We have $F(J_r^B) = B^{-1}0$, $(\nabla f)^{-1}0 = \mathrm{VI}(H_1, \nabla f)$, and $P_{H_1} = I$; by Theorem 1, we obtain the desired result. □

If in Theorems 3 and 1 we assume that T is nonexpansive, then we have that T is strictly pseudocontractive with $k = 1$, and hence, we get the following corollaries.

Corollary 1. *Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A_i\colon H_1 \longrightarrow H_2$ be bounded linear operators for $i = 1, 2$, $S\colon C \longrightarrow C$ be a nonexpansive mapping, and $T\colon C \longrightarrow C$ be a nonexpansive mapping such that $\Gamma = F(T) \cap F(S) \cap_{i=1,2}^{2} \Omega_i \neq \varnothing$. Suppose $\{x_n\}$ and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = \gamma_n P_C\left(x_n - \lambda_n \nabla f_{1\alpha_n} x_n\right) \\ +(1-\gamma_n) P_C\left(x_n - \lambda_n \nabla f_{2s_n} x_n\right), \\ x_{n+1} = a_n x_n + b_n\left((1-\beta_n) z_n + \beta_n T z_n\right) \\ + c_n\left((1-\delta_n) z_n + \delta_n S z_n\right), \quad \forall n \geq 0. \end{cases} \tag{107}$$

If conditions $(a)-(d)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Corollary 2. *Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A\colon H_1 \longrightarrow H_2$ be a bounded linear operator, $S\colon C \longrightarrow C$ be a nonexpansive mapping, and $T\colon C \longrightarrow C$ be a nonexpansive mapping such that $\Gamma = F(T) \cap F(S) \cap \Omega \neq \varnothing$. Suppose that $\{x_n\}$, $\{v_n\}$, and $\{z_n\}$ are sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ v_n = x_n + \varepsilon_n (x_n - x_{n-1}), \\ z_n = P_C\left(I - \lambda_n \nabla f_{\alpha_n}\right) v_n, \\ x_{n+1} = a_n x_n + b_n\left((1-\beta_n) z_n + \beta_n T z_n\right) \\ + c_n S P_C\left(v_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \quad \forall n \geq 0. \end{cases} \tag{108}$$

If conditions $(a)-(e)$ are satisfied, then $\{x_n\}$ converges weakly to an element $u \in \Gamma$.

Corollary 3. *Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A\colon H_1 \longrightarrow H_2$ be a bounded linear operator, $S\colon C \longrightarrow C$ be a nonexpansive map, and $T\colon C \longrightarrow C$ be nonexpansive such that $\Gamma = F(T) \cap F(S) \cap \Omega \neq \varnothing$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C, \\ z_n = P_C\left(I - \lambda_n \nabla f_{\alpha_n}\right) x_n, \\ y_n = P_C\left(x_n - \lambda_n \nabla f_{\alpha_n} z_n\right), \\ x_{n+1} = a_n x_0 + b_n x_n + c_n\left((1-\beta_n) S y_n + \beta_n T_n z_n\right), \quad \forall n \geq 0, \end{cases} \tag{109}$$

where $T_n = (1-\gamma_n)I + \gamma_n T$ and $\gamma_n \subset (k, 1)$. If conditions $(a)-(d)$ are satisfied, then $\{x_n\}$ converges strongly to the point $u = P_\Gamma(x_0)$.

Data Availability

All data required for this paper are included within this paper.

Conflicts of Interest

The authors declare no conflicts of interest.

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