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Research Article

Maximum Principle for the Space-Time Fractional Conformable Differential System Involving the Fractional Laplace Operator

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In this paper, the authors consider a IBVP for the time-space fractional PDE with the fractional conformable derivative and the fractional Laplace operator. A fractional conformable extremum principle is presented and proved. Based on the extremum principle, a maximum principle for the fractional conformable Laplace system is established. Furthermore, the maximum principle is applied to the linear space-time fractional Laplace conformable differential system to obtain a new comparison theorem. Besides that, the uniqueness and continuous dependence of the solution of the above system are also proved.

1. Introduction

Many fractional partial differential equations were used for modeling complex dynamic systems of engineering, physics, biology, and many other fields [1–4]. As a significant tool, the maximum principle plays an important role in the study of the complex dynamic systems without certain knowledge of the solutions [5–13]. In 2016, by using the maximum principle, Luchko and Yamamoto [14] obtained the uniqueness of both the strong and the weak solutions of the IBVP for a general time-fractional distributed order diffusion equation. In 2016, Jia and Li [15] applied the maximum principle to the classical solution and weak solution of a time-space fractional diffusion equation. Furthermore, they also deduced the maximum principle for a full fractional diffusion equation other than time-fractional and spatial-integer order diffusion equations. In 2019, Wang et al. [16] investigated the IBVP for Hadamard fractional differential equations with fractional Laplace operator $(-\Delta)^{\beta}$ by using the maximum principle.

There are diverse fractional derivatives, such as the Riemann–Liouville derivative, the Caputo fractional derivative, the left and right conformable derivatives, and other fractional derivatives [17–40]. In 2015, Abdeljawad [34] defined the left and right conformable derivatives. Depending on [34], Jarad et al. [35] introduced the fractional conformable derivatives and presented the fractional conformable derivative in the sense of Caputo. The extremum principle of the Caputo fractional conformable derivative is seldom regarded in the existing literature. In addition, the papers which mentioned the fractional conformable derivative do not include the fractional Laplace operator.

Motivated by the above works, in this context, the authors investigate the IBVP for a space-time Caputo fractional conformable diffusion system with the fractional Laplace operator. First, we provide a detailed proof of the Caputo fractional conformable extremum principle. Then, the new maximum principle is obtained by applying the extreme principle. As some applications of the maximum principle, a comparison principle for the space-time fractional Laplace conformable differential system is developed, and the properties of the solution of the system are given, such as the uniqueness and continuous dependence on the initial and boundary condition.

The article is organized as follows: in Section 2, the extremum principle for the Caputo fractional conformable derivative is established. In Section 3, the maximum principle of the space-time fractional Laplace conformable differential system is derived, which is used to obtain the comparison principle for the space-time fractional Laplace

conformable differential system, and the properties of the solution of the above system are given in Section 4.

2. Problem Formulation and Extremum Principles

In this paper, we focus on a space-time Caputo fractional conformable system with the fractional Laplace operator:

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha}u(x,t)+(-\Delta)^{\gamma}u(x,t)-a(x,t)u(x,t)=g(x,t), & (x,t)\in\Omega\times(a,b],\\ u(x,t)=0, & x\in\mathbb{R}^N\backslash\overline{\Omega},\ t\geq a,\end{cases} \tag{1}$$

where Ω represents an open and bounded domain in $\mathbb{R}^N\,(N\geq 1)$ in which boundary Γ is smooth and $a(x,t)\in\overline{\Omega}\times[a,b]$ is a bounded function. Here, ${}_a^{C\beta}D_t^{\alpha}$ is the left Caputo fractional conformable derivative. For a function $f\in C_{\alpha,a}^n$, the left Caputo fractional conformable derivative of order β is defined by

$${}_a^{C\beta}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\beta)}\int_a^t\left(\frac{(t-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{n-\beta-1}\frac{{}_a^nT^{\alpha}f(\tau)}{(\tau-a)^{1-\alpha}}\mathrm{d}\tau, \tag{2}$$

with $0<\beta<1$, $0<\alpha<1$, $n=[\beta]+1$, ${}_aT^{\alpha}f(t)=(t-a)^{1-\alpha}f'(t)$, ${}_a^nT^{\alpha}=\underbrace{{}_aT^{\alpha}\,{}_aT^{\alpha},\ldots,{}_aT^{\alpha}}_{n\text{ times}}$, and $C_{\alpha,a}^n[a,b]=\{f:[a,b]\longrightarrow\mathbb{R}|{}_a^{n-1}T^{\alpha}f\in I_{\alpha}[a,b]\}$ (where $I_{\alpha}[a,b]$ is defined in Definition 1 of [34]). For detailed information of the Caputo fractional conformable derivative, see [35].

When $\phi\in C_{\mathrm{loc}}^{1,1}(\mathbb{R}^N)\cap L^{\gamma}$, the fractional Laplace operator could be given by

$$(-\Delta)^{\gamma}\phi(x)=C_{N,\gamma}\int_{\mathbb{R}^N}\frac{\phi(x)-\phi(y)}{|x-y|^{N+2\gamma}}\mathrm{d}y, \tag{3}$$

with $C_{N,\gamma}=(\gamma 2^{2\gamma}\Gamma(N+2\gamma/2)/\pi^{N/2}\Gamma(1-\gamma))$, and

$$L_{\gamma}=\left\{\phi:\mathbb{R}^N\longrightarrow\mathbb{R}|\int_{\mathbb{R}^N}\frac{|;\phi(x)|}{1+|x|^{N+\gamma}}\mathrm{d}x<\infty.\right\}. \tag{4}$$

Denote

$$H(\overline{\Omega})=\left\{u(x,t)|u(x,t)\in C^{2,1}(\Omega\times(a,b)),u(x,t)\in C(\overline{\Omega}\times[a,b])\right\}. \tag{5}$$

Firstly, we can state two Caputo fractional conformable extremum principles.

Lemma 1. *If $f\in C_{\alpha,a}^1([a,b])$ reaches its maximum at a point $t_0\in[a,b]$, then*

$${}_a^{C\beta}D_{t_0}^{\alpha}f(t_0)\geq 0, \tag{6}$$

holds.

Proof. First, we introduce an auxiliary function

$$g(t)=f(t_0)-f(t)\geq 0,\quad t\in[a,b]. \tag{7}$$

Concurrently, $g(t)\in C_{\alpha,a}^1([a,b])$, $g(t_0)=0$, and ${}_a^{C\beta}D_t^{\alpha}g(t)=-{}_a^{C\beta}D_t^{\alpha}f(t)$.

By calculation, we notice that

$$\begin{aligned}{}_a^{C\beta}D_{\alpha}^{t_0}g(t_0)&=\frac{1}{\Gamma(1-\beta)}\int_a^{t_0}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta}\\&\quad\cdot\frac{(\tau-a)^{1-\alpha}g'(\tau)}{(\tau-a)^{1-\alpha}}\mathrm{d}\tau\\&=\frac{1}{\Gamma(1-\beta)}\int_a^{t_0}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta}g'(\tau)\mathrm{d}\tau\\&=\frac{1}{\Gamma(1-\beta)}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta}g(\tau)\Big|_a^{t_0}\\&\quad-\frac{\beta}{\Gamma(1-\beta)}\int_a^{t_0}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta-1}\\&\quad\cdot(\tau-a)^{\alpha-1}g(\tau)\mathrm{d}\tau.\end{aligned} \tag{8}$$

This is because

$$\begin{aligned}&\lim_{\tau\longrightarrow t_0}\frac{1}{\Gamma(1-\beta)}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta}g(\tau)\\&=\frac{\alpha^{\beta}}{\Gamma(1-\beta)}\lim_{\tau\longrightarrow t_0}\frac{g'(\tau)}{\beta((t_0-a)^{\alpha}-(\tau-a)^{\alpha})^{\beta-1}(-\alpha)(\tau-a)^{\alpha-1}}\\&=0.\end{aligned} \tag{9}$$

Therefore, formula (9) becomes

$$\begin{aligned}{}_a^{C\beta}D_{t_0}^{\alpha}g(t_0)&=-\frac{1}{\Gamma(1-\beta)}\left(\frac{(t_0-a)^{\alpha}}{\alpha}\right)^{-\beta}g(a)\\&\quad-\frac{\beta}{\Gamma(1-\beta)}\int_a^{t_0}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta-1}\\&\quad\cdot(\tau-a)^{\alpha-1}g(\tau)\mathrm{d}\tau\\&\leq 0.\end{aligned} \tag{10}$$

We can obtain ${}_{a}^{C\beta}D_{t_0}^{\alpha} f(t_0) \le 0$.

The lemma is proved.

Using the same method, it is easy to obtain the following lemma. □

Lemma 2. *If $f \in C_{\alpha,a}^{1}([a,b])$ reaches its minimum at a point $t_0 \in [a,b]$, then*

$$ {}_{a}^{C\beta}D_{t_0}^{\alpha} f(t_0) \le 0, \tag{11}$$

holds.

3. Maximum Principle

In this section, we focus on linear space-time Caputo fractional conformable Laplace system (1) with the initial-boundary condition:

$$u(x,a) = \varphi(x), \quad x \in \Omega, \tag{12}$$

$$u(x,t) = \mu(x,t), \quad (x,t) \in \Gamma \times [a,b]. \tag{13}$$

Theorem 1. *Let a function $u \in H(\overline{\Omega})$ satisfy linear space-time Caputo fractional conformable Laplace system (1), (12), and (13). Suppose $g(x,t) \le 0$, $\forall (x,t) \in \Omega \times (a,b]$. Then, we have*

$$u(x,t) \le \max\left\{\max_{x\in\Omega} \varphi(x), \max_{(x,t)\in\Gamma\times[a,b]} \mu(x,t), 0\right\}, \quad \forall (x,t) \in \overline{\Omega} \times [a,b]. \tag{14}$$

Proof. We first suppose that inequality (14) is false; then, there exists a point $(x_0,t_0) \in \Omega \times (a,b]$ such that

$$u(x_0,t_0) > \max\left\{\max_{x\in\Omega} \varphi(x), \max_{(x,t)\in\Gamma\times[a,b]} \mu(x,t), 0\right\} = M > 0. \tag{15}$$

Denote $\varepsilon = u(x_0,t_0) - M > 0$ and

$$w(x,t) = u(x,t) + \frac{\varepsilon}{2}\frac{b-(t-a)}{b}, \quad (x,t) \in \overline{\Omega} \times [a,b]. \tag{16}$$

Besides, w implies

$$w(x,t) \le u(x,t) + \frac{\varepsilon}{2}, \quad (x,t) \in \overline{\Omega} \times [a,b],$$

$$\begin{aligned} w(x_0,t_0) &\ge u(x_0,t_0) = \varepsilon + M \ge \varepsilon + u(x,t) \ge \varepsilon + w(x,t) - \frac{\varepsilon}{2} \\ &\ge \frac{\varepsilon}{2} + w(x,t), \ (x,t) \in (\Gamma \times [a,b]) \cup (\Omega \times \{a\}). \end{aligned} \tag{17}$$

The latter property implies that the maximum of w cannot be attained on $(\Gamma \times [a,b]) \cup (\Omega \times \{a\})$. Let $w(x_1,t_1) = \max_{(x,t)\in\overline{\Omega}\times[a,b]} w(x,t)$; then,

$$w(x_1,t_1) \ge u(x_0,t_0) \ge \varepsilon + M > \varepsilon, \tag{18}$$

$$(-\Delta)^{\gamma} w(x,t)|_{(x_1,t_1)} = C_{N,\gamma} \int_{\mathbb{R}^N} \frac{w(x_1,t_1) - w(x,t_1)}{|x_1 - x|^{N+2\gamma}} \mathrm{d}x \ge 0. \tag{19}$$

By Lemma 1, we know

$$ {}_{a}^{C\beta}D_{t}^{\alpha} w(x,t)|_{(x_1,t_1)} \ge 0. \tag{20}$$

By calculation, we can show

$$ {}_{a}^{C\beta}D_{t}^{\alpha}\left(\frac{\varepsilon}{2}\frac{b-(t-a)}{b}\right) = -\frac{1}{\Gamma(1-\beta)}\frac{\varepsilon}{2b}\int_a^t \left(\frac{(t-a)^{\alpha} - (\tau-a)^{\alpha}}{\alpha}\right)^{-\beta} \mathrm{d}\tau. \tag{21}$$

Assuming $u = (\tau - a/t - a)^{\alpha}$ and substituting into formula (21), we get

$$\begin{aligned} {}_{a}^{C\beta}D_{t}^{\alpha}\left(\frac{\varepsilon}{2}\frac{b-(t-a)}{b}\right) &= -\frac{1}{\Gamma(1-\beta)}\frac{\varepsilon}{2b}\alpha^{\beta-1}\int_0^1 (t-a)^{1-\alpha\beta} \\ &\quad \cdot (1-u)^{-\beta} u^{1-\alpha} \mathrm{d}u \\ &= -\alpha^{\beta-1}(t-a)^{1-\alpha\beta}\frac{\varepsilon}{2b}\frac{\Gamma(2-\alpha)}{\Gamma(3-\alpha-\beta)}. \end{aligned} \tag{22}$$

Applying (19)–(22), it holds that

$$\begin{aligned} &{}_{a}^{C\beta}D_{t}^{\alpha} u(x,t)|_{(x_1,t_1)} + (-\Delta)^{\gamma} u(x,t)|_{(x_1,t_1)} - a(x_1,t_1)u(x_1,t_1) - g(x_1,t_1) \\ &= {}_{a}^{C\beta}D_{t}^{\alpha} w(x,t)|_{(x_1,t_1)} - {}_{a}^{C\beta}D_{t}^{\alpha}\left(\frac{\varepsilon}{2}\frac{b-(t_1-a)}{b}\right) + (-\Delta)^{\gamma} w(x_1,t_1) \\ &\quad - a(x_1,t_1)\left(w(x_1,t_1) - \frac{\varepsilon}{2}\frac{b-(t_1-a)}{b}\right) - g(x_1,t_1) \\ &\ge \alpha^{\beta-1}(t_1-a)^{1-\alpha\beta}\frac{\varepsilon}{2b}\frac{\Gamma(2-\alpha)}{\Gamma(3-\alpha-\beta)} - a(x_1,t_1)\varepsilon\left(1 - \frac{b-(t_1-a)}{2b}\right) \\ &\ge 0. \end{aligned} \tag{23}$$

Equation (23) is in contradiction with (1).

The proof of the theorem is completed.

Similarly, the minimum principle can be obtained as follows. □

Theorem 2. *Let a function $u \in H(\overline{\Omega})$ satisfy linear space-time Caputo fractional conformable Laplace system (1), (12), and (13). Suppose $g(x,t) \geq 0$, $\forall (x,t) \in \Omega \times (a,b]$. Then, we have*

$$u(x,t) \geq \min\left\{\min_{x\in\Omega} \varphi(x), \min_{(x,t)\in\Gamma\times[a,b]} \mu(x,t), 0\right\}, \quad \forall (x,t) \in \overline{\Omega} \times [a,b]. \tag{24}$$

4. Some Applications of the Maximum Principle

Theorem 3. *Let $u(x,t) \in H(\overline{\Omega})$ be a solution of system (1) with initial-boundary values (12) and (13). Then,*

$$\|u\|_{C(\overline{\Omega}\times[a,b])} \leq \max\left\{\max_{x\in\Omega} \|\varphi(x)\|, \max_{(x,t)\in\Gamma\times[a,b]} \|\mu(x,t)\|\right\} + 2M\frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(b-a)^{\alpha\beta}, \tag{25}$$

where

$$M = \|g\|_{C(\overline{\Omega}\times[a,b])}. \tag{26}$$

Proof. We first present a function

$$w(x,t) = u(x,t) - M\frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(t-a)^{\alpha\beta}, \quad (x,t) \in \overline{\Omega} \times [a,b]. \tag{27}$$

If $u(x,t)$ is a solution of system (1), (12), and (13), then $w(x,t)$ is a solution of problem (1) with

$$\begin{aligned} g_1(x,t) &= g(x,t) - M\frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}\,{}_{a}^{C\beta}D_{t}^{\alpha}(t-a)^{\alpha\beta} \\ &= g(x,t) - M, \\ \mu_1(x,t) &= \mu(x,t) - M\frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(t-a)^{\alpha\beta}. \end{aligned} \tag{28}$$

Substitute $g_1(x,t)$ and $\mu_1(x,t)$ for $g(x,t)$ and $\mu(x,t)$, respectively. Owing to $g_1(x,t) \leq 0$, applying Theorem 1 (maximum principle), we have

$$w(x,t) \leq \max\left\{\max_{x\in\Omega} \|\varphi(x)\|, \max_{(x,t)\in\Gamma\times[a,b]} \|\mu(x,t)\| + M\frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(b-a)^{\alpha\beta}\right\}. \tag{29}$$

Therefore,

$$u(x,t) \leq \max\left\{\max_{x\in\Omega} \|\varphi(x)\|, \max_{(x,t)\in\Gamma\times[a,b]} \|\mu(x,t)\|\right\} + 2M\frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(b-a)^{\alpha\beta}. \tag{30}$$

In a similar manner, we can get

$$u(x,t) \geq -\max\left\{\max_{x\in\Omega} \|\varphi(x)\|, \max_{(x,t)\in\Gamma\times[a,b]} \|\mu(x,t)\|\right\} - 2M\frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(b-a)^{\alpha\beta}. \tag{31}$$

Combining (30) and (31), the theorem is proved. □

Theorem 4. *Let $u(x,t)$ satisfy IBVP (1), (12), and (13). $u(x,t)$ is continuous depending on the data given. That is, if*

$$\|g - g_1\|_{C(\overline{\Omega}\times[a,b])} \leq \varepsilon, \ \|\varphi(x) - \varphi_1(x)\|_{C(\overline{\Omega})} \leq \varepsilon_0, \|\mu(x,t) - \mu_1(x,t)\|_{C(\Gamma\times[a,b])} \leq \varepsilon_1, \tag{32}$$

then the estimation of the classical solution of $u(x,t)$ and $u_1(x,t)$,

$$\|u - u_1\|_{C(\overline{\Omega}\times[a,b])} \leq \max\{\varepsilon_0, \varepsilon_1\} + 2\frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(b-a)^{\alpha\beta}\varepsilon, \tag{33}$$

holds.

The demonstration process is similar to Theorem 3.

Theorem 5. *Let $u \in H(\overline{\Omega})$ be a solution of IBVP (1), (12), and (13). Assume $g(x,t) \leq 0$ and $a(x,t) \leq 0, \forall (x,t) \in \Omega\times(a,b]$. Then, it follows that*

$$u(x,t) \leq 0, \quad (x,t) \in \overline{\Omega} \times [a,b], \tag{34}$$

if $\varphi(x) \leq 0$, and $\mu(x,t) \leq 0$.

Theorem 6. *Let $u \in H(\overline{\Omega})$ satisfy IBVP (1), (12), and (13). Assume $g(x,t) \geq 0$ and $a(x,t) \geq 0, \forall (x,t) \in \Omega \times (a,b]$. Then, it follows that*

$$u(x,t) \geq 0, \quad (x,t) \in \overline{\Omega} \times [a,b], \tag{35}$$

if $\varphi(x) \geq 0$, and $\mu(x,t) \geq 0$.

The conclusion of Theorem 5 and Theorem 6 is obtained by Theorem 1.

Remark 1. Let $u \in H(\overline{\Omega})$ satisfy IBVP (1), (12), and (13). Assume $g(x,t) = a(x,t) = 0, \forall (x,t) \in \Omega \times (a,b]$. Then, it follows that

$$u(x,t) = 0, \quad \forall (x,t) \in \overline{\Omega} \times [a,b], \tag{36}$$

if $\varphi(x) = \mu(x,t) = 0$.

Theorem 7 (comparison theorem). Suppose $a(x,t) \geq 0$, $b(x,t) \geq 0$, and $b(x,t) > a(x,t)$, $\forall (x,t) \in \Omega \times (a,b]$. Assume $(u,v) \in H(\overline{\Omega}) \times H(\overline{\Omega})$ satisfies the following linear space-time fractional Laplace conformable differential system:

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha} u(x,t) + (-\Delta)^{\gamma} u(x,t) - a(x,t)v(x,t) - b(x,t)u(x,t) \geq 0, & (x,t) \in \Omega \times (a,b], \\ {}_a^{C\beta}D_t^{\alpha} v(x,t) + (-\Delta)^{\gamma} v(x,t) - a(x,t)u(x,t) - b(x,t)v(x,t) \geq 0, & (x,t) \in \Omega \times (a,b], \\ u(x,t) = 0, \quad v(x,t) = 0, \ x \in \mathbb{R}^N \backslash \overline{\Omega}, \ t \geq a, \\ u(x,a) \geq 0, \quad v(x,a) \geq 0, \ x \in \Omega, \\ u(x,t) \geq 0, \quad v(x,t) \geq 0, \ (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{37}$$

Then, it follows that

$$u(x,t) \geq 0, v(x,t) \geq 0, (x,t) \in \overline{\Omega} \times [a,b]. \tag{38}$$

Proof. Let $p(x,t) = u(x,t) + v(x,t)$, $\forall (x,t) \in \overline{\Omega} \times [a,b]$. Then, by (37), we have

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha} p(x,t) + (-\Delta)^{\gamma} p(x,t) - a(x,t)p(x,t) + b(x,t)p(x,t) \geq 0, & (x,t) \in \Omega \times (a,b], \\ p(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \overline{\Omega}, \ t \geq a, \\ p(x,a) \geq 0, \quad x \in \Omega, \\ p(x,t) \geq 0, \quad (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{39}$$

Thus, by (39) and Theorem 6, we obtain

$$p(x,t) \geq 0, \quad \forall (x,t) \in \overline{\Omega} \times [a,b], \text{i.e.u}(x,t) + v(x,t) \geq 0, \cdot (x,t) \in \overline{\Omega} \times [a,b]. \tag{40}$$

Using (37) and (40), we have that

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha} u(x,t) + (-\Delta)^{\gamma} u(x,t) - (b(x,t) - a(x,t))u(x,t) \geq 0, & (x,t) \in \Omega \times (a,b], \\ u(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \overline{\Omega}, \ t \geq a, \\ u(x,a) \geq 0, \quad x \in \Omega, \\ u(x,t) \geq 0, \quad (x,t) \in \Gamma \times [a,b], \end{cases} \tag{41}$$

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha} v(x,t) + (-\Delta)^{\gamma} v(x,t) - (b(x,t) - a(x,t))v(x,t) \geq 0, & (x,t) \in \Omega \times (a,b], \\ v(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \overline{\Omega}, \ t \geq a, \\ v(x,a) \geq 0, \quad x \in \Omega, \\ v(x,t) \geq 0, \quad (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{42}$$

Applying Theorem 6 to (41) and (42), we can get

$$u(x,t) \geq 0, v(x,t) \geq 0, (x,t) \in \overline{\Omega} \times [a,b]. \tag{43}$$

Thus, the conclusion holds.

Similarly, the following theorem holds. □

Theorem 8. *Suppose* $a(x,t) \leq 0$, $b(x,t) \leq 0$, *and* $b(x,t) < a(x,t)$, $\forall (x,t) \in \Omega \times (a,b]$. *Assume* $(u,v) \in H(\overline{\Omega}) \times H(\overline{\Omega})$

satisfies the following linear space-time fractional Laplace conformable differential system:

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha}u(x,t)+(-\Delta)^{\gamma}u(x,t)-a(x,t)v(x,t)-b(x,t)u(x,t)\le 0, & (x,t)\in\Omega\times(a,b],\\ {}_a^{C\beta}D_t^{\alpha}v(x,t)+(-\Delta)^{\gamma}v(x,t)-a(x,t)u(x,t)-b(x,t)v(x,t)\le 0, & (x,t)\in\Omega\times(a,b],\\ u(x,t)=0, v(x,t)=0, & x\in\mathbb{R}^N\backslash\overline{\Omega},\ t\ge a,\\ u(x,a)\le 0, v(x,a)\le 0, & x\in\Omega,\\ u(x,t)\le 0,\ v(x,t)\le 0, & (x,t)\in\Gamma\times[a,b]. \end{cases} \tag{44}$$

Then, it follows that

$$u(x,t)\le 0, v(x,t)\le 0,\ (x,t)\in\overline{\Omega}\times[a,b]. \tag{45}$$

Remark 2. Let $(u,v)\in H(\overline{\Omega})\times H(\overline{\Omega})$ satisfy the following linear space-time fractional Laplace conformable differential system:

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha}u(x,t)+(-\Delta)^{\gamma}u(x,t)-a(x,t)v(x,t)-b(x,t)u(x,t)=0, & (x,t)\in\Omega\times(a,b],\\ {}_a^{C\beta}D_t^{\alpha}v(x,t)+(-\Delta)^{\gamma}v(x,t)-a(x,t)u(x,t)-b(x,t)v(x,t)=0, & (x,t)\in\Omega\times(a,b],\\ u(x,t)=0, v(x,t)=0, & x\in\mathbb{R}^N\backslash\overline{\Omega},\ t\ge a,\\ u(x,a)=\varphi_1(x), v(x,a)=\varphi_2(x), & x\in\Omega,\\ u(x,t)=\mu_1(x,t),\ v(x,t)=\mu_2(x,t), & (x,t)\in\Gamma\times[a,b]. \end{cases} \tag{46}$$

Suppose $a(x,t)=b(x,t)=0, \forall(x,t)\in\Omega\times(a,b]$. Then, it follows that

$$u(x,t)=0, v(x,t)=0, \quad \forall(x,t)\in\overline{\Omega}\times[a,b], \tag{47}$$

if $\varphi_1(x)=\varphi_2(x)=\mu_1(x,t)=\mu_2(x,t)=0$.

Next, we focus on the following linear space-time fractional Laplace conformable differential system:

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha}u(x,t)+(-\Delta)^{\gamma}u(x,t)-a(x,t)v(x,t)-b(x,t)u(x,t)=g_1(x,t), & (x,t)\in\Omega\times(a,b],\\ {}_a^{C\beta}D_t^{\alpha}v(x,t)+(-\Delta)^{\gamma}v(x,t)-a(x,t)u(x,t)-b(x,t)v(x,t)=g_2(x,t), & (x,t)\in\Omega\times(a,b],\\ u(x,t)=0, v(x,t)=0, & x\in\mathbb{R}^N\backslash\overline{\Omega},\ t\ge a,\\ u(x,a)=\varphi_1(x), v(x,a)=\varphi_2(x), & x\in\Omega,\\ u(x,t)=\mu_1(x,t),\ v(x,t)=\mu_2(x,t), & (x,t)\in\Gamma\times[a,b]. \end{cases} \tag{48}$$

Theorem 9. *Suppose $a(x,t)\le 0$, $b(x,t)\le 0$, $b(x,t)<a(x,t)$, $g_1(x,t)\le 0$, and $g_2(x,t)\le 0$, $\forall(x,t)\in\Omega\times(a,b]$; then, IBVP (48) has a unique solution on $H(\overline{\Omega})\times H(\overline{\Omega})$.*

Proof. Let (u_1,v_1) and (u_2,v_2) be two solutions of IBVP (48). Denote

$$u(x,t)=u_1(x,t)-u_2(x,t), v(x,t)=v_1(x,t)-v_2(x,t), \quad \forall(x,t)\in\overline{\Omega}\times[a,b], \tag{49}$$

satisfies the system

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha}u(x,t)+(-\Delta)^{\gamma}u(x,t)-a(x,t)v(x,t)-b(x,t)u(x,t)=0, & (x,t)\in\Omega\times(a,b],\\ {}_a^{C\beta}D_t^{\alpha}v(x,t)+(-\Delta)^{\gamma}v(x,t)-a(x,t)u(x,t)-b(x,t)v(x,t)=0, & (x,t)\in\Omega\times(a,b],\\ u(x,t)=0, v(x,t)=0, & x\in\mathbb{R}^N\backslash\overline{\Omega},\ t\ge a,\\ u(x,a)=0, v(x,a)=0, & x\in\Omega,\\ u(x,t)=0, v(x,t)=0, & (x,t)\in\Gamma\times[a,b]. \end{cases} \tag{50}$$

Let $p(x,t) = u(x,t) + v(x,t)$, $\forall (x,t) \in \overline{\Omega} \times [a,b]$. By (50), we have

$$\begin{cases} {}_{a}^{C\beta}D_{t}^{\alpha} p(x,t) + (-\Delta)^{\gamma} p(x,t) - (b(x,t) - a(x,t))p(x,t) = 0, \quad (x,t) \in \Omega \times (a,b]. \\ p(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \overline{\Omega},\ t \geq a, \\ p(x,a) = 0, \quad x \in \Omega, \\ p(x,t) = 0, \quad (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{51}$$

Applying Theorem 8, we get

$$u(x,t) \leq 0, v(x,t) \leq 0, (x,t) \in \overline{\Omega} \times [a,b]. \tag{52}$$

By the same way, using Theorem 8 to $-u(x,t)$ and $-v(x,t)$, we have

$$u(x,t) \geq 0, v(x,t) \geq 0, (x,t) \in \overline{\Omega} \times [a,b]. \tag{53}$$

Combining (52) and (53), we can get

$$u(x,t) = 0, v(x,t) = 0, \quad \forall (x,t) \in \overline{\Omega} \times [a,b]. \tag{54}$$

Thus, the conclusion holds. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

Both authors contributed equally and approved the final manuscript.

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