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Research Article

Hadamard-Type k-Fractional Integral Inequalities for Exponentially $(\alpha, h-m)$-Convex Functions

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The class of exponentially $(\alpha, h-m)$-convex functions has been discovered to unify different kinds of convexities. This paper finds new Riemann–Liouville fractional Hadamard-type inequalities for this generalized class of convex functions. These results further have their consequences which are presented in the form of corollaries. Moreover, some known results are identified in the form of remarks.

1. Introduction

Fractional calculus has opened a new era in the theoretic and application point of view. Many subjects of science and engineering get wonderful advancements in modeling and solving complex systems with the help of fractional integral and derivative operators. Besides making advancements in other fields, fractional integral and derivative operators have been proved very useful in generalizing and extending mathematical inequalities. In this paper, we will extend the Hadamard-type inequalities for Riemann–Liouville (RL) fractional integral operators.

Definition 1. Let $f \in L_1[a,b]$. The left and right RL fractional integrals $I_{a^+}^{\alpha} f$ and $I_{b^-}^{\alpha} f$ of order $\alpha \in \mathbb{C}$, $\mathfrak{R}(\alpha) > 0$, of f are defined by

$$
\begin{aligned}
I_{a^+}^{\alpha} f(x) &= \left(\frac{1}{\Gamma(\alpha)}\right) \int_a^x (x-t)^{1-\alpha} f(t)\mathrm{d}t, x > a, \\
I_{b^-}^{\alpha} f(x) &= \left(\frac{1}{\Gamma(\alpha)}\right) \int_x^b (t-x)^{1-\alpha} f(t)\mathrm{d}t, x < b,
\end{aligned}
\tag{1}
$$

respectively, where $\Gamma(\alpha)$ is the Euler's gamma function and $I_{a^+}^{0} f(x) = I_{b^-}^{0} f(x) = f(x)$.

The k-analogue RL fractional integral operators are studied by Mubeen and Habibullah in [1].

Definition 2. Let $f \in L_1[a,b]$. The left and right RL k-fractional integrals $I_{a^+}^{\alpha,k} f$ and $I_{b^-}^{\alpha,k} f$ of order $\alpha \in \mathbb{C}, \mathfrak{R}(\alpha) > 0, k > 0$, of f are defined by

$$
\begin{aligned}
I_{a^+}^{\alpha,k} f(x) &= \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{(\alpha/k)-1} f(t)\mathrm{d}t, \quad x > a, \\
I_{b^-}^{\alpha,k} f(x) &= \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{(\alpha/k)-1} f(t)\mathrm{d}t, \quad x < b,
\end{aligned}
\tag{2}
$$

where $\Gamma_k(\alpha)$ is the k-gamma function and $I_{a^+}^{0,1} f(x) = I_{b^-}^{0,1} f(x) = f(x)$.

Fractional integral inequalities are generalizations of classical inequalities which play a key role in the solutions and their uniqueness in fractional boundary value problems. The Hadamard inequality is one of the classical inequalities for convex functions, and it has been studied by a lot of researchers for different kinds of fractional integral and derivative operators, see [2–11] and references therein. It is stated in the undermentioned theorem.

Theorem 1 (see [11]). *If $f\colon I \longrightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$ then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

Definition 3. A function $f\colon I \longrightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, is said to be convex function if

$$f(rx + (1-r)y) \leq rf(x) + (1-r)f(y), \tag{4}$$

holds for all $x, y \in I$ and $r \in [0, 1]$.

The Hadamard inequality is the geometric interpretation of convex functions which has been analyzed by many researchers for fractional integral and differentiation operators. The objective of this paper is to obtain k-fractional integral inequalities for a generalized class of convex functions, namely, exponentially $(\alpha, h-m)$-convex functions. Convex functions proved very useful for the establishment of very known and vital inequalities. An important and significant generalization of convex functions is exponentially $(\alpha, h-m)$-convex functions. It is defined by He et al. in [12] as follows.

Definition 4. Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h\colon J \longrightarrow \mathbb{R}$ be a nonnegative function. A function $f\colon [0, b] \longrightarrow \mathbb{R}$ is called an exponentially $(\alpha, h-m)$-convex function, if f is nonnegative, and for all $x, y \in [0, b]$, $t \in (0, 1)$, $\alpha, m \in [0, 1]^2$, and $\eta \in \mathbb{R}$, one has

$$f(tx + m(1-t)y) \leq h(t^\alpha)\frac{f(x)}{e^{\eta x}} + mh(1-t^\alpha)\frac{f(y)}{e^{\eta y}}. \tag{5}$$

Remark 1. By selecting a suitable function h and a particular value of parameters m and η, the above definition produces the functions comprised in the following remarks:

(i) By setting $\eta = 0$, $(\alpha, h-m)$-convex function [4] can be obtained

(ii) By setting $\eta = 0$ and $\alpha = 1$, $(h-m)$-convex function [13] can be obtained

(iii) By setting $\eta = 0$ and $h(t^\alpha) = t^\alpha$, (α, m)-convex function [14] can be obtained

(iv) By setting $\eta = 0, \alpha = 1$, and $m = 1$, h-convex function [15] can be obtained

(v) By setting $\eta = 0$, $\alpha = 1$, and $h(t) = t$, m-convex function [16] can be obtained

(vi) By setting $\eta = 0, \alpha = 1, m = 1$, and $h(t) = t$, convex function [17] can be obtained

(vii) By setting $\eta = 0$, $m = 1$, $\alpha = 1$, and $h(t) = 1$, p-function [18] can be obtained

(viii) By setting $\alpha = 1$ and $h(t) = t^s$, exponentially (s, m)-convex function [19] can be obtained

(ix) By setting $\alpha = 1$, $m = 1$, and $h(t) = t^s$, exponentially s-convex function [20] can be obtained

(x) By setting $\alpha = 1$ and $h(t) = t$, exponentially m-convex function [21] can be obtained

(xi) By setting $\alpha = 1, m = 1$, and $h(t) = t$, exponentially convex function [22] can be obtained

(xii) By setting $\eta = 0$, $\alpha = 1$, and $h(t) = t^s$, (s, m)-convex function [23] can be obtained

(xiii) By setting $\alpha = 1$, $\eta = 0$, $m = 1$, and $h(t) = t^s$, s-convex function [20] can be obtained

(xiv) By setting $\alpha = 1$, $\eta = 0$, $m = 1$, and $h(t) = (1/t)$, Godunova–Levin function [24] can be obtained

(xv) By setting $\alpha = 1$, $\eta = 0$, $m = 1$, and $h(t) = (1/t^s)$, s-Godunova–Levin function of the second kind [25] can be obtained

The article is organized in the following manner: in Section 2, we prove the k-fractional integral inequality of Hadamard type for exponentially $(\alpha, h-m)$-convex functions and deduce some related results. In section 3, we prove a version of the k-fractional integral inequality of Hadamard type for differentiable function f so that $|f'|$ is exponentially $(\alpha, h-m)$-convex. In section 4, we prove the k-fractional integral inequality of Hadamard type for the product of two exponentially $(\alpha, h-m)$-convex functions.

2. Main Results

First, we give a k-fractional integral inequality for exponentially $(\alpha, h-m)$-convex functions as follows.

Theorem 2. *Let $f\colon [0, \infty) \longrightarrow \mathbb{R}$ be an exponentially $(\alpha, h-m)$-convex function with $(\alpha, m) \in [0, 1] \times (0, 1]$, also let $f \in L_1[a, b]$, $0 \leq a < mb$. Then, we will have*

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}} I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)\\
&\leq \left(\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{(\alpha/k)-1}h(t^\alpha)\mathrm{d}t + m\left(\frac{f(b/m)}{e^{\eta b/m}} + \frac{f(a/m)}{e^{\eta a/m}}\right)\int_0^1 t^{(\alpha/k)-1}h(1-t^\alpha)\mathrm{d}t \qquad (6)\\
&\leq \frac{1}{((\alpha/k)p - p + 1)^{1/p}}\left[\left(\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}}\right)\left(\int_0^1 (h(t^\alpha))^q\mathrm{d}t\right)^{1/q} + m\left(\frac{f(b/m)}{e^{\eta b/m}} + \frac{f(a/m)}{e^{\eta a/m}}\right)\left(\int_0^1 (h(t^\alpha))^q\mathrm{d}t\right)^{1/q}\right],
\end{aligned}$$

where $p^{-1}+q^{-1}=1, p>1,$ *and* $\eta\in\mathbb{R}$.

Proof. Since f is exponentially $(\alpha, h-m)$-convex on $[a,b]$, for $t\in[0,1]$, we have

$$f(ta+(1-t)b)+f((1-t)a+tb)\le h(t^\alpha)\left[\frac{f(a)}{e^{\eta a}}+\frac{f(b)}{e^{\eta b}}\right]+mh(1-t^\alpha)\left[\frac{f(b/m)}{e^{(\eta b/m)}}+\frac{f(a/m)}{e^{(\eta a/m)}}\right], \tag{7}$$

from which multiplying both sides with $t^{(\alpha/k)=-1}$ and integrating over $[0,1]$, we will have

$$\int_0^1 t^{(\alpha/k)-1}[f(ta+(1-t)b)+f((1-t)a+tb)]\mathrm{d}t \le\left[\frac{f(a)}{e^{\eta a}}+\frac{f(b)}{e^{\eta b}}\right]\int_0^1 t^{(\alpha/k)-1}h(t^\alpha)\mathrm{d}t +m\left[\frac{f(b/m)}{e^{\eta b/m}}+\frac{f(a/m)}{e^{\eta a/m}}\right]\int_0^1 t^{(\alpha/k)-1}h(1-t^\alpha)\mathrm{d}t. \tag{8}$$

By changing variables, we will have

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)\right] \le\left[\frac{f(a)}{e^{\eta a}}+\frac{f(b)}{e^{\eta b}}\right]\int_0^1 t^{(\alpha/k)-1}h(t^\alpha)\mathrm{d}t +m\left[\frac{f(b/m)}{e^{\eta b/m}}+\frac{f(a/m)}{e^{\eta a/m}}\right]\int_0^1 t^{(\alpha/k)-1}h(1-t^\alpha)\mathrm{d}t. \tag{9}$$

This completes the proof of first inequality in (6). The second inequality in (6) follows by using Holder's inequality:

$$\int_0^1 t^{(\alpha/k)-1}h(t^\alpha)\mathrm{d}t\le\frac{1}{((\alpha/k)p-p+1)^{(1/p)}}\left(\int_0^1 (h(t^\alpha))^q\mathrm{d}t\right)^{(1/q)},$$
$$\int_0^1 t^{(\alpha/k)-1}h(1-t^\alpha)\mathrm{d}t\le\frac{1}{((\alpha/k)p-p+1)^{(1/p)}}\left(\int_0^1 (h(1-t^\alpha))^q\mathrm{d}t\right)^{(1/q)}. \tag{10}$$

Thus from (9), we will get (6). □

Some special cases of the above theorem are discussed in the following corollaries.

Corollary 1. *The following inequality holds for an exponentially* $(\alpha, h-m)$*-convex function via an RL fractional integral by setting* $k=1$ *in inequality (6):*

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha}I_{a^+}^\alpha f(b)+I_{b^-}^\alpha f(a) \le\left(\frac{f(a)}{e^{\eta a}}+\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\alpha-1}h(t^\alpha)\mathrm{d}t+m\left(\frac{f(b/m)}{e^{(\eta b/m)}}+\frac{f(a/m)}{e^{(\eta a/m)}}\right)\int_0^1 t^{\alpha-1}h(1-t^\alpha)\mathrm{d}t \le\frac{1}{(\alpha p-p+1)^{1/p}}\left[\left(\frac{f(a)}{e^{\eta a}}+\frac{f(b)}{e^{\eta b}}\right)\left(\int_0^1 (h(t^\alpha))^q\mathrm{d}t\right)^{(1/q)}+m\left(\frac{f(b/m)}{e^{(\eta b/m)}}+\frac{f(a/m)}{e^{(\eta a/m)}}\right)\left(\int_0^1 (h(1-t^\alpha))^q\mathrm{d}t\right)^{(1/q)}\right]. \tag{11}$$

Corollary 2. *The following inequality holds for the* $(\alpha, h-m)$*-convex function via an RL k-fractional integral by setting* $\eta=0$ *in inequality (6):*

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{\alpha/k}}I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a) \le(f(a)+f(b))\int_0^1 t^{(\alpha/k)-1}h(t^\alpha)\mathrm{d}t+m\left(f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right)\int_0^1 t^{(\alpha/k)-1}h(1-t^\alpha)\mathrm{d}t \le\frac{1}{((\alpha/k)p-p+1)^{(1/p)}}\left[(f(a)+f(b))\left(\int_0^1 (h(t^\alpha))^q\mathrm{d}t\right)^{(1/q)}+m\left(f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right)\left(\int_0^1 (h(1-t^\alpha))^q\mathrm{d}t\right)^{(1/q)}\right]. \tag{12}$$

Corollary 3. *The following inequality holds for the* $(h-m)$*-convex functions via RL k-fractional integrals by setting* $\eta=0$ *and* $\alpha=1$ *in inequality (6):*

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}} I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)$$
$$\leq (f(a)+f(b)) \int_0^1 t^{(\alpha/k)-1} h(t)\mathrm{d}t + m\left[f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right]\int_0^1 t^{(\alpha/k)-1} h(1-t)\mathrm{d}t \tag{13}$$
$$\leq \frac{1}{((\alpha/k)p-p+1)^{(1/p)}}\left(\int_0^1 (h(t))^q \mathrm{d}t\right)^{(1/q)}\left[(f(a)+f(b)) + m\left(f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right)\right].$$

Corollary 4. *The following inequality holds for the h-convex function via RL fractional integrals by setting $\eta = 0$, $k = 1$, $m = 1$, and $\alpha = 1$ in inequality (6):*

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[I_{a^+}^{\alpha} f(b) + I_{b^-}^{\alpha} f(a)\right]$$
$$\leq [f(a)+f(b)]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]\mathrm{d}t \leq \frac{2[f(a)+f(b)]}{(\alpha p - p+1)^{(1/p)}}\left(\int_0^1 (h(t))^q \mathrm{d}t\right)^{(1/q)}. \tag{14}$$

The next two special cases are already proved in [26].

Corollary 5. *The following inequality holds for the exponentially (s, m)-convex function via RL k-fractional integrals:*

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)\right]$$
$$\leq \left[\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}}\right]\frac{k}{\alpha+ks} + m\left[\frac{f(b/m)}{e^{(\eta b/m)}} + \frac{f(a/m)}{e^{\eta a/m}}\right]\beta\left(\frac{\alpha}{k}, s+1\right) \tag{15}$$
$$\leq \left[\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}}\right]\frac{k}{\alpha+ks} + \left[\frac{f(b/m)}{e^{(\eta b/m)}} + \frac{f(a/m)}{e^{\eta a/m}}\right]\frac{m}{((\alpha/k)p-p+1)^{(1/p)}(qs+1)^{1/q}}.$$

Proof. By setting $\alpha = 1$ and $h(1/2) = (1/2)^s$ in inequality (6) of Theorem 2, we get the above inequality (15) which is given in [26] (Theorem 1). □

Corollary 6. *The following inequality holds for the (s, m)-convex function via RL k-fractional integrals:*

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)\right]$$
$$\leq [f(a)+f(b)]\frac{k}{\alpha+ks} + m\left[f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right]\beta\left(\frac{\alpha}{k}, s+1\right) \tag{16}$$
$$\leq [f(a)+f(b)]\frac{k}{\alpha+ks} + \left[f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right]\frac{m}{((\alpha/k)p-p+1)^{(1/p)}(qs+1)^{(1/q)}}.$$

Proof. By setting $\eta = 0$, $\alpha = 1$, and $h(1/2) = (1/2)^s$ in inequality (6) of Theorem 2, we get the above inequality (16) which is given in [26] (Theorem 4). □

The next result holds for exponentially $(\alpha, h-m)$-convex functions, whereas the function h is superadditive. □

Theorem 3. *Let $f\colon [0,\infty) \longrightarrow \mathbb{R}$ be exponentially $(\alpha, h-m)$-convex functions with $(\alpha, m) \in [0,1] \times (0,1]$, also let h be superadditive and $f \in L_1[a,b]$, $0 \le a < mb$. Then, for RL k-fractional integrals, we have*

$$\frac{\Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)\right] \le \frac{h(1)}{2}\left[\left(\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}}\right) + m\left(\frac{f(b/m)}{e^{(\eta b/m)}} + \frac{f(a/m)}{e^{(\eta a/m)}}\right)\right], \eta \in \mathbb{R}. \tag{17}$$

Proof. Since f is an exponentially $(\alpha, h-m)$-convex function on $[a,b]$, for $t \in [0,1]$, we get

$$f(ta+(1-t)b) + f((1-t)a+tb) \le \frac{[h(t^\alpha) + h(1-t^\alpha)]}{2}\left[\left(\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}}\right) + m\left(\frac{f(b/m)}{e^{(\eta b/m)}} + \frac{f(a/m)}{e^{(\eta a/m)}}\right)\right], \tag{18}$$

since h is superadditive; therefore,

$$f(ta+(1-t)b) + f((1-t)a+tb) \le \frac{h(1)}{2}\left[\left(\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}}\right) + m\left(\frac{f(b/m)}{e^{(\eta b/m)}} + \frac{f(a/m)}{e^{(\eta a/m)}}\right)\right]. \tag{19}$$

By multiplying both sides of the above inequality with $t^{(\alpha/k)-1}$ and integrating over $[0,1]$, we will have

$$\int_0^1 t^{(\alpha/k)-1}[f(ta+(1-t)b) + f((1-t)a+tb)]\mathrm{d}t \le \frac{h(1)}{2}\left[\left(\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}}\right) + m\left(\frac{f(b/m)}{e^{(\eta b/m)}} + \frac{f(a/m)}{e^{(\eta a/m)}}\right)\right]\int_0^1 t^{(\alpha/k)-1}\mathrm{d}t. \tag{20}$$

By substituting $w = ta + (1-t)b$ in the left side of the above inequality, it leads to (17). □

Some special cases are given in the form of following corollaries.

Corollary 7. *The following inequality holds for the exponentially $(\alpha, h-m)$-convex function via RL fractional integrals by setting $k=1$ in inequality (17):*

$$\frac{\Gamma_k(\alpha+k)}{(b-a)^{\alpha/k}}\left[I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)\right] \le \frac{h(1)}{2}\left[\left(\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}}\right) + m\left(\frac{f(b/m)}{e^{(\eta b/m)}} + \frac{f(a/m)}{e^{(\eta a/m)}}\right)\right]. \tag{21}$$

Corollary 8. *The following inequality holds for the $(\alpha, h-m)$-convex function via RL k-fractional integrals by setting $\eta = 0$ in inequality (17):*

$$\frac{\Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)\right] \le \frac{h(1)}{2}\left[(f(a)+f(b)) + m\left(f\left(\frac{b}{m}\right) + f\left(\frac{a}{m}\right)\right)\right]. \tag{22}$$

Corollary 9. *The following inequality holds for $(h-m)$-convex functions via RL k-fractional integrals:*

$$\frac{\Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)\right] \le h(1)\left[\frac{f(a)+f(b)}{2} + m\left(\frac{f(b/m)+f(a/m)}{2}\right)\right]. \tag{23}$$

Proof. By setting $\eta = 0$ and $\alpha = 1$ in inequality (17) of Theorem 3, we get the above inequality (22) which is given in [4] (Theorem 2.6). □

3. Fractional Hadamard-Type Inequalities for Functions Whose Derivatives in Absolute Values are Exponentially $(\alpha, h-m)$-Convex

In the following, k-fractional integral inequalities of Hadamard type for exponentially $(\alpha, h-m)$-convex functions in terms of the first derivatives have been obtained. For the next result, we use the following lemma.

Lemma 1 (see [4]). *Let $f\colon [a, mb] \longrightarrow \mathbb{R}$ be a differentiable mapping on the interval (a, mb) with $a < mb$. If $f' \in L_1[a, mb]$, then for k-fractional integrals, we will have*

$$\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k((\alpha/k)+k)}{2(mb-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)\right]=\frac{mb-a}{2}\int_0^1\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]f'(m(1-t)b+ta)\mathrm{d}t. \tag{24}$$

Theorem 4. *Let $f\colon [a, mb] \longrightarrow \mathbb{R}$ be a function such that $[a, mb] \subseteq [0, \infty)$ and $f \in L_1[a, mb]$. If $|f'|$ is exponentially $(\alpha, h-m)$-convex functions with $(\alpha, m) \in [0,1]^2$ and $h^q \in L_1[0,1]$, $q > 1$. Then, for RL k-fractional integrals, we have*

$$\begin{aligned}&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k((\alpha/k)+k)}{2(mb-a)^{\alpha/k}}\left[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)\right]\right|\\&\leq\frac{mb-a}{2}\left(\left[\frac{2^{(\alpha/k)p+1}-1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}-\left[\frac{1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}\right)\\&\times\left[\frac{|f'(a)|}{e^{\eta a}}\left(\left[\int_0^{(1/2)}(h(t^\alpha))^q\mathrm{d}t\right]^{(1/q)}+\left[\int_{(1/2)}^1(h(t^\alpha))^q\mathrm{d}t\right]^{(1/q)}\right)+m\frac{|f'(b)|}{e^{\eta b}}\left(\left[\int_0^{(1/2)}(h(1-t^\alpha))^q\mathrm{d}t\right]^{(1/q)}+\left[\int_{(1/2)}^1(h(1-t^\alpha))^q\mathrm{d}t\right]^{(1/q)}\right),\end{aligned} \tag{25}$$

where $(1/p)+(1/q)=1, \eta \in \mathbb{R}$.

Proof. By using the property of modulus from Lemma 1, we will get

$$\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k((\alpha/k)+k)}{2(mb-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)\right]\right|\leq\frac{mb-a}{2}\int_0^1\left|(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right|\left|f'(m(1-t)b+ta)\right|\mathrm{d}t. \tag{26}$$

By exponentially $(\alpha, h-m)$-convexity of $|f'|$, we have

$$\begin{aligned}&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k((\alpha/k)+k)}{2(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)\right]\right|\\&\leq\frac{mb-a}{2}\int_0^{(1/2)}\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\left[mh(1-t^\alpha)\frac{|f'(b)|}{e^{\eta b}}+h(t^\alpha)\frac{|f'(a)|}{e^{\eta a}}\right]\mathrm{d}t\\&+\int_{(1/2)}^0\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\left[mh(1-t^\alpha)\frac{|f'(b)|}{e^{\eta b}}+h(t^\alpha)\frac{|f'(a)|}{e^{\eta a}}\right]\mathrm{d}t\\&=\frac{mb-a}{2}\left\{\frac{|f'(a)|}{e^{\eta a}}\left[\int_0^{(1/2)}(1-t)^{(\alpha/k)}h(t^\alpha)\mathrm{d}t-\int_0^{(1/2)}t^{(\alpha/k)}h(t^\alpha)\mathrm{d}t\right]\right.\\&+m\frac{|f'(b)|}{e^{\eta b}}\left[\int_0^{(1/2)}(1-t)^{(\alpha/k)}h(1-t^\alpha)\mathrm{d}t-\int_0^{(1/2)}t^{(\alpha/k)}h(1-t^\alpha)\mathrm{d}t\right]\\&+\frac{|f'(a)|}{e^{\eta a}}\left[\int_{(1/2)}^1t^{(\alpha/k)}h(t^\alpha)\mathrm{d}t-\int_{(1/2)}^1(1-t)^{(\alpha/k)}h(t^\alpha)\mathrm{d}t\right]\\&\left.+m\frac{|f'(b)|}{e^{\eta b}}\left[\int_{(1/2)}^1t^{(\alpha/k)}h(1-t)^{(\alpha/k)}\mathrm{d}t-\int_{(1/2)}^1(1-t)^{(\alpha/k)}h(1-t^\alpha)\mathrm{d}t\right]\right\}.\end{aligned} \tag{27}$$

Now, by using Holder inequality in the right-hand side of (27), we will get

$$\begin{aligned}&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k((\alpha/k)+k)}{2(mb-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)\right]\right|\\&\leq\frac{mb-a}{2}\left\{\frac{|f'(a)|}{e^{\eta a}}\left[\left(\left[\frac{2^{(\alpha/k)p+1}-1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}-\left[\frac{1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}\right)\left[\int_0^{(1/2)}[h(t^\alpha)]^q\mathrm{d}t\right]^{(1/q)}\right.\right.\\&+\left.\left(\left[\frac{2^{(\alpha/k)p+1}-1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}-\left[\frac{1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}\right)\left[\int_{(1/2)}^1[h(t^\alpha)]^q\mathrm{d}t\right]^{(1/q)}\right]\\&+m\frac{|f'(b)|}{e^{\eta b}}\left[\left(\left[\frac{2^{(\alpha/k)p+1}-1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}-\left[\frac{1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}\right)\left[\int_0^{(1/2)}[h(1-t^\alpha)]^q\mathrm{d}t\right]^{(1/q)}\right.\\&+\left.\left.\left(\left[\frac{2^{(\alpha/k)p+1}-1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{\frac{1}{p}}\right)\left[\int_{\frac{1}{2}}^1[h(1-t^\alpha)]^q\mathrm{d}t\right]^{\frac{1}{q}}\right]\right\}.\end{aligned}\tag{28}$$

By some manipulation, one can get inequality (25). □

Corollary 10. *The following inequality holds for the exponentially $(\alpha, h-m)$-convex function via RL fractional integrals by setting $k=1$ in inequality (25):*

$$\begin{aligned}&\left|\frac{f(mb)+f(a)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left[I_{a^+}^\alpha f(mb)+I_{mb^-}^\alpha f(a)\right]\right|\\&\leq\frac{mb-a}{2}\left(\left[\frac{2^{\alpha p+1}-1}{2^{\alpha p+1}(\alpha p+1)}\right]^{(1/p)}-\left[\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right]^{(1/p)}\right)\\&\times\left[\frac{|f'(a)|}{e^{\eta a}}\left(\left[\int_0^{(1/2)}(h(t^\alpha))^q\mathrm{d}t\right]^{(1/q)}+\left[\int_{(1/2)}^1(h(1-t^\alpha))^q\mathrm{d}t\right]^{(1/q)}\right)\right.\\&+\left.m\frac{|f'(b)|}{e^{\eta b}}\left(\left[\int_0^{(1/2)}(h(1-t^\alpha))^q\mathrm{d}t\right]^{(1/q)}+\left[\int_{(1/2)}^1(h(1-t^\alpha))^q\mathrm{d}t\right]^{(1/q)}\right)\right].\end{aligned}\tag{29}$$

Corollary 11. *The following inequality holds for the $(\alpha, h-m)$-convex function via RL k-fractional integrals by setting $\eta=0$ in inequality (25):*

$$\begin{aligned}&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k((\alpha/k)+k)}{2(mb-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)\right]\right|\\&\leq\frac{mb-a}{2}\left(\left[\frac{2^{(\alpha/k)p+1}-1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}-\left[\frac{1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}\right)\\&\times\left[|f'(a)|\left(\left[\int_0^{(1/2)}(h(t^\alpha)^q)\mathrm{d}t\right]^{(1/q)}+\left[\int_{(1/2)}^1(h(t^\alpha)^q)\mathrm{d}t\right]^{(1/q)}\right)\right.\\&+\left.m|f'(a)|\left(\left[\int_0^{\frac{1}{2}}(h(1-t^\alpha)^q)\mathrm{d}t\right]^{(1/q)}+\left[\int_{\frac{1}{2}}^1(h(1-t^\alpha)^q)\mathrm{d}t\right]^{(1/q)}\right)\right].\end{aligned}\tag{30}$$

The following special cases are proved in [4, 26].

Corollary 12. *The following inequality holds for the $(h-m)$-convex function via RL k-fractional integrals:*

$$\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k((\alpha/k)+k)}{2(mb-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)\right]\right| \leq \frac{mb-a\left[\left|f'(a)\right|+m\left|f'(b)\right|\right]}{2}\left(\left[\frac{2^{(\alpha/k)p+1}-1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}-\left[\frac{1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}\right) \times\left[\left(\left[\int_0^{(1/2)}(h(t^\alpha)^q)\mathrm{d}t\right]^{(1/q)}+\left[\int_{(1/2)}^1(h(t^\alpha)^q)\mathrm{d}t\right]^{(1/q)}\right)\right]. \quad (31)$$

Proof. By setting $\eta=0$ and $\alpha=1$ in inequality (25) of Theorem 4, we get the above inequality (31) which is given in [4] (Theorem 3.6). □

Corollary 13. *The following inequality holds for the exponentially (s,m)-convex function via RL k-fractional integrals:*

$$\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k((\alpha/k)+k)}{2(mb-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)\right]\right| \leq \frac{(mb-a)\left(m\left|f'(b)\right|/e^{\eta b}+\left|f'(a)\right|/e^{\eta a}\right)}{2}\left\{\left[\frac{2^{(\alpha/k)+s+1}-2}{2^{(\alpha/k)+s+1}((\alpha/k)+s+1)}-\left[\frac{1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}\right.\right. \left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{(1/q)}+\left[\frac{2^{(\alpha/k)p+1}-1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{(1/q)}. \quad (32)$$

Proof. By setting $\alpha=1$ and $h(t)=t^s$ in inequality (25) of Theorem 4, we get the above inequality (32) which is given in [26] (Theorem 2). □

Corollary 14. *The following inequality holds for the exponentially (s,m)-convex function via RL k-fractional integrals:*

$$\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k((\alpha/k)+k)}{2(mb-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)\right]\right| \leq \frac{(mb-a)\left(m\left|f'(b)\right|-\left|f'(a)\right|\right)}{2}\left\{\left[\frac{2^{(\alpha/k)+s+1}-2}{2^{(\alpha/k)+s+1}((\alpha/k)+s+1)}\right.\right. -\left[\frac{1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}\left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{(1/q)}+\left[\frac{2^{(\alpha/k)p+1}-1}{2^{(\alpha/k)p+1}((\alpha/k)p+1)}\right]^{(1/p)}\left.\left.\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{(1/q,)}\right]\right\}. \quad (33)$$

Proof. By setting $\eta=0$, $\alpha=1$, and $h(t)=t^s$ in inequality (25) of Theorem 4, we get the above inequality (33) which is given in [26] (Theorem 5). □

4. Fractional Hadamard-Type Inequalities for Product of Two Exponentially $(\alpha, h-m)$-Convex Functions

Now, we obtain some Hadamard-type inequalities for the product of two exponentially $(\alpha, h-m)$-convex functions via RL k-fractional integrals.

Theorem 5. *Let $f, g\colon [0,\infty) \longrightarrow \mathbb{R}$ be functions such that $fg \in L_1[a,b]$, $0 \le a < mb$. If function f is exponentially $(\alpha, h-m_1)$-convex and function g is exponentially $(\alpha, h-m_2)$-convex on $[0,\infty)$ with $(\alpha, m_1), (\alpha, m_2) \in [0,1)^2$, then the following inequalities hold for RL k-fractional integrals:*

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k} f(b)g(b) + I_{b^-}^{\alpha,k} f(a)g(a)\right] \le \left[\frac{f(a)g(a)}{e^{2\eta a}} + \frac{f(b)g(b)}{e^{2\eta b}}\right]\int_0^1 t^{(\alpha/k)-1}h^2(t^\alpha)\mathrm{d}t \\
&+\left\{m_2\left[\frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} + \frac{f(b)g(a/m_2)}{e^{\eta(b+(a/m_2))}}\right] + m_1\left[\frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}} + \frac{g(b)f((a/m_1))}{e^{\eta(b+(a/m_1))}}\right]\right\} \\
&\times \int_0^1 t^{(\alpha/k)-1}h(t^\alpha)h(1-t^\alpha)\mathrm{d}t \\
&+ m_1m_2\left[\frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1m_2)}} + \frac{f(a/m_1)g(a/m_2)}{e^{\eta a(m_1+m_2/m_1m_2)}}\right]\int_0^1 t^{(\alpha/k)-1}h^2(1-t^\alpha)\mathrm{d}t \\
&\le \frac{1}{((\alpha/k)p-p+1)^{(1/p)}}\left\{\left(\int_0^1 h^{2q}(t^\alpha)\mathrm{d}t\right)^{(1/q)}\left[\frac{f(a)g(a)}{e^{2\eta a}} + \frac{f(b)g(b)}{e^{2\eta b}}\right]\right. \\
&+\left[m_2\frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} + m_1\frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}} + m_2\frac{f(b)g(a/m_2)}{e^{\eta(b+(a/m_2))}} + m_1\frac{g(b)f(a/m_1)}{e^{\eta(b+(a/m_1))}}\right] \\
&\times\left(\int_0^1 (h(t^\alpha)h(1-t^\alpha))^q\mathrm{d}t\right)^{(1/q)} \\
&\left.+\left(\int_0^1 h^{2q}(1-t^\alpha)\mathrm{d}t\right)^{(1/q)}\left[m_1m_2\frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1m_2)}} + \frac{f(a/m_1)g(a/m_2)}{e^{\eta a(m_1+m_2/m_1m_2)}}\right]\right\},
\end{aligned} \tag{34}$$

where $p > 1$ and $(1/p) + (1/q) = 1$.

Proof. Since function f is exponentially $(\alpha, h-m_1)$-convex and function g is exponentially $(\alpha, h-m_2)$-convex, for $t \in [0,1]$, we have

$$\begin{aligned}
f(ta+(1-t)b)g(ta+(1-t)b) &\le h^2(t^\alpha)\frac{f(a)g(a)}{e^{2\eta a}} + m_2h(t^\alpha)h(1-t^\alpha)\frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} \\
&+ m_1h(t^\alpha)h(1-t^\alpha)\frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}} + m_1m_2h^2(1-t^\alpha)\frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1m_2)}},
\end{aligned} \tag{35}$$

by multiplying both sides of the above inequality with $t^{(\alpha/k)-1}$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{(\alpha/k)-1} f(ta+(1-t)b)g(ta+(1-t)b)\mathrm{d}t \\
&\le \frac{f(a)g(a)}{e^{2\eta a}}\int_0^1 t^{(\alpha/k)-1}h^2(t^\alpha)\mathrm{d}t + m_2\frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}}\int_0^1 t^{(\alpha/k)-1}h(t^\alpha)h(1-t^\alpha)\mathrm{dtn} \\
&+ m_1\frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}}\int_0^1 t^{(\alpha/k)-1}h(t^\alpha)h(1-t^\alpha)\mathrm{d}t + m_1m_2\frac{f(b/m_2)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1m_2)}}\int_0^1 t^{(\alpha/k)-1}h^2(1-t^\alpha)\mathrm{d}t.
\end{aligned} \tag{36}$$

By substituting $z = ta + (1-t)b$ in the left-hand side of the above inequality, we get

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}} I_{a^+}^{\alpha,k} f(b)g(b) \\
&\leq \frac{f(a)g(a)}{e^{2\eta a}} \int_0^1 t^{(\alpha/k)-1} h^2(t^\alpha)\mathrm{d}t + m_2 \frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} \int_0^1 t^{(\alpha/k)-1} h(t^\alpha)h(1-t^\alpha)\mathrm{d}t \\
&+ m_1 \frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}} \int_0^1 t^{(\alpha/k)-1} h(t^\alpha)h(1-t^\alpha)\mathrm{d}t + m_1 m_2 \frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1 m_2)}} \int_0^1 t^{(\alpha/k)-1} h^2(1-t^\alpha)\mathrm{d}t.
\end{aligned} \tag{37}$$

By using Holder inequality, we will have

$$\begin{aligned}
\int_0^1 t^{(\alpha/k)-1} h^2(t^\alpha)\mathrm{d}t &\leq \frac{1}{((\alpha/k)p - p + 1)^{(1/p)}} \left(\int_0^1 h^{2q}(t^\alpha)\mathrm{d}t \right)^{(1/q)}, \\
\int_0^1 t^{(\alpha/k)-1} h(t^\alpha)h(1-t^\alpha)\mathrm{d}t &\leq \frac{1}{((\alpha/k)p - p + 1)^{(1/p)}} \left(\int_0^1 (h(t^\alpha)h(1-t^\alpha))^q \mathrm{d}t \right)^{(1/q)}, \\
\int_0^1 t^{(\alpha/k)-1} h^2(1-t^\alpha)\mathrm{d}t &\leq \frac{1}{((\alpha/k)p - p + 1)^{(1/p)}} \left(\int_0^1 h^{2q}(1-t^\alpha)\mathrm{d}t \right)^{(1/q)}.
\end{aligned} \tag{38}$$

Thus, we will get

$$\begin{aligned}
\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}} I_{a^+}^{\alpha,k} f(b)g(b) &\leq \frac{f(a)g(a)}{e^{2\eta a}} \int_0^1 t^{(\alpha/k)-1} h^2(t^\alpha)\mathrm{d}t + m_2 \frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} \int_0^1 t^{(\alpha/k)-1} h(t^\alpha)h(1-t^\alpha)\mathrm{d}t \\
&+ m_1 \frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_2))}} \int_0^1 t^{(\alpha/k)-1} h(t^\alpha)h(1-t^\alpha)\mathrm{d}t + m_1 m_2 \frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1 m_2)}} \int_0^1 t^{(\alpha/k)-1} h^2(1-t^\alpha)\mathrm{d}t \\
&\leq \frac{1}{((\alpha/k)p - p + 1)^{(1/p)}} \left\{ \left(\int_0^1 h^{2q}(t^\alpha)\mathrm{d}t \right)^{(1/p)} \frac{f(a)g(a)}{e^{2\eta a}} \right. \\
&+ \left(\int_0^1 (h(t^\alpha)h(1-t^\alpha))^q \mathrm{d}t \right)^{(1/q)} \left[m_2 \frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} + m_1 \frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}} \right] \\
&\left. + \left(\int_0^1 h^{2q}(1-t^\alpha)\mathrm{d}t \right)^{(1/q)} m_1 m_2 \frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1 m_2)}} \right\}.
\end{aligned} \tag{39}$$

Similarly, by changing the roles of a and b, after a little computation, one can get

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}} I_{b^-}^{\alpha,k} f(a)g(a) \leq \frac{f(b)g(b)}{e^{2\eta b}} \int_0^1 t^{(\alpha/k)-1} h^2(t^\alpha)\mathrm{d}t + m_2 \frac{f(b)g(a/m_2)}{e^{\eta(b+(a/m_2))}} \int_0^1 t^{(\alpha/k)-1} h(t^\alpha)h(1-t^\alpha)\mathrm{d}t$$

$$+ m_1 \frac{g(b)f(a/m_1)}{e^{\eta(b+(a/m_1))}} \int_0^1 t^{(\alpha/k)-1} h(t^\alpha)h(1-t^\alpha)\mathrm{d}t + m_1 m_2 \frac{f(a/m_1)g(a/m_2)}{e^{\eta a(m_1+m_2/m_1 m_2)}} \int_0^1 t^{(\alpha/k)-1} h^2(1-t^\alpha)\mathrm{d}t$$

$$\leq \frac{1}{((\alpha/k)p-p+1))^{(1/p)}} \left\{ \left(\int_0^1 h^{2q}(t^\alpha)\mathrm{d}t \right)^{(1/q)} \frac{f(b)g(b)}{e^{2\eta b}} + \left(\int_0^1 (h(t^\alpha)h(1-t^\alpha))^q \mathrm{d}t \right)^{(1/q)} \right.$$

$$\left. \cdot \left[m_2 \frac{f(b)g(a/m_2)}{e^{\eta(b+(a/m_2))}} + m_1 \frac{g(b)f(a/m_1)}{e^{\eta(b+(a/m_1))}} \right] + \left(\int_0^1 h^{2q}(1-t^\alpha)\mathrm{d}t \right)^{(1/q)} m_1 m_2 \frac{f(a/m_1)g(a/m_2)}{e^{\eta a(m_1+m_2/m_1 m_2)}}. \right. \tag{40}$$

Adding (39) and (40), we get the required result. □

Corollary 15. *The following inequality holds for the exponentially $(\alpha, h-m)$-convex function via RL fractional integrals:*

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[I_{a^+}^\alpha f(b)g(b) + I_{b^-}^\alpha f(a)g(a)\right] \leq \left[\frac{f(a)g(a)}{e^{2\eta a}} + \frac{f(b)g(b)}{e^{2\eta b}}\right] \int_0^1 t^{\alpha-1} h^2(t^\alpha)\mathrm{d}t$$

$$+ \left\{ m_2 \left[\frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} + \frac{f(b)g(a/m_2)}{e^{\eta(b+(a/m_2))}} \right] + m_1 \left[\frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}} + \frac{g(b)f(a/m_1)}{e^{\eta(b+(a/m_1))}} \right] \right\}$$

$$\times \int_0^1 t^{\alpha-1} h(t^\alpha)h(1-t^\alpha)\mathrm{d}t + m_1 m_2 \left[\frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1 m_2)}} + \frac{f(a/m_1)g(a/m_2)}{e^{\eta a(m_1+m_2/m_1 m_2)}} \right]$$

$$\cdot \int_0^1 t^{\alpha-1} h^2(1-t^\alpha)\mathrm{d}t$$

$$\leq \frac{1}{((\alpha/k)p-p+1)^{(1/p)}} \left\{ \left(\int_0^1 h^{2q}(t^\alpha)\mathrm{d}t \right)^{(1/q)} \left[\frac{f(a)g(a)}{e^{2\eta a}} + \frac{f(b)g(b)}{e^{2\eta b}} \right] \right.$$

$$+ \left[m_2 \frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} + m_1 \frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}} + m_2 \frac{f(b)g(a/m_2)}{e^{\eta(b+(a/m_2))}} + m_1 \frac{g(b)f(a/m_1)}{e^{\eta(b+(a/m_1))}} \right]$$

$$\times \left(\int_0^1 h((t^\alpha)h(1-t^\alpha))^q \mathrm{d}t \right)^{(1/q)}$$

$$\left. + \left(\int_0^1 h^{2q}(1-t^\alpha)\mathrm{d}t \right)^{(1/q)} \left[m_1 m_2 \frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1 m_2)}} + \frac{f(a/m_1)g(a/m_2)}{e^{\eta a(m_1+m_2/m_1 m_2)}} \right] \right\}. \tag{41}$$

Proof. By setting $k=1$ in inequality (34) of Theorem 5, we get the above inequality (41). □

Corollary 16. *The following inequality holds for the $(\alpha, h-m)$-convex function via RL k-fractional integrals:*

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(b)g(b)+I_{b^-}^{\alpha,k}f(a)g(a)\right]\leq[f(a)g(a)+f(b)g(b)]\int_0^1 t^{(\alpha/k)-1}h^2(t^\alpha)\mathrm{d}t$$
$$+\left\{m_2\left[f(a)g\left(\frac{b}{m_2}\right)+f(b)g\left(\frac{a}{m_2}\right)\right]+m_1\left[g(a)f\left(\frac{b}{m_1}\right)+g(b)f\left(\frac{a}{m_1}\right)\right]\right\}$$
$$\times\int_0^1 t^{(\alpha/k)-1}h(t^\alpha)h(1-t^\alpha)\mathrm{d}t+m_1m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]$$
$$\cdot\int_0^1 t^{(\alpha/k)-1}h^2(1-t^\alpha)\mathrm{d}t$$
$$\leq\frac{1}{((\alpha/k)p-p+1)^{(1/p)}}\left\{\left(\int_0^1 h^{2q}(t^\alpha)\mathrm{d}t\right)^{(1/q)}[f(a)g(a)]\right.$$
$$+\left[m_2f(a)g\left(\frac{b}{m_2}\right)+m_1g(a)f\left(\frac{b}{m_1}\right)+m_2f(b)g\left(\frac{a}{m_2}\right)+m_1g(b)f\left(\frac{a}{m_1}\right)\right]$$
$$\times\left(\int_0^1(h(t^\alpha)h(1-t^\alpha))^q\mathrm{d}t\right)^{(1/q)}$$
$$\left.+\left(\int_0^1 h^{2q}(1-t^\alpha)\mathrm{d}t\right)^{(1/q)}\left[m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\right\}. \tag{42}$$

Proof. By setting $\eta=0$ in inequality (34) of Theorem 5, we get the above inequality (42). □

Corollary 17. *The following inequality holds for the $(h-m)$-convex function via RL k-fractional integrals:*

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(b)g(b)+I_{b^-}^{\alpha,k}f(a)g(a)\right]\leq[f(a)g(a)+f(b)g(b)]\int_0^1 t^{(\alpha/k)-1}h^2(t)\mathrm{d}t$$
$$+\left\{m_2\left[f(a)g\left(\frac{b}{m_2}\right)+f(b)g\left(\frac{a}{m_2}\right)\right]+m_1\left[g(a)f\left(\frac{b}{m_1}\right)+g(b)f\left(\frac{a}{m_1}\right)\right]\right\}$$
$$\times\int_0^1 t^{(\alpha/k)-1}h(t)h(1-t)\mathrm{d}t$$
$$+m_1m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\int_0^1 t^{(\alpha/k)-1}h^2(1-t)\mathrm{d}t$$
$$\leq\frac{1}{((\alpha/k)p-p+1)^{(1/p)}}\left\{\left(\int_0^1 h^{2q}(t)\mathrm{d}t\right)^{(1/q)}\left[f(a)g(a)+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right.\right.$$
$$\left.+f(b)g(b)+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right]+\left(\int_0^1(h(t)h(1-t))^q\mathrm{d}t\right)^{(1/q)}$$
$$\left.\cdot\left[m_2f(a)g\left(\frac{b}{m_2}\right)+m_1g(a)f\left(\frac{b}{m_1}\right)+m_2f(b)g\left(\frac{a}{m_2}\right)+m_1g(b)f\left(\frac{a}{m_1}\right)\right]\right\}. \tag{43}$$

Proof. By setting $\eta = 0$ and $\alpha = 1$ in inequality (34) of Theorem 5, we get the above inequality (43) which is given in [4] (Theorem 4.3). □

Corollary 18. *The following inequality holds for the exponentially (s, m)-convex function via RL k-fractional integrals:*

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k} f(b)g(b) + I_{b^-}^{\alpha,k} f(a)g(a)\right] \\
&\leq \left(\frac{k}{\alpha+2ks}\right)\left[\frac{f(a)g(a)}{e^{2\eta a}} + \frac{f(b)g(b)}{e^{2\eta b}}\right] + \left[m_1\left(\frac{g(b)f(a/m_1)}{e^{\eta(b+(a/m_1))}} + \frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}}\right)\right. \\
&\left.+ m_2\left(\frac{f(b)g(a/m_2)}{e^{\eta(b+(a/m_2))}} + \frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}}\right)\right]\beta((\alpha/k)+s, s+1) \\
&+ m_1 m_2\left[\frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1 m_2)}} + \frac{f(a/m_1)g(a/m_2)}{e^{\eta a(m_1+m_2/m_1 m_2)}}\right]\beta((\alpha/k), 2s+1) \\
&\leq \left(\frac{k}{\alpha+2ks}\right)\left[\frac{f(a)g(a)}{e^{2\eta a}} + \frac{f(b)g(b)}{e^{2\eta b}}\right] + \left[m_1\left(\frac{g(b)f(a/m_1)}{e^{\eta(b+(a/m_1))}} + \frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}}\right)\right. \\
&\left.+ m_2\left(\frac{f(b)g(a/m_2)}{e^{\eta(b+(a/m_2))}} + \frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}}\right)\right] \times \left(\frac{k}{p(\alpha-k+ks)+k}\right)^{(1/p)}\left(\frac{1}{qs+1}\right)^{(1/q)} \\
&+ m_1 m_2\left[\frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1 m_2)}} + \frac{f(a/m_1)g(a/m_2)}{e^{\eta a(m_1+m_2/m_1 m_2)}}\right]\left(\frac{k}{\alpha p - kp + k}\right)^{(1/p)}\left(\frac{1}{2qs+1}\right)^{(1/q)}.
\end{aligned} \tag{44}$$

Proof. By setting $\alpha = 1$ and $h(t) = t^s$ in inequality (34) of Theorem 5, we get the above inequality (44) which is given in [26] (Theorem 3). □

Corollary 19. *The following inequality holds for the (s, m)-convex function via RL k-fractional integrals:*

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k} f(b)g(b) + I_{b^-}^{\alpha,k} f(a)g(a)\right] \\
&\leq \left(\frac{k}{\alpha+2ks}\right)[f(a)g(a) + f(b)g(b)] + \left[m_1\left(g(b)f\left(\frac{a}{m_1}\right) + g(a)f\left(\frac{b}{m_1}\right)\right)\right. \\
&\left.+ m_2\left(f(b)g\left(\frac{a}{m_2}\right) + f(a)g\left(\frac{b}{m_2}\right)\right)\right]\beta\left(\left(\frac{\alpha}{k}\right)+s, s+1\right) \\
&+ m_1 m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right) + f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\beta\left(\left(\frac{\alpha}{k}\right), 2s+1\right) \\
&\leq \left(\frac{k}{\alpha+2ks}\right)[f(a)g(a) + f(b)g(b)] + \left[m_1\left(g(b)f\left(\frac{a}{m_1}\right) + g(a)f\left(\frac{b}{m_1}\right)\right)\right. \\
&\left.+ m_2\left(f(b)g\left(\frac{a}{m_2}\right) + f(a)g\left(\frac{b}{m_2}\right)\right)\right] \times \left(\frac{k}{p(\alpha-k+ks)+k}\right)^{(1/p)}\left(\frac{1}{qs+1}\right)^{(1/q)} \\
&+ m_1 m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right) + f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\left(\frac{k}{\alpha p - kp + k}\right)^{(1/p)}\left(\frac{1}{2qs+1}\right)^{(1/q)}.
\end{aligned} \tag{45}$$

Proof. By setting $\eta = 0$, $\alpha = 1$, and $h(t) = t^s$ in inequality (34) of Theorem 5, we get the above inequality (45) which is given in [26] (Theorem 6). □

Theorem 6. *Let $f, g\colon [0,\infty) \longrightarrow \mathbb{R}$ be functions such that $fg \in L_1[a,b]$, $0 \le a < mb$. If function f is exponentially $(\alpha_1, h-m_1)$-convex and function g is exponentially $(\alpha_2, h-m_2)$-convex on $[0,\infty)$ with $(\alpha, m_1), (\alpha, m_2) \in (0,1]^2$, then the following inequalities hold for RL k-fractional integrals:*

$$\begin{aligned}\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k} f(b)g(b) + I_{b^-}^{\alpha,k} f(a)g(a)\right] &\le \left[\frac{f(a)g(a)}{e^{2\eta a}} + \frac{f(b)g(b)}{e^{2\eta b}}\right] \\ &\int_0^1 t^{(\alpha/k)-1} h(t^{\alpha_1})h(t^{\alpha_2})\mathrm{d}t + m_1 m_2\left[\frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1m_2)}} + \frac{f(a/m_1)g(a/m_2)}{e^{\eta a(m_1+m_2/m_1m_2)}}\right] \\ &\times \int_0^1 t^{(\alpha/k)-1} h(1-t^{\alpha_1})h(1-t^{\alpha_2})\mathrm{d}t + \left\{m_2\left[\frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} + \frac{f(b)g(a/m_2)}{e^{\eta(b+(a/m_2))}}\right]\right. \\ &\left. + m_1\left[\frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}} + \frac{g(b)f(a/m_1)}{e^{\eta(b+(a/m_1))}}\right]\right\} \int_0^1 t^{(\alpha/k)-1} h(t^{\alpha_2})h(1-t^{\alpha_1})\mathrm{d}t.\end{aligned} \tag{46}$$

Proof. Since function f is exponentially $(\alpha_1, h-m_1)$-convex and function g is exponentially $(\alpha_2, h-m_2)$-convex, then for $t \in [0,1]$, we have

$$\begin{aligned}&f(ta+(1-t)b)g(ta+(1-t)b) \\ &\le h(t^{\alpha_1})h(t^{\alpha_2})\frac{f(a)g(a)}{e^{2\eta a}} + m_2 h(t^{\alpha_1})h(1-t^{\alpha_2})\frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} \\ &+ m_1 h(t^{\alpha_2})h(1-t^{\alpha_1})\frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}} + m_1 m_2 h(1-t^{\alpha_1})h(1-t^{\alpha_2})\frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1m_2)}}.\end{aligned} \tag{47}$$

By multiplying both sides of the above inequality with $t^{(\alpha/k)-1}$ and integrating over $[0,1]$, we have

$$\begin{aligned}\int_0^1 t^{(\alpha/k)-1} f(ta+(1-t)b)g(ta+(1-t)b)\mathrm{d}t &\le \frac{f(a)g(a)}{e^{2\eta a}}\int_0^1 t^{(\alpha/k)-1}h(t^{\alpha_1})h(t^{\alpha_2})\mathrm{d}t \\ &+ m_2\frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}}\int_0^1 t^{(\alpha/k)-1}h(t^{\alpha_1})h(1-t^{\alpha_2})\mathrm{d}t + m_1\frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}} \\ &\cdot \int_0^1 t^{(\alpha/k)-1}h(t^{\alpha_2})h(1-t^{\alpha_1})\mathrm{d}t + m_1 m_2\frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1m_2)}} \\ &\cdot \int_0^1 t^{(\alpha/k)-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})\mathrm{d}t.\end{aligned} \tag{48}$$

By substituting $w = ta + (1-t)b$ in the left-hand side of the above inequality, we get

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}} I_{a^+}^{\alpha,k} f(b)g(b) \le \frac{f(a)g(a)}{e^{2\eta a}} \int_0^1 t^{(\alpha/k)-1} h(t^{\alpha_1}) h(t^{\alpha_2}) dt$$
$$+ m_2 \frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} \int_0^1 t^{(\alpha/k)-1} h(t^{\alpha_1}) h(1-t^{\alpha_2}) dt$$
$$+ m_1 \frac{g(a)f(b/m_1)}{e^{\eta(a+(b/m_1))}} \int_0^1 t^{(\alpha/k)-1} h(t^{\alpha_2}) h(1-t^{\alpha_1}) dt$$
$$+ m_1 m_2 \frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1 m_2)}} \int_0^1 t^{(\alpha/k)-1} h(1-t^{\alpha_1}) h(1-t^{\alpha_2}) dt. \quad (49)$$

Similarly, by changing the rules of a and b, after a little computation, one can get

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{\alpha/k}} I_{b^-}^{\alpha,k} f(a)g(a) \le \frac{f(b)g(b)}{e^{2\eta b}} \int_0^1 t^{(\alpha/k)-1} h(t^{\alpha_1}) h(t^{\alpha_2}) dt$$
$$+ m_2 \frac{f(b)g(a/m_2)}{e^{\eta(b+(a/m_2))}} \int_0^1 t^{(\alpha/k)-1} h(t^{\alpha_1}) h(1-t^{\alpha_2}) dt + m_1 \frac{g(b)f(a/m_1)}{e^{\eta(b+(a/m_1))}} \int_0^1 t^{(\alpha/k)-1} h(t^{\alpha_2}) h(1-t^{\alpha_1}) dt$$
$$+ m_1 m_2 \frac{f(a/m_1)g(a/m_2)}{e^{\eta a(m_1+m_2/m_1 m_2)}} \int_0^1 t^{(\alpha/k)-1} h(1-t^{\alpha_1}) h(1-t^{\alpha_2}) dt. \quad (50)$$

Adding (49) and (50), we get the required result. □

Corollary 20. *The following inequality holds for the exponentially $(\alpha, h-m)$-convex function via RL fractional integrals:*

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[I_{a^+}^{\alpha} f(b)g(b) + I_{b^-}^{\alpha} f(a)g(a) \right] \le \left[\frac{f(a)g(a)}{e^{2\eta a}} + \frac{f(b)g(b)}{e^{2\eta b}} \right] \int_0^1 t^{\alpha-1} h(t^{\alpha_1}) h(t^{\alpha_2}) dt$$
$$+ m_1 m_2 \left[\frac{f(b/m_1)g(b/m_2)}{e^{\eta b(m_1+m_2/m_1 m_2)}} + \frac{f(a/m_1)g(a/m_2)}{e^{\eta a(m_1+m_2/m_1 m_2)}} \right]$$
$$\times \int_0^1 t^{\alpha-1} h(1-t^{\alpha_1}) h(1-t^{\alpha_2}) dt + \left\{ m_2 \left[\frac{f(a)g(b/m_2)}{e^{\eta(a+(b/m_2))}} + \frac{f(b)g(a/m_2)}{e^{\eta(b+(a/m_2))}} \right] \right.$$
$$\left. + m_1 \left[\frac{g(a)f(b/m_2)}{e^{\eta(a+(b/m_2))}} + \frac{g(b)f(a/m_1)}{e^{\eta(b+(a/m_1))}} \right] \int_0^1 t^{\alpha-1} h(t^{\alpha_2}) h(1-t^{\alpha_1}) dt \right\}. \quad (51)$$

Proof. By setting $k=1$ in inequality (49) of Theorem 6, we get the above inequality (51). □

Corollary 21. *The following inequality holds for the $(\alpha, h-m)$-convex function via RL k-fractional integrals:*

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{(\alpha/k)}}\left[I_{a^+}^{\alpha,k}f(b)g(b)+I_{b^-}^{\alpha,k}f(a)g(a)\right]\leq[f(a)g(a)+f(b)g(b)]\int_0^1 t^{(\alpha/k)-1}h(t^{\alpha_1})h(t^{\alpha_2})dt$$
$$+m_1m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]$$
$$\times\int_0^1 t^{(\alpha/k)-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt+\left\{m_2\left[f(a)g\left(\frac{b}{m_2}\right)+f(b)g\left(\frac{a}{m_2}\right)\right]\right.$$
$$\left.+m_1\left[g(a)f\left(\frac{b}{m_1}\right)+g(b)f\left(\frac{a}{m_1}\right)\right]\int_0^1 t^{(\alpha/k)-1}h(t^{\alpha_2})h(1-t^{\alpha_1})dt\right\}. \quad (52)$$

Proof. By setting $\eta=0$ in inequality (49) of Theorem 6, we get the above inequality (52) which is given in [4] (Theorem 4.6). □

Corollary 22. *The following inequality holds for the (α,m)-convex function via RL fractional integrals:*

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}I_{a^+}^{\alpha}f(b)g(b)$$
$$\leq\frac{1}{\alpha_1+\alpha_2+\alpha}f(a)g(a)+\frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)}m_2f(a)g\left(\frac{b}{m_2}\right)$$
$$+\frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)}m_1g(a)f\left(\frac{b}{m_1}\right) \quad (53)$$
$$+\left(\left(\frac{1}{\alpha}\right)-\left(\frac{1}{\alpha+\alpha_1}\right)-\left(\frac{1}{\alpha+\alpha_2}\right)+\left(\frac{1}{\alpha_1+\alpha_2+\alpha}\right)\right)m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right),$$

and

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}I_{b^-}^{\alpha}f(a)g(a)$$
$$\leq\frac{1}{\alpha_1+\alpha_2+\alpha}f(b)g(b)+\frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)}m_2f(b)g\left(\frac{b}{m_2}\right)$$
$$+\frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)}m_1g(b)f\left(\frac{b}{m_1}\right) \quad (54)$$
$$+\left(\left(\frac{1}{\alpha}\right)-\left(\frac{1}{\alpha+\alpha_1}\right)-\left(\frac{1}{\alpha+\alpha_2}\right)+\left(\frac{1}{\alpha_1+\alpha_2+\alpha}\right)\right)m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right).$$

Proof. From (49) by setting $\eta=0, k=1$, and $h(t)=t$, we get (53). Similarly, using $\eta=0$, $k=1$, and $h(t)=t$ in (50), we get (54) which is given in [27] (Theorem 12). □

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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