Hindawi
Journal of Mathematics
Volume 2020, Article ID 7416893, 6 pages
https://doi.org/10.1155/2020/7416893

Research Article

Ring Extensions with Finitely Many Non-Artinian Intermediate Rings

Noômen Jarboui ,[1] Naseam Al-Kuleab,[1] and Omar Almallah [2]

[1]*Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia*
[2]*Al-Balqa University, Amman, Jordan*

Correspondence should be addressed to Noômen Jarboui; njarboui@kfu.edu.sa

Received 30 June 2020; Accepted 3 November 2020; Published 12 November 2020

Academic Editor: Elena Guardo

The commutative ring extensions with exactly two non-Artinian intermediate rings are characterized. An initial step involves the description of the commutative ring extensions with only one non-Artinian intermediate ring.

1. Introduction

All rings and algebras considered in this paper are assumed to be commutative with the identity element; all subrings, ring extensions, algebras, and ring (resp., algebra) homomorphisms are assumed unital. If $R \subset S$ is a ring extension, it is convenient to let $[R, S]$ denote the set of intermediate rings (that is, the set of rings T such that $R \subseteq T \subseteq S$). We shall call a ring T in $[R, S]$ an S-overring of R. Such a ring is said to be a proper S-overring of R if $T \neq R$. When S is the total quotient ring of R, then each ring $T \in [R, S]$ is called an overring of R. Such a ring T is termed a proper overring of R if $T \neq R$. If $\mathscr{B}$ is a ring-theoretic property and $R \subset S$ is a ring extension, then (R, S) is said to be a $\mathscr{B}$-pair if each S-overring of R has a property $\mathscr{B}$ (see [1], p. 34). In 1992, Gilmer and Heinzer have studied Artinian pairs of rings (cf. [1]). Recall that a ring R is called Artinian if R satisfies the descending chain condition for ideals. Examples of Artinian rings are finite dimensional algebras over a field (recall that an algebra A over a field K is said to be finite dimensional or infinite dimensional according to whether the K-vector space A is finite dimensional or infinite dimensional). Artinian rings, and especially local Artinian rings, play an important role in the algebraic geometry, for example, in deformation theory. It is worth noticing that one of the most important trends in commutative ring theory is the study of the influence of given systems of intermediate rings of a ring extension $R \subset S$ on the structure of the extension itself. Examples of such systems are the family of all S-overrings, that of all proper S-overrings, etc. (see [1–17]). For a ring-theoretic property $\mathscr{B}$ and a ring extension $R \subset S$, let $\mathfrak{C}$ denote the family of S-overrings T of R such that T does not satisfy $\mathscr{B}$. Recently, many authors have investigated the behaviour of ring extensions $R \subset S$ for which $\mathfrak{C} = \{R\}$. In this case, the ring R is called a maximal non-$\mathscr{B}$ subring of S. These ring extensions have been studied for various properties $\mathscr{B}$ such as $\mathscr{B} :=$ Noetherian, ACCP, Jaffard, universally catenarian, local, valuation, pseudovaluation, integrally closed, and Prüfer (cf. [3–5, 9–15, 17, 18]). We are interested in this paper in the property $\mathscr{B} :=$ Artinian and the family $\mathfrak{C}$ of cardinality 1 or 2. Our work is motivated on the one hand by [5], in which the authors have studied maximal non-Noetherian subrings of a domain, and on the other hand by the abovementioned work of Gilmer and Heinzer concerning Artinian pairs (cf. [1]) and also by the increasing interest in ring extensions with many $\mathscr{B}$ intermediate rings as explained above. In Section 2, we study ring extensions $R \subset S$ with only one non-Artinian intermediate ring. We show in Theorem 1 that there exists a unique intermediate ring T between R and S such that T is not Artinian if and only if R is a maximal non-Artinian subring of S if and only if $R \subset S$ is a closed minimal extension and S is Artinian. As a consequence, if S is an integral domain, then R is a maximal non-Artinian subring of S if and only if R is a rank one valuation domain with quotient field S (see Corollary 1). In Section 3, we study ring extensions $R \subset S$ having exactly two non-Artinian intermediate rings. We give full characterizations of these extensions in Theorems 2 and 3.

Let $R \subset S$ be a ring extension. Throughout this paper, $\overline{R}_S$ denotes the integral closure of R in S and R' denotes the integral closure of R (in its total quotient ring). We use "$\subseteq$" for inclusion and "$\subset$" for strict inclusion. Any undefined notation or terminology is standard, as in [19, 20].

2. Ring Extensions with Only One Non-Artinian Intermediate Ring

We start with the following result which is an easy consequence of [1] (Theorem 2) or [21] (Theorem 3.8).

Proposition 1. *Let $R \subset S$ be a ring extension, and suppose that there is at least one S-overring of R which is not Artinian. Assume, moreover, that the class $\mathfrak{C}$ of non-Artinian S-overrings of R is finite. Then, $R \in \mathfrak{C}$.*

We start our investigations by recalling some results about minimal ring extensions and normal pairs of rings. A ring extension $R \subset S$ is said to be minimal if R is a proper subring of S and $[R, S] = \{R, S\}$. If $R \subset S$ is a minimal extension, then either $\overline{R}_S = R$, in which case, $R \subset S$ is called a closed minimal extension, or $\overline{R}_S = S$, in which case $R \subset S$ is called a minimal integral extension (see [22]). If $R \subset S$ is a ring extension, then (R, S) is called a normal pair if T is integrally closed in S for each S-overring T of R. The concept of a normal pair (R, S) was introduced in case S is a (integral) domain by Davis [23]. The most natural example of a normal pair (R, S) arises when R is an arbitrary Prüfer domain and S is its quotient field (cf. [23] (Theorem 1) or [19] (Theorems 23.4(1) and 26.1(1))). In [24], Ayache and Jaballah have pursued the study of normal pairs of integral domains. So, several characterizations of such pairs have been obtained. In [25–27], the authors have studied normal pairs of rings with zero divisors, so many results are generalized from the domain-theoretic case to arbitrary rings.

Recalling from [28] that given rings $R \subseteq S$ and an element $s \in S$, we say that s is primitive over R in case s is a root of a polynomial $f \in R[X]$ with unit content, that is, the coefficients of f generate the unit ideal of R. If each element of S is primitive over R, then $R \subseteq S$ is said to be a P-extension. Following [20] (p. 28), we let INC denote the incomparability property of ring extensions (so a ring extension $R \subseteq S$ satisfies INC if and only if distinct comparable prime ideals of S must contract to distinct prime ideals of R). As in [29], if $R \subseteq S$ is a ring extension, we say that (R, S) is an INC pair if $R \subseteq T$ satisfies INC for each S-overring T of R. It was proved in [30] (Theorem) that $R \subseteq S$ is a P-extension if and only if (R, S) is an INC pair. The authors in [25] (Theorem 1) ensure that (R, S) is a normal pair if and only if $R \subset S$ is a P-extension and R is integrally closed in S.

The next theorem characterizes ring extensions with only one non-Artinian intermediate ring.

Theorem 1. *Let $R \subset S$ be a ring extension. Then, the following statements are equivalent:*

(1) There exists a unique intermediate ring T between R and S such that T is not Artinian

(2) R is a maximal non-Artinian subring of S

(3) $R \subset S$ is a (closed) minimal extension and S is Artinian

Proof.

(1) $\Rightarrow$ (2). Proposition 1 asserts that R is not Artinian, and as, by assumption, there exists a unique intermediate ring between R and S which is not Artinian, it follows that R is a maximal non-Artinian subring of S.

(2)$\Rightarrow$(3). First, we note that $R \subset S$ is a P-extension or equivalently (R, S) is an INC pair. Indeed, if A is a proper S-overring of R, then A is zero dimensional (since it is Artinian), so clearly the ring extension $R \subset A$ satisfies incomparability. Now, we claim that R is integrally closed in S. Let $a \in S$ be an integral over R and suppose that $a \notin R$. Then, $R[a]$ is a proper S-overring of R. Hence, $R[a]$ is an Artinian ring. As $R \subset R[a]$ is an integral extension and $R[a]$ is zero dimensional, we infer that so too is R. It is also evident that R is Noetherian, and hence R is Artinian, which is a contradiction. We conclude using [25] (Theorem 1), see also the last comments in our introduction, that (R, S) is a normal pair. We will demonstrate that $R \subset S$ is a minimal ring extension. To this end, suppose that B is a proper S-overring of R, then (B, S) is a zero dimensional pair. Thus, the authors in [31] (Corollary 4.2) ensure that S is an integral over B. But, by what we have already observed, (B, S) must be a normal pair; hence, $B = S$, and we are done.

(3)$\Rightarrow$(1). It is enough to show that the ring R is not Artinian. To this end, assume the contrary. Then, as S is Artinian, $R \subset S$ would be an integral extension by virtue of [31] (Corollary 4.2), which is a contradiction. This completes the proof. □

Next, we treat the particular case, where S is an integral domain.

Corollary 1. *Let $R \subset S$ be an extension of integral domains. Then, the following statements are equivalent:*

(1) R is a maximal non-Artinian subring of S

(2) R is a rank one valuation domain with quotient field S

Proof.

(1)$\Rightarrow$(2). As S is an Artinian integral domain, then S is a field. According to Theorem 1, the ring extension $R \subset S$ is a closed minimal extension. Thus, S is the quotient field of R and R is a rank one valuation domain

(2)$\Rightarrow$(1). Trivial □

3. Ring Extensions with Exactly Two Non-Artinian Intermediate Rings

We start with the following result.

Proposition 2. *Let $R \subset S$ be a ring extension having exactly two non-Artinian intermediate rings. Then, either $R \subset S$ is a minimal extension and S is not Artinian or there is an intermediate ring T such that $R \subset T$ and $T \subset S$ are minimal extensions, T is integrally closed in S, S is Artinian, and T is not Artinian.*

Proof. According to Proposition 1, R is not Artinian. Let T be the second non-Artinian ring such that $R \subset T \subseteq S$. Notice that $R \subset T$ is a minimal ring extension. Indeed, assume the contrary and let T_1 be a ring such that $R \subset T_1 \subset T$. As each T-overring of T_1 properly contained in T is Artinian, then T would be Artinian by virtue of [1] (Theorem 2), a contradiction. If $T = S$, then $R \subset S$ is a minimal extension with both R and S non-Artinian. Suppose now that $T \neq S$. As the ring extension $R \subset S$ has exactly two non-Artinian intermediate rings, then S is Artinian. Since T is not Artinian and each proper S-overring of T is Artinian, then T would be the maximal non-Artinian subring of S. Thus, Theorem 1 guarantees that $T \subset S$ is a closed minimal extension. □

In the next theorem, we identify ring extensions $R \subset S$ with exactly two non-Artinian intermediate rings in case R is integrally closed in S.

Theorem 2. *Let $R \subset S$ be a ring extension such that R is integrally closed in S. Then, the following statements are equivalent:*

(1) There are exactly two non-Artinian intermediate rings between R and S

(2) Either $R \subset S$ is a (closed) minimal extension with S non-Artinian or $[R, S]$ is a chain of length 2 and S is Artinian

Proof.

(1)⇒(2). If $R \subset S$ is a minimal extension, then we are done. Hence, suppose now that $R \subset S$ is not a minimal extension. Then, according to Proposition 2, there exists an intermediate ring T such that $R \subset T$ and $T \subset S$ are minimal extensions, S is Artinian, and T is not Artinian. In view of [25] (Theorems 1 and 2), (R, S) is a normal pair. Now, we claim that $[R, S] = \{R, T, S\}$. Indeed, suppose that $A \in [R, S] \backslash \{R, T, S\}$. Then, (A, S) is an Artinian pair. Hence, S is integral over A ([31], Corollary 4.2). But, A is integrally closed in S since (R, S) is a normal pair. Thus, $A = S$, a contradiction.

(2)⇒(1). If $R \subset S$ is a minimal extension with S non-Artinian, then according to [1] (Theorem 2), R is not Artinian. Hence, we are done. Assume now that $[R, S]$ is a chain of length 2 and S is Artinian. Then, $[R, S] = \{R, T, S\}$, where $R \subset T$ and $T \subset S$ are (closed) minimal extensions. The ring T is not Artinian since otherwise (T, S) would be an Artinian pair and so S would be integral over T according to [31] (Corollary 4.2), which is impossible since T is integrally closed in S. Moreover, R is not Artinian since otherwise T would be Artinian by virtue of [1] (Theorem 2). Therefore, there are exactly two non-Artinian intermediate rings between R and S, namely, R and T. This completes the proof. □

In the following theorem, we determine all ring extensions $R \subset S$ with exactly two non-Artinian intermediate rings in case R is not integrally closed in S. But, first some facts about minimal ring extensions are recalled. According to [22] (Théorème 1(i) and Lemme 1.3), if $R \subset S$ is a minimal extension and R is not a field, then there exists a unique maximal ideal M of R called the crucial maximal ideal of $R \subset S$ such that the canonical injective ring homomorphism $R_M \longrightarrow S_M$ can be viewed as a minimal ring extension, while the canonical ring homomorphism $R_Q \longrightarrow S_Q$ is an isomorphism for all prime ideals Q of R, except M. If in addition $R \subset S$ is an integral extension, then M is precisely the conductor $(R\colon S)\colon = \{x \in R | xS \subseteq R\}$ (cf. [22], Théorème 1(ii)).

Theorem 3. *Let $R \subset S$ be a ring extension such that R is not integrally closed in S. Then, the following statements are equivalent:*

(1) There are exactly two non-Artinian intermediate rings between R and S

(2) Either $R \subset S$ is a minimal integral extension with S non-Artinian, or $[R, S]$ is a chain of length 2 such that S is Artinian and R is not Artinian, or $[R, S]$ consists of two chains of length 2 such that S is Artinian and R is not Artinian

Proof.

(1)⇒(2). If $R \subset S$ is a minimal extension, then it must be integral since R is not integrally closed in S. Moreover, as there are exactly two non-Artinian intermediate rings between R and S, then R and S should be non-Artinian. Now, assume that $R \subset S$ is not a minimal extension. Then, according to Proposition 2, there exists an intermediate ring T such that $R \subset T$ and $T \subset S$ are minimal extensions, S is Artinian, and T is not Artinian. Moreover, $T \subset S$ is a closed minimal extension. It follows that the minimal ring extension $R \subset T$ is an integral. In this case, $\overline{R}_S \in [T, S]$ and hence $T = \overline{R}_S$. Consider the set:

$$[R] \coloneqq \{B \in [R, S] | B \cap \overline{R}_S = R\}. \tag{1}$$

Claim 1. $[R]$ has a maximal element

Indeed, it is obvious that $[R]$ is nonempty since $R \in [R]$. The set $[R]$ equipped with the inclusion relation is a partially ordered set. Let now $\{B_i | i \in \Lambda\}$ be a totally ordered subfamily of $[R]$, and let $B = \cup_{i\in\Lambda} B_i$.

One can easily check that B is an intermediate ring between R and S, and $B\cap\overline{R}_S=\cup_{i\in\Lambda}(B_i\cap\overline{R}_S)=R$. Thus, $B\in[R]$. It follows, by virtue of Zorn's lemma, that $[R]$ has a maximal element.

Claim 2. $[R]$ has a greatest element

According to claim 1, $[R]$ has a maximal element, say Ω. We will show that Ω is the greatest element of $[R]$. If $[R]=\{R\}$, then clearly $\Omega=R$ is the greatest element of $[R]$. Assume now that $[R]\neq\{R\}$ and then Ω contains R properly. It is worth noticing that if $A\in[R]\backslash\{R\}$, then A and $\overline{R}_S$ are incomparable under inclusion. So, $A\in[R,S]\backslash\{R,\overline{R}_S,S\}$. Hence, A is Artinian. Clearly, R is a maximal non-Artinian subring of A. It follows that $R\subset A$ is a closed minimal extension, by Theorem 1. In particular, $R\subset\Omega$ is a closed minimal extension. It is not difficult to check that $\Omega\subset S$ is a minimal integral extension. Let $A\in[R]\backslash\{R\}$. We need to prove that $A\subseteq\Omega$. If $\Omega\subseteq A$, then by maximality of Ω, we get $A=\Omega$. Assume now that $\Omega\nsubseteq A$ and $A\nsubseteq\Omega$. Let $H=A\Omega$. As H is a proper S-overring of Ω and $\Omega\subset S$ is minimal, then necessarily $H=S$. Let M be the crucial maximal ideal of the minimal extension $R\subset\overline{R}_S$. It follows from [32] (Lemma 2.3) that $[R_M,S_M]=\{R_M,(\overline{R}_S)_M,S_M\}$. As R_M is integrally closed in A_M and in Ω_M, then a fortiori $R_M=A_M=\Omega_M$. Hence, $S_M=H_M=(A\Omega)_M=A_M\Omega_M=R_M$, which is the desired contradiction. We deduce that $A\subseteq\Omega$ and so Ω is the greatest element of $[R]$.

If $[R]=\{R\}$, then $[R,S]$ is a chain of length 2. More precisely, $[R,S]=\{R,\overline{R}_S,S\}$. Indeed, let $A\in[R,S]$. If $A\cap\overline{R}_S=R$, then $A\in[R]$. So $A=R$. If $A\cap\overline{R}_S=\overline{R}_S$, then $A\in[\overline{R}_S,S]$. So $A=\overline{R}_S$ or $A=S$. Now, if $[R]\neq\{R\}$, we claim that $[R,S]=\{R,\overline{R}_S,S,\Omega\}$. Indeed, let $B\in[R,S]$. If $B\cap\overline{R}_S=R$, then $B\in[R]$. So $B=R$ or $B=\Omega$ because $R\subset\Omega$ is minimal as noted above. If $B\cap\overline{R}_S=\overline{R}_S$, then B contains $\overline{R}_S$ and so $B=\overline{R}_S$ or $B=S$, which is the desired conclusion.

(2)⇒(1). If $R\subset S$ is a minimal extension and S is not Artinian, then we are done, by Proposition 1. Suppose now that $[R,S]$ is a chain of length 2, S is Artinian, and R is not Artinian. We claim that $[R,S]=\{R,\overline{R}_S,S\}$. Indeed, as R is not integrally closed in S, then $R\neq\overline{R}_S$. Now, assume that $\overline{R}_S=S$. As $[R,S]$ is a chain of length 2, then there exists a ring T such that $R\subset T\subset\overline{R}_S=S$. The ring T cannot be Artinian, since otherwise R would be a maximal non-Artinian subring of T and so R would be integrally closed in T by virtue of Theorem 1. Thus, $T=R$, which is absurd. It follows that $[R,S]=\{R,\overline{R}_S,S\}$ as claimed. The ring $\overline{R}_S$ cannot be Artinian, by Theorem 1. Therefore, there are exactly two non-Artinian intermediate rings between R and S, namely, R and $\overline{R}_S$. Now, assume that $[R,S]$ consists of two chains of length 2, S is Artinian and R is not Artinian. Then, there exist two incomparable rings T_1 and T_2 distinct from R and from S such that $[R,S]=\{R,T_1,T_2,S\}$. First, we handle the case where $\overline{R}_S=S$. Suppose that T_i for some i is not Artinian, then T_i would be a maximal non-Artinian subring of S. So, T_i would be integrally closed in S, by Theorem 1, which is impossible since S is integral over T_i and $S\neq T_i$. Thus, T_1 and T_2 are Artinian. It follows that R is a maximal non-Artinian subring of T_1. Thus, Theorem 1 ensures that R is integrally closed in T_1, a contradiction since T_1 is integral over R and $T_1\neq R$. Thus, we conclude that $\overline{R}_S\neq S$. Without loss of generality, we can suppose that $\overline{R}_S=T_1$. The ring $\overline{R}_S$ cannot be Artinian since otherwise by [31] (Corollary 4.2), we get $\overline{R}_S=S$, which is absurd. The ring T_2 is Artinian. Indeed, assuming the contrary, then T_2 would be a maximal non-Artinian subring of S and hence $T_2\subset S$ would be a closed minimal extension according to Theorem 1. Thus, $R\subset T_2$ is a minimal integral extension, for otherwise R would be integrally closed in S, which contradicts the assumption made on R. Hence, $T_2\subseteq\overline{R}_S$, a contradiction with the fact that T_2 and $T_1=\overline{R}_S$ are incomparable. Therefore, there are exactly two non-Artinian intermediate rings between R and S, namely, R and $\overline{R}_S$. The proof is complete. □

The following corollary treats the particular case, where S is an integral domain. It is worth noticing that an important step, toward the classification of minimal extensions of integral domains, was taken by Sato–Sugatani–Yoshida, who showed in [33] (page 1738, lines 8–13) that if $R\subset S$ is a minimal extension such that R is not a field, then S is an overring of R.

Corollary 2. *Let $R\subset S$ be an extension of integral domains. Then, the following statements are equivalent:*

(1) There are exactly two non-Artinian intermediate rings between R and S

(2) Either $R\subset S$ is a minimal extension and S is not a field, or R is a rank two valuation domain with quotient field S, or $R\subset R'$ is a minimal extension and R' is a rank one valuation domain with quotient field S

Proof.

(1)⇒(2). If $R\subset S$ is a minimal extension, then we are done. Thus, suppose that $R\subset S$ is not a minimal extension. It follows from Theorems 2 and 3 that S is a field and R is not a field. If R is integrally closed in S, then $[R,S]$ is a chain of length 2. According to [33] (page 1738, lines 8–13), $S=qf(R)$. Thus, R is a rank two valuation domain with quotient field S. If R is not integrally closed in S, then Theorem 3 and [33] (page 1738, lines 8–13) guarantee that $R\subset R'\subset S=qf(R)$ is a chain of length 2. Hence, $R\subset R'$ is a minimal extension and R' is a rank one valuation domain with quotient field S.

(2)⇒(1). If $R\subset S$ is a minimal extension and S is not a field, then R cannot be a field (see [22], Théorème 1). If R is a rank two valuation domain with quotient field S, then $[R,S]=\{R,V,S\}$, where V is a rank one valuation overring of R. Finally, if $R\subset R'$ is a minimal extension and R' is a rank one valuation domain with quotient field S, then the authors in [34] (Theorem 2.4) ensure that $[R, S] = \{R, R', S\}$. Therefore, in all cases, there are exactly two non-Artinian intermediate rings between R and S. This completes the proof. □

We close the paper with the following example. The authors would like to thank Professor Gabriel Picavet for providing them this example.

Example 1. This example provides a ring extension $R \subset S$ such that $[R, S]$ consists of two chains of length 2, namely, $R \subset \overline{R}_S \subset S$ and $R \subset T \subset S$, such that S and T are Artinian, whereas R and $\overline{R}_S$ are not Artinian.

Let (V, P) be a discrete valuation domain with quotient field K, so that $V \subset K$ is a minimal closed extension, and let $K \subset L$ be a minimal field extension (and then minimal integral). Set $R{:} = V \times K, T : = K \times K, S{:} = K \times L$, and $U{:} = V \times L$. Clearly, T and S are Artinian since they are products of two fields. Moreover, one can easily check that $(R{:}\, U) = V \times \{0\}$, $(U{:}\, S) = \{0\} \times L$, $(R{:}\, T) = \{0\} \times K$, and $(T{:}\, S) = K \times \{0\}$. In view of [35] (Proposition 4.7), we get that $R \subset U$ and $T \subset S$ are minimal integral extensions, while $R \subset T$ and $U \subset S$ are minimal closed extensions. This leads to $U = \overline{R}_S$. Now, it is not difficult to check that $P \times K$ (resp., $K \times \{0\}$) is the crucial maximal ideal of $R \subset T$ (resp., $T \subset S$). Since $(K \times \{0\}) \cap R = V \times \{0\} \nsubseteq P \times K$, crosswise exchange lemma (cf. [36], Lemma 2.7) asserts that $[R, S] = \{R, T, \overline{R}_S, S\}$. In particular, $[R, S]$ consists of two chains of length 2: $R \subset \overline{R}_S \subset S$ and $R \subset T \subset S$. The ring R (resp., $\overline{R}_S$) is not Artinian because $\{0\} \times K$ (resp., $\{0\} \times L$) is a prime nonmaximal ideal of R (resp., $\overline{R}_S$).

Data Availability

All data required for this paper are included within this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors acknowledge the Deanship of Scientific Research at King Faisal University for the financial support under the annual research project (Grant no. 180087).

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