Research Article

On Optimal M -Sets Related to Motzkin's Problem

Quan-Hui Yang ¹, Ting Pan ¹ and Jian-Dong Wu ²

¹School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

²School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, China

Correspondence should be addressed to Quan-Hui Yang; yangquanhui01@163.com

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Let M be a set of positive integers. A set S of nonnegative integers is called an M -set if a and $b \in S$, then $a - b \notin M$. If $S \subseteq \{0, 1, \dots, n\}$ is an M -set with the maximal cardinality, then S is called a maximal M -set of $\{0, 1, \dots, n\}$. If $S \cap \{0, 1, \dots, n\}$ is a maximal M -set of $\{0, 1, \dots, n\}$ for all integers $n \geq 0$, then we call S an optimal M -set. In this paper, we study the existence of an optimal M -set.

1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For a positive real number x and $S \subseteq \mathbb{N}$, we denote by $S(x)$ the number of elements $n \in S$ such that $n \leq x$. The upper and lower densities of S denoted by $\bar{\delta}(S)$ and $\underline{\delta}(S)$, respectively, are given by

$$\bar{\delta}(S) := \limsup_{x \rightarrow \infty} \frac{S(x)}{x}, \quad (1)$$

$$\underline{\delta}(S) := \liminf_{x \rightarrow \infty} \frac{S(x)}{x}.$$

If $\bar{\delta}(S) = \underline{\delta}(S)$, let $\delta(S) = \bar{\delta}(S) = \underline{\delta}(S)$, then we say that S has density $\delta(S)$.

Given a set M of positive integers, we call $S \subseteq \mathbb{N}$ an M -set if $a \in S$ and $b \in S$, then $a - b \notin M$. Motzkin [1] presented the problem of determining the quantity

$$\mu(M) := \sup_S \bar{\delta}(S), \quad (2)$$

where the supremum is taken over all M -sets S . Cantor and Gordon [2] proved that if $|M| = 1$, $\mu(M) = (1/2)$ and that if $M = \{m_1, m_2\}$, then $\mu(M) = (\lfloor (m_1 + m_2)/2 \rfloor / (m_1 + m_2))$. Later, Haralambis [3] determined $\mu(M)$ for many members of the families $\{1, j, k\}$ and $\{1, 2, j, k\}$. In 1999, Gupta and Tripathi [4] completely determined $\mu(M)$ when M is a finite

arithmetic progression. In 2011, Pandey and Tripathi [5] investigated this quantity when M is related to an arithmetic progression. After that, Pandey and Tripathi [6] and the first author and Tang [7] determined $\mu(M)$ when M is a geometric progression. For other results, one may refer to [8–11].

In 2010, Chen and Yang [12] and Khovanova and Konyagin [13] studied the upper density among sets of nonnegative integers in which no two elements have quotient belonging to M .

Suppose G is a graph and T is a set of nonnegative integers. A T -coloring of G is an assignment of a positive integer $f(x)$ to each vertex x of G so that if x and y are joined by an edge of G , then $f(x) - f(y)$ is not in T . T -colorings were introduced by Hale [14] in connection with the channel assignment problem in communications. Motzkin's problem has connections with the T -colouring problem. For more results on this topic, one can refer to [15–19].

Motzkin's problem also has connections with some other problems, such as problems related to the fractional chromatic number of distance graphs and the Lonely Runner Conjecture. One can refer to [9, 20–29].

In this paper, for any integers a and b , we define $[a, b] = \{a \leq n \leq b : n \in \mathbb{Z}\}$. For an M -set $S \subseteq [0, n]$, if $|S| \geq |S'|$ for any M -set $S' \subseteq [0, n]$, then S is said to be a

maximal M -set of $[0, n]$. If $S \cap [0, n]$ is a maximal M -set of $[0, n]$ for all integers $n \geq 0$, then we call S an optimal M -set. If M is finite, then an optimal M -set is infinite. For a finite set M , let $G(M)$ be the infinite sequence $0, a_1, a_2, \dots$ recursively by choosing a_{k+1} to be the least integer exceeding a_k such that $\{0, a_1, a_2, \dots, a_{k+1}\}$ is also an M -set. It is easy to see that, for a given set M , if there exists an optimal M -set, then this set must be $G(M)$.

In this paper, we determine all the optimal M -set when $|M| \leq 2$.

Theorem 1. *For any set M with $|M| = 1$, there exists an optimal M -set.*

Theorem 2. *Let $M = \{m_1, m_2\}$, where $m_1 < m_2$, and let $\gcd(m_1, m_2) = d$. Then, there exists an optimal M -set if and only if $m_1 | m_2$ or $m_2 = km_1 \pm d$ with an odd k .*

Remark 1. By Theorem 2, we know that there exists an optimal M -set when $m_1 = 2$.

2. Proofs of Theorem 1

Before presenting the proof of Theorem 1, we first give some lemmas here. Lemma 1 is a small generalization of Corollary 1, which is used by Cantor and Gordon in [2].

Lemma 1. *Let $M_1 = \{m_1, m_2, \dots\}$ and $M_2 = \{dm_1, dm_2, \dots\}$, where d is a positive integer. For any M_1 -sets $S_0, S_1, \dots, S_{d-1}$,*

$$S = \bigcup_{i=0}^{d-1} \{da + i : a \in S_i\}, \quad (3)$$

is an M_2 -set. Conversely, for any M_2 -set T and $i \in \{0, 1, \dots, d-1\}$,

$$T_i = \left\{ \frac{a-i}{d} : a \in T, a \equiv i \pmod{d} \right\}, \quad (4)$$

is an M_1 -set.

Proof. Suppose that there exist two elements $u, v \in S$ such that $u - v = dm_i$ for some positive integer i . Let $u = da + i$ and $v = db + j$ with $a, b \in S_j$ and $0 \leq i, j \leq d-1$. Then, by $d|u - v$, we obtain $i = j$. Hence, $a, b \in S_i$ and $a - b = m_i \in M_1$. This contradicts that S_i is an M_1 -set.

Conversely, suppose that there exists an integer i with $0 \leq i \leq d-1$ such that T_i is not an M_1 -set. Then, there exist $x, y \in T_i$ such that $x - y = m_i$ for some positive integer i . Let $x = ((m-i)/d)$, $y = ((n-i)/d)$, where m and $n \in T$. Then, $m - n = dm_i \in M_2$. This contradicts that T is an M_2 -set. $\square$

Corollary 1. *Let $M_1 = \{m_1, m_2, \dots\}$ and $M_2 = \{dm_1, dm_2, \dots\}$, where d is a positive integer. For any M_1 -sets S , the set $\bigcup_{i=0}^{d-1} \{da + i : a \in S\}$ is an M_2 -set.*

Corollary 1 follows from Lemma 1 immediately by taking $S_0 = S_1 = \dots = S_{d-1} = S$.

For two sets A and B and an integer d , let $A + B = \{a + b : a \in A, b \in B\}$,

$A - B = \{a - b : a \in A, b \in B\}$, $d \cdot A = \{da : a \in A\}$, and $d + A = \{d + a : a \in A\}$.

Lemma 2. *Let $M_1 = \{m_1, m_2, \dots\}$ and $M_2 = \{dm_1, dm_2, \dots\}$, where d is a positive integer. Then, $G(M_1)$ is an optimal M_1 -set if and only if $G(M_2)$ is an optimal M_2 -set.*

Proof. We first prove the relationship between $G(M_1)$ and $G(M_2)$, i.e.,

$$G(M_2) = d \cdot G(M_1) + D, \quad \text{where } D = \{0, 1, \dots, d-1\}. \quad (5)$$

Suppose that equality (5) does not hold. Let n be the smallest integer such that $G(M_2) \cap [0, n] \neq (d \cdot G(M_1) + D) \cap [0, n]$. For any integer $m \leq n$, let $m = q_m d + r_m$, where $0 \leq r_m \leq d-1$. Then, $n \in G(M_2)$ is equivalent to $n - m \notin M_2$ for all $m < n$ and $m \in G(M_2)$. If $r_m \neq r_n$, then $n - m \notin M_2$ must hold. Hence, we only need to consider the case $r_m = r_n$. So, $n \in G(M_2)$ is equivalent to $q_n - q_m \notin M_1$ for all integers q_m with $q_m < q_n$ and $q_m \in G(M_1)$, that is, $q_n \in G(M_1)$. Hence, $n \in G(M_2)$ is equivalent to $n \in d \cdot G(M_1) + D$, a contradiction.

Now, we shall prove $G(M_1)$ is an optimal M_1 -set if and only if $G(M_2)$ is an optimal M_2 -set.

Suppose that $G(M_2)$ is an optimal M_2 -set. If $G(M_1)$ is not an optimal M_1 -set, then there exists an integer n_1 and an M_1 -set $S_1 \subseteq [0, n_1]$ such that $|S_1| > |G(M_1) \cap [0, n_1]|$. Let $S_2 = d \cdot S_1 + D$. Then, $S_2 \subseteq [0, dn_1 + d - 1]$. By (5), we have $|S_2| > |G(M_2) \cap [0, dn_1 + d - 1]|$. By Lemma 1, S_2 is an M_2 -set. Thus, $G(M_2)$ is not an optimal M_2 -set.

Now, suppose that $G(M_1)$ is an optimal M_1 -set. If $G(M_2)$ is not an optimal M_2 -set, then there exists an integer n_2 and an M_2 -set $S_2 \subseteq [0, n_2]$ such that $|S_2| > |G(M_2) \cap [0, n_2]|$.

Suppose that $n_2 \in G(M_2)$. Let n'_2 be the maximal integer such that $n'_2 \leq n_2$ and $n'_2 \equiv d-1 \pmod{d}$. Then, by (5), we have $[n'_2 + 1, n_2] \subseteq G(M_2)$. Thus,

$$|S_2 \cap [0, n'_2]| > |G(M_2) \cap [0, n'_2]|. \quad (6)$$

Suppose that $n_2 \notin G(M_2)$. Let n''_2 be the minimal integer such that $n''_2 \geq n_2$ and $n''_2 \equiv d-1 \pmod{d}$. Then, by (5), we have $[n_2, n''_2] \cap G(M_2) = \emptyset$. Thus,

$$|S_2 \cap [0, n''_2]| > |G(M_2) \cap [0, n''_2]|. \quad (7)$$

Hence, without loss of generality, we may assume $n_2 \equiv d-1 \pmod{d}$. By (5), we have

$$G(M_2) \cap [0, n_2] = d \cdot \left(G(M_1) \cap \left[0, \frac{(n_2 - d + 1)}{d} \right] \right) + D. \quad (8)$$

For $0 \leq i \leq d-1$, let $T_i = \{((a-i)/d) : a \in S_2, a \equiv i \pmod{d}\}$. Then, $T_i \subseteq [0, ((n_2 - d + 1)/d)]$. Since $|S_2| > |G(M_2) \cap [0, n_2]|$, by (8), there exists an integer i_0 with $0 \leq i_0 \leq d-1$ such that $|T_{i_0}| > |G(M_1) \cap [0, ((n_2 - d + 1)/d)]|$. By Lemma 1, T_{i_0} is an M_1 -set. Thus, $G(M_1)$ is not an optimal M_1 -set.

Therefore, $G(M_1)$ is an optimal M_1 -set if and only if $G(M_2)$ is an optimal M_2 -set.

Proof of Theorem 1: it suffices to prove that $G(M)$ is an optimal M -set when $|M| = 1$. By Lemma 2, we may assume that $M = \{1\}$. Clearly, $G(M)$ is the set of all even numbers. Since any M -set S contains no consecutive integers, it follows that $G(M)$ is an optimal M -set. $\square$

3. Proofs of Theorem 2

In order to prove Theorem 2, we first prepare some lemmas. Let $|x|_m$ denote the absolute value of the absolutely least remainder of $x \pmod{m}$. The proof of Lemma 3 follows proofs of Theorem 1 and Theorem 4 (ii) in [2]. For the convenience of the reader, we present it here.

Lemma 3 (see [2], Theorem 1 and Theorem 4 (ii)). *Let $M = \{m_1, m_2\}$, where $\gcd(m_1 + m_2) = 1$ and $2|m_1 + m_2$. Then, there exists an M -set $S \subseteq [0, m_1 + m_2 - 1]$ such that $|S| = ((m_1 + m_2 - 1)/2)$.*

Proof. By $\gcd(m_1 + m_2) = 1$ and $2|m_1 + m_2$, we have $\gcd(m_2 - m_1, m_1 + m_2) = 1$, and then there exists c such that $c(m_2 - m_1) \equiv 1 \pmod{m_1 + m_2}$. Thus, $cm_2 - cm_1 \equiv 1 \pmod{m_1 + m_2}$. Noting that $cm_2 + cm_1 \equiv 0 \pmod{m_1 + m_2}$, we obtain $2cm_2 \equiv 1 \pmod{m_1 + m_2}$ and $2cm_1 \equiv -1 \pmod{m_1 + m_2}$. Hence, $2cm_1 \equiv -2cm_2 \equiv -1 \equiv m_1 + m_2 - 1 \pmod{m_1 + m_2}$, and then,

$$cm_1 \equiv -cm_2 \equiv \frac{m_1 + m_2 - 1}{2} \pmod{m_1 + m_2}. \quad (9)$$

Let

$$S = \bigcup_{r=0}^{(m_1+m_2-1)/2-1} \{n \in \mathbb{Z}: 0 \leq n \leq m_1 + m_2 - 1, cn \equiv r \pmod{m_1 + m_2}\}. \quad (10)$$

Then, $S \subseteq [0, m_1 + m_2 - 1]$ and $|S| = (m_1 + m_2 - 1)/2$. Next, we shall prove that S is an M -set. That is, if a and $b \in S$, then $a - b \notin M$. We obtain $ca = q_1(m_1 + m_2) + r_1$ and $cb = q_2(m_1 + m_2) + r_2$, where $0 \leq r_1, r_2 < (m_1 + m_2 - 1)/2$. Then, if $a - b \in M$, we have

$$c(a - b) = (q_1 - q_2)(m_1 + m_2) + (r_1 - r_2) = cm_1 \text{ or } cm_2. \quad (11)$$

It follows that $r_1 - r_2 \equiv cm_1$ or $cm_2 \pmod{m_1 + m_2}$. However, by $0 \leq r_1, r_2 < (m_1 + m_2 - 1)/2$ and (9), we have

$$|r_1 - r_2| < \frac{m_1 + m_2 - 1}{2} = |cm_1|_{m_1+m_2} = |cm_2|_{m_1+m_2}, \quad (12)$$

a contradiction.

In the next lemma, we study the period of $G(M)$. $\square$

Lemma 4. *Let $M = \{m_1, m_2\}$. Then, $G(M)$ is periodic with period $m_1 + m_2$.*

Proof. We shall prove that $n \in G(M)$ if and only if $n + m_1 + m_2 \in G(M)$ for all $n \geq 0$.

If $n \in G(M)$, then $n + m_1$ and $n + m_2 \notin G(M)$. By the definition of $G(M)$, we have $n + m_1 + m_2 \in G(M)$.

If $n \notin G(M)$, then we shall prove $n + m_1 + m_2 \notin G(M)$. Suppose that $n + m_1 + m_2 \in G(M)$. Then, $n + m_1$ and $n + m_2 \notin G(M)$. If $n + m_2 - m_1 \notin G(M)$, by $n \notin G(M)$ and the definition of $G(M)$, we have $n + m_2 \in G(M)$, a contradiction. Thus, we have $n + m_2 - m_1 \in G(M)$, and then $n - m_1 \notin G(M)$. Similarly, we also have $n + m_1 - m_2 \in G(M)$ and $n - m_2 \notin G(M)$. Since $n - m_1 \notin G(M)$ and $n - m_2 \notin G(M)$, by the definition of $G(M)$, it follows that $n \in G(M)$, a contradiction. Hence, $n + m_1 + m_2 \notin G(M)$.

Therefore, $G(M)$ is a periodic set with period $m_1 + m_2$.

In 2012, the authors of [30] study the difference sets containing integer powers. Lemma 5 is the main theorem in [30], which is used to prove Lemma 6. $\square$

Lemma 5 (see [30], Theorem 1). *Let $n > m \geq 2$ be positive integers and $n = (m + 1)l + r$, where $0 \leq r \leq m$. Let*

$$\tau(m, n) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 1, & \text{if } m \text{ is odd,} \\ \left(\frac{ml}{2} \right) + \delta, & \text{if } m \text{ is even,} \end{cases} \quad (13)$$

where δ denotes the cardinality of even numbers in the interval $[0, \min\{r, m - 2\}]$. Then, we have the following:

- (i) *If C is a subset of $[0, n]$ with $|C| > \tau(m, n)$, then $C - C$ contains either 1 or m*
- (ii) *There exists a subset C of $[0, n]$ with $|C| = \tau(m, n)$ such that $C - C$ contains no power of m*

Lemma 6. *Let $M = \{1, m_2\}$. Then, $G(M)$ is an optimal M -set.*

Proof. It is easy to see that, if m_2 is odd, then $G(M)$ is the set of all even integers; if m_2 is even, then

$$G(M) = \bigcup_{k=0}^{\infty} A_k, \quad \text{where } A_k = k(m_2 + 1) + 2 \cdot \left[0, \frac{m_2}{2} - 1 \right]. \quad (14)$$

Hence, $|G(M) \cap [0, n]| = \tau(m_2, n)$. By Lemma 5 (i), there is no set $C \subseteq [0, n]$ such that $|C| > |G(M) \cap [0, n]|$ and C is also an M -set. Hence, $G(M)$ is an optimal M -set.

Proof of Theorem 2: it suffices for us to determine M such that $G(M)$ is an optimal M -set. By Lemma 2, we may assume $d = 1$. If $m_1 = 1$, by Lemma 6, then $G(M)$ is an optimal M -set. Now, we suppose that $m_1 > 1$. Noting that $(m_1, m_2) = 1$, we write $m_2 = km_1 + r$, where $1 \leq r \leq m_1 - 1$. If k is odd, then

$$G(M) \cap [0, m_1 + m_2 - 1] = \bigcup_{j=0}^{((k-1)/2)} \{2jm_1 + s: 0 \leq s \leq m_1 - 1\}, \quad (15)$$

and so $|G(M) \cap [0, m_1 + m_2 - 1]| = ((k + 1)m_1/2)$. If k is even, then

$$G(M) \cap [0, m_1 + m_2 - 1] = \bigcup_{j=0}^{(k/2)-1} \{2jm_1 + s : 0 \leq s \leq m_1 - 1\} \cup [km_1, km_1 + r - 1]. \quad (16)$$

It follows that $|G(M) \cap [0, m_1 + m_2 - 1]| = (km_1/2) + r$. $\square$

Case 1. $2|m_1, 2|m_2$. Let M' be the set of all even numbers contained in $[0, m_1 + m_2 - 1]$. Clearly, M' is an M -set and $|M'| = ((m_1 + m_2)/2)$. Since $((m_1 + m_2)/2) > ((k+1)m_1/2)$ and $((m_1 + m_2)/2) > (km_1/2) + r$, we have $|M'| > |G(M) \cap [0, m_1 + m_2 - 1]|$. Hence, $G(M)$ is not an optimal M -set.

Case 2. $2|m_1 + m_2$. By Lemma 3, there exists an M -set $S \subseteq [0, m_1 + m_2 - 1]$ such that $|S| = ((m_1 + m_2 - 1)/2)$. It is easy to check that $|S| > |G(M) \cap [0, m_1 + m_2 - 1]|$ except for the following two cases: (a) k is odd and $r = 1$ and (b) k is even and $r = m_1 - 1$. Hence, $G(M)$ is not an optimal M -set except for these two cases.

Now, we shall prove that $G(M)$ is an optimal M -set in cases (a) and (b). By Lemma 4, $G(M)$ is periodic with period $m_1 + m_2$. Hence, it suffices for us to prove that $G(M) \cap [0, n]$ is a maximal M -set of $[0, n]$ for any integer $n \in [0, m_1 + m_2 - 1]$.

Subcase 1: k is odd and $r = 1$. Then, $m_2 = km_1 + 1$. Suppose that $T \subseteq [0, m_1 + m_2 - 1]$ is an M -set. Noting that $m_1 \in M$, we have

$$|T \cap \{2jm_1 + i, (2j+1)m_1 + i\}| \leq 1, \quad (17)$$

for all integers i, j with $0 \leq i \leq m_1 - 1$ and $0 \leq j \leq ((k-1)/2)$. Since

$$\left\{ 2jm_1 + i : 0 \leq i \leq m_1 - 1, 0 \leq j \leq \frac{(k-1)}{2} \right\} \subseteq G(M). \quad (18)$$

It follows that $G(M) \cap [0, n]$ is a maximal M -set of $[0, n]$ for all $n \leq (k+1)m_1 - 1$.

Now, we consider the case $n = (k+1)m_1 = m_1 + m_2 - 1$. If $|T| > |G(M) \cap [0, m_1 + m_2 - 1]|$, then we have $(k+1)m_1 \in T$ and

$$|T \cap \{2jm_1 + i, (2j+1)m_1 + i\}| = 1, \quad (19)$$

for all integers i, j with $0 \leq i \leq m_1 - 1$ and $0 \leq j \leq ((k-1)/2)$. Hence,

$$\begin{aligned} (k+1)m_1 \in T &\stackrel{m_2 \in M}{\Rightarrow} m_1 - 1 \notin T \stackrel{(3.2)}{\Rightarrow} 2m_1 - 1 \in T \\ &\stackrel{(3.2)}{\Rightarrow} \dots \stackrel{(3.2)}{\Rightarrow} (k+1)m_1 - 1 \in T \\ m_2 \in M &\stackrel{(3.2)}{\Rightarrow} m_1 - 2 \notin T \stackrel{(3.2)}{\Rightarrow} 2m_1 - 2 \in T \\ &\stackrel{(3.2)}{\Rightarrow} \dots \stackrel{(3.2)}{\Rightarrow} (k+1)m_1 - 2 \in T. \end{aligned} \quad (20)$$

Similarly, we have $\{(k+1)m_1 - 3, (k+1)m_1 - 4, \dots, km_1\} \subseteq T$. On the other hand, by $(k+1)m_1 \in T$ and $m_1 \in M$, we have $km_1 \notin T$, a contradiction. Thus, $G(M)$ is an optimal M -set in this case.

Subcase 2: k is even and $r = m_1 - 1$. Then, $m_2 = (k+1)m_1 - 1$. Suppose that $T' \subseteq [0, m_1 + m_2 - 1]$ is an M -set. By $m_1 \in M$, we have

$$|T' \cap \{2jm_1 + i, (2j+1)m_1 + i\}| \leq 1, \quad (21)$$

for all integers i, j with $0 \leq i \leq m_1 - 1$ and $0 \leq j \leq (k/2) - 1$. However,

$$\bigcup_{j=0}^{k/2-1} \{2jm_1 + s : 0 \leq s \leq m_1 - 1\} \cup [km_1, m_2 - 1] \subseteq G(M). \quad (22)$$

It follows that $G(M) \cap [0, n]$ is a maximal M -set of $[0, n]$ for all $n \leq m_2 - 1$.

Now, we consider the case $m_2 \leq n \leq m_1 + m_2 - 1$. If $G(M) \cap [0, m_1 + m_2 - 1]$ is a maximal M -set of $[0, m_1 + m_2 - 1]$, by $[n+1, m_1 + m_2 - 1] \cap G(M) = \emptyset$, then $G(M) \cap [0, n]$ is a maximal M -set of $[0, n]$. Hence, it is enough for us to prove $G(M) \cap [0, m_1 + m_2 - 1]$ is a maximal M -set of $[0, m_1 + m_2 - 1]$. If $|T'| > |G(M) \cap [0, m_1 + m_2 - 1]|$, then we have

$$|T' \cap \{2jm_1 + i, (2j+1)m_1 + i\}| = 1, \quad (23)$$

for all integers i, j with $0 \leq i \leq m_1 - 1$ and $0 \leq j \leq (k/2)$. In particular, $(k+1)m_1 - 1 \in T'$, since $(k+2)m_1 - 1 = m_1 + m_2 \notin T'$. Hence,

$$\begin{aligned} (k+1)m_1 - 1 \in T' &\stackrel{m_2 \in M}{\Rightarrow} 0 \notin T' \stackrel{(3.3)}{\Rightarrow} m_1 \in T' \stackrel{(3.3)}{\Rightarrow} \dots \\ &\stackrel{(3.3)}{\Rightarrow} km_1 \notin T' \stackrel{(3.3)}{\Rightarrow} (k+1)m_1 \in T' \\ &\stackrel{m_2 \in M}{\Rightarrow} 1 \notin T' \stackrel{(3.3)}{\Rightarrow} \dots \stackrel{(3.3)}{\Rightarrow} km_1 + 1 \notin T'. \end{aligned} \quad (24)$$

Similarly, we obtain $[km_1 + 2, (k+1)m_1 - 1] \cap T' = \emptyset$. It is a contradiction with $(k+1)m_1 - 1 \in T'$. Hence, $G(M)$ is also an optimal M -set in this case.

Therefore, $G(M)$ is an optimal M -set if and only if $m_1 = 1$ or $m_2 = km_1 \pm 1$, where k is odd.

At the end of this paper, we pose a problem for further research.

Problem 1. Given an integer $k \geq 3$, determine all sets M with $|M| = k$ such that there exists an optimal M -set.

Remark 2. If $G(M)$ is optimal, then $G(M)$ gives the value of $\mu(M)$. If $G(M)$ is not optimal, we cannot find any information about the value of $\mu(M)$. Thus, Problem 1 and Motzkin's problem are incomparable.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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