Research Article

Solutions of a Class of Degenerate Kinetic Equations Using Steepest Descent in Wasserstein Space

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We use the steepest descent method in an Orlicz–Wasserstein space to study the existence of solutions for a very broad class of kinetic equations, which include the Boltzmann equation, the Vlasov–Poisson equation, the porous medium equation, and the parabolic p -Laplacian equation, among others. We combine a splitting technique along with an iterative variational scheme to build a discrete solution which converges to a weak solution of our problem.

1. Introduction

The general model describing the kinetic equations is about an evolution equation of unknown function f , representing a time-depending density of probability distribution of a material in a given domain of the space. In the present work, f may measure the density distribution of a system of identical particles of a bulk material. The density f depends

on the time t and the position x and the velocity v of some particles at t . Roughly speaking, the equation is considered as the evolution of the density function in the phase space $\Omega \times \mathbb{R}^d$, with Ω as an open bounded domain with periodic boundary. As a probability density, f remains positive in the court of time and satisfies the mass conservation principle: $\int_{\Omega \times \mathbb{R}^d} f(t, x, v) dx dv = 1$, for all $t \geq 0$:

$$\begin{cases} \frac{\partial f(t, x, v)}{\partial t} + v \cdot \nabla_x f(t, x, v) = \operatorname{div}_v \{f(t, x, v) \nabla c^* [\nabla_v (G'(f(t, x, v)))]\}, & \text{in } [0, \infty) \times \Omega \times \mathbb{R}^d, \\ f(0, x, v) = f_0(x, v), & \text{in } \Omega \times \mathbb{R}^d, \end{cases} \quad (1)$$

where the initial datum $f_0(x, v)$ is a probability density on $\Omega \times \mathbb{R}^d$. Here, Ω is an open, bounded, convex, and smooth domain of $\mathbb{R}^d$, ($d \geq 1$) with $\partial\Omega$ periodic, c^* is the Legendre transform of a cost function c , and $G: [0, \infty) \rightarrow \mathbb{R}$ is a convex function.

Equation (1) can be viewed as a balance result of a streaming phenomenon with a general nonlinear interaction phenomenon between the particles described, respectively, as

$$\frac{\partial f(t, x, v)}{\partial t} + v \cdot \nabla_x f(t, x, v) = 0, \quad (2)$$

$$\frac{\partial f(t, x, v)}{\partial t} - \operatorname{div}_v \{f(t, x, v) \nabla c^* [\nabla_v (G'(f(t, x, v)))]\} = 0. \quad (3)$$

Accordingly, the transport equation (2) can somewhat be interpreted as a relaxation of (1) at the absence of the

interaction phenomena, whereas it reduced to (3) in absence of streaming.

One of the interests in considering (1) under a general nonlinearity is that it covers a very broad range of problems which occurred in physics and it is a purely mathematical challenge. Of course, it has been motivated by some previous works in the literature, namely, the works in [1–7], where (1) is investigated in some particular cases. Indeed, in [3], the authors dealt with the heat equation:

$$\frac{\partial \rho(t, v)}{\partial t} = \Delta_v \rho(t, v), \quad \text{in } [0, \infty) \times \mathbb{R}^d. \quad (4)$$

By fixing an probability density ρ_0 with $\int_{\mathbb{R}^d} |v|^2 \rho_0(v) dv$ finite and a time step $h > 0$, they define the mass density ρ_k^h as a discrete solution of (4) at time $t_k = hk$, which minimizes the functional

$$I(\rho) := \int_{\mathbb{R}^d} \rho \ln \rho dv + \frac{1}{h} W_2^2(\rho, \rho_{k-1}^h), \quad (5)$$

on $P_2(\mathbb{R}^d)$, where $P_2(\mathbb{R}^d)$ is the set of all probability density on $\mathbb{R}^d$ having finite second moments and W_2 is the 2-Wasserstein metric defined as

$$W_2: P_2(\mathbb{R}^d) \times P_2(\mathbb{R}^d) \longrightarrow \mathbb{R}, \quad (\rho_1, \rho_2) \longmapsto \inf_{T: \rho_1 \rightarrow \rho_2} \left(\int_{\mathbb{R}^d} |v - T(v)|^2 \rho_1(v) dv \right)^{1/2}. \quad (6)$$

By defining ρ^h as follows: $\{\rho^h(t, v) = \rho_k^h(v), \text{ if } t \in [hk, h(k+1)), \rho^h(0, v) = \rho_0(v)\}$, they tend h to 0 and then show that the sequence $(\rho^h)_h$ converges to a nonnegative function ρ , which solves (4) in a weak sense.

In [1], the existence of solutions for the spatially homogeneous equations associated with (1), that is, the equation for fixed x

$$\frac{\partial f}{\partial t} = \text{div}_v \{ f \nabla c^* (\nabla_v (G'(f))) \}, \quad \text{in } [0, \infty) \times \Omega', \quad f = f(t, v), \quad (7)$$

has been proved by M. Agueh, using a similar variational scheme as in [6]. Here, Ω' is an bounded and convex domain (see [6] for more details).

A particular case of (1), namely, the kinetic equation

$$\frac{\partial f(t, x, v)}{\partial t} + v \cdot \nabla_x f(t, x, v) = \Delta_v f(t, x, v), \quad (8)$$

obtained by choosing $c^*(x) = |x|^2/2$ and $G(x) = x \ln x$, has been studied in [5] by using a discretization scheme basing on the “splitting method.” This enables the authors to decompose a discrete solution f_k of the kinetic equation (8) in the form $f_k = \rho_k^h F_k^{xh}$, where ρ_k^h stands for a discrete solution of the free transport equation

$$\frac{\partial f(t, x, v)}{\partial t} + v \cdot \nabla_x f(t, x, v) = 0, \quad (9)$$

when v is fixed and F_k^{xh} a discrete solution of the diffusion equation in (8) when x is fixed.

Defining f^h as

$$\begin{aligned} f^h(t, x, v) &= f_k((x - (t - t_k)v, v), \quad \text{if } t \in [kh, (k+1)h), \\ f^h(0, x, v) &= f_0(x, v), \quad \text{with } t_k = kh, \end{aligned} \quad (10)$$

they show that $(f^h)_h$ converges to a nonnegative function f which solves the kinetic equation (8) in a weak sense.

Such a decomposition is not suitable in the case of problem (1) because of its nonlinear structure. To deal with the more general class of kinetic equation (1), we combine some ideas from the splitting method in [5] along with some techniques developed in [1] for the spatially homogeneous equations:

$$\frac{\partial \rho(t, v)}{\partial t} = \text{div}_v (\rho \nabla c^* (\nabla_v G'(\rho))), \quad \text{in } [0, \infty) \times \mathbb{R}^d. \quad (11)$$

For the best of our knowledge, our technique is new and is stated in a more general setting. It is worth mentioning that the class of the kinetic equation (1) also includes the Vlasov–Poisson equation

$$\frac{\partial f(t, x, v)}{\partial t} + v \cdot \nabla_x f(t, x, v) = \text{div}_v (\omega f(t, x, v)), \quad (12)$$

obtained when

$$c^*(x) = \sum_{i=1}^d x_i, \quad \omega = (1, 1, \dots, 1) \in \mathbb{R}^d, \quad (13)$$

and the parabolic p -Laplacian equation

$$\frac{\partial f(t, x, v)}{\partial t} + v \cdot \nabla_x f(t, x, v) = \text{div}_v \{ |\nabla f|^{p-2} \nabla f \}, \quad (14)$$

in the case $c^*(x) = |x|^p/p$ and $G(x) = x^\gamma/\gamma(\gamma-1)$ with $\gamma = (2p-3/p-1)$.

In order to facilitate the reading of the paper, we summarize below the main steps and technical schemes according to which ours results will be carried out:

- (1) First of all, we fix a time step $h > 0$ and define f_k as a discrete solution of the kinetic equation (1) at time $t_k = hk$, for $k \in \mathbb{N}$ (see Section 2.1).
- (2) Next, we prove that the solution T_k of the Monge problem

$$(M): \inf_{S: F_k^{xh} \rightarrow G_k^{xh}} \left\{ \int_{\mathbb{R}^d} c \left(\frac{v - S(v)}{h} \right) F_k^{xh} dv \right\}, \quad (15)$$

is defined by

$$T_k(v) := v + h \nabla c^* (\nabla_v G'(\rho_k F_k^{xh})), \quad (16)$$

where F_k^{xh} and G_k^{xh} are as in Section 2.1. We use (16) to show that the sequence $(f_k)_k$ satisfies the time-discretization equation of the kinetic equation (1)

$$\frac{f_k - f_{k-1}}{h} + \nu \cdot \nabla_x f_{k-1} = \text{div}_\nu (f_k \nabla c^* (\nabla_\nu G'(f_k))) + A_k^h, \quad (17)$$

weakly, for $k \in \mathbb{N}$, where A_k^h tends to 0 and when h tends to 0.

- (3) Then, we define an approximate solution f^h of the kinetic equation (1) (see (118)), and we prove that the sequence $(f^h)_h$ converges to a nonnegative function f which solves the kinetic equation (1) in a weak sense when h tends to 0.

The convergence result has been achieved as follows:

- (a) The weak convergence of $(f^h)_h$ to f in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ for $0 < T < \infty$ follows from the displacement convexity of the functional $H(F) = \int_{\mathbb{R}^d} |F|^p d\nu$ on the set of all probability density (see Proposition 3), and its strong convergence in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ is obtained, thanks to a diagonal method combined with a result obtained in [1].
- (b) The convexity of c^* and the boundedness of $(\nabla_\nu (G'(f^h)))_h$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ help to prove a weak convergence of the nonlinear term $(\text{div}_\nu (f^\nabla c^* (\nabla_\nu G'(f^h))))_h$ to $\text{div}_\nu (f \nabla c^* (\nabla_\nu G'(f)))$ in $[C_c^\infty([0, T] \times \Omega \times \mathbb{R}^d)]'$.
- (c) Finally, the strong convergence of $(f^h)_h$ to f in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ and the weak convergence of the nonlinear term $(\text{div}_\nu (f^\nabla c^* (\nabla_\nu G'(f^h))))_h$ to $\text{div}_\nu (f \nabla c^* (\nabla_\nu G'(f)))$ in $[C_c^\infty([0, T] \times \Omega \times \mathbb{R}^d)]'$ enable us to establish that f is a weak solution of the kinetic equation (1).

The paper is structured as follows. In Section 2, we state the required hypotheses and set some tools relevant for our problem. In Section 3, we set the variational formulation of the discrete problem related to our problem and construct the discrete solution. Section 4 concludes our main result by proving the convergence of the discrete problem to the considered problem. Section 5 ends the paper by giving an illustration example followed by an appendix on some regularity results.

2. Preliminaries

Throughout this work, we will assume the following:

$(H_G)G: [0, \infty) \rightarrow \mathbb{R}$ is a convex function of class $C^2((0, \infty))$ such that $t \rightarrow t^d G(t^{-d})$ is convex.

$(H_c)c \in C^1(\mathbb{R}^d)$ is even and convex function such that $c(0) = 0$ and $A_1 |x|^q \leq c(x) \leq A_2 (|x|^q + 1)$, for all $x \in \mathbb{R}^d$, with $1 < q < \infty$.

$(H_{f_0})f_0$ is a probability density on $\Omega \times \mathbb{R}^d$ such that

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^d} |v|^q f_0(x, v) dx dv &< \infty, \\ \int_{\Omega \times \mathbb{R}^d} G(f_0(x, v)) dx dv &< \infty. \end{aligned} \quad (18)$$

$f_0 \in L^p(\Omega \times \mathbb{R}^d)$ with $p = (q/q - 1)$ and $N \leq f_0 \leq M$ with $0 < N$.

Remark 1. Typical examples satisfying assumption (H_G) are the functions $G(s) = s \ln s$, $s > 0$ and $G(s) = s^r$, $r > 1$.

Proposition 1. Assume that c satisfies H_c . Let $H: [0, \infty) \rightarrow \mathbb{R}$ be a convex function such that $x \mapsto x^d H(x^{-d})$ is convex and decreasing.

Then, the functional

$$\begin{aligned} E: P_q(\mathbb{R}^d) &\rightarrow \mathbb{R}, \\ F &\mapsto \int_{\mathbb{R}^d} H(F) d\nu, \end{aligned} \quad (19)$$

is displacement convex.

Proof. Let $F_0, F_1 \in P_q(\mathbb{R}^d)$ and $T_\# F_1 = F_0$ be a c -optimal map that pushes F_1 forward to F_0 . We define $T_t = (1 - t)\text{id} + tT$ and $F_t = T_{t\#} F_1$, where $t \in [0, 1]$.

Then, we have

$$E(F_t) = \int_{\mathbb{R}^d} H(F_t) d\nu = \int_{F_t > 0} \frac{H(F_t)}{F_t} F_t d\nu = \int_{\mathbb{R}^d} \frac{H(F_t \circ T_t)}{F_t \circ T_t} F_1 d\nu. \quad (20)$$

From [1] and Proposition 3, we have that

$$F_1 = \det(\nabla T_t) F_t \circ T_t. \quad (21)$$

Replacing (21) in (20), we obtain

$$E(F_t) = \int_{\mathbb{R}^d} \frac{H(F_1 / \det \nabla T_t)}{F_1 / \det \nabla T_t} F_1 d\nu. \quad (22)$$

Recalling again [1] and Proposition 3, we get that ∇T_t is diagonalizable with positive eigenvalues. So, using the fact that the map $A \mapsto (\det A)^{1/d}$ is concave on the set of $d \times d$ diagonalizable matrices with positive eigenvalues, we get

$$(\det \nabla T_t)^{1/d} \geq (1 - t) + t(\det \nabla T)^{1/d}. \quad (23)$$

Since $x \mapsto x^d H(x^{-d})$ is decreasing, then

$$\frac{H(F_1 / \det \nabla T_t)}{F_1 / \det \nabla T_t} \leq ((1 - t)F_1^{-1/d} + tF_1^{-1/d} (\det \nabla T)^{1/d})^d \quad (24)$$

$$H\left(\left((1 - t)F_1^{-1/d} + tF_1^{-1/d} (\det \nabla T)^{1/d}\right)^{-d}\right).$$

From (23) and (24) and the fact that $x \mapsto x^d H(x^{-d})$ is convex, we obtain

$$\begin{aligned} E(F_t) &\leq (1 - t)E(F_1) + t \int_{\mathbb{R}^d} \frac{H(F_1 / \det \nabla T)}{F_1 / \det \nabla T} F_1 d\nu \\ &= (1 - t)E(F_1) + t \int_{\mathbb{R}^d} H(F_0) d\nu \\ &= (1 - t)E(F_1) + tE(F_0). \end{aligned} \quad (25)$$

Hence, we conclude that the functional E is displacement convex. $\square$

Corollary 1. *Since the functional E is displacement convex, we have that*

$$E(F_0) - E(F_1) \geq \frac{dE(F_t)}{dt} \Big|_{t=0} = \int_{\mathbb{R}^d} \left\langle \nabla_v \left(H'(F_1) \right), T(v) - v \right\rangle F_1 dv. \quad (26)$$

$F_1, F_0 \in P_q(\mathbb{R}^d)$ and $T_{\#}F_1 = F_0$ is the c -optimal map that pushes F_1 forward to F_0 .

Definition 1. Let $f_0 \in L^1(\Omega \times \mathbb{R}^d)$ be nonnegative. We say that a nonnegative function $f(t, x, v)$ on $[0, \infty) \times \Omega \times \mathbb{R}^d$ is a weak solution of (1) in the time's interval $[0, T]$ for some $T > 0$, if for every test function $\psi \in C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)$ with time support $[-T, T]$, we have

$$\int_0^T \int_{\Omega \times \mathbb{R}^d} [\partial_t \psi + v \cdot \nabla_x \psi - \nabla_v \psi \cdot \nabla c^*(\nabla_v G'(f))] f(t, x, v) dt dx dv = - \int_{\Omega \times \mathbb{R}^d} f_0(x, v) \psi(0, x, v) dx dv. \quad (27)$$

2.1. The Flow and Descend Algorithm. Assume that the probability density $f_0(x, v)$ satisfies (H_{f_0}) and fix $h > 0$ a time step, then we define the following:

- (1) $g_1^h(x, v) := f_0^h(x - hv, v) = f_0(x - hv, v)$
- (2) $\rho_1^h(x) := \int_{\mathbb{R}^d} g_1^h(x, v) dv$
- (3) $G_1^{xh}(v) := g_1^h(x, v) / \rho_1^h(x)$ for $x \in \Omega$ fixed such that $\rho_1^h(x) > 0$
- (4) $f_1^h(x, v) := \rho_1^h(x) F_1^{xh}(v)$

For each x fixed, F_1^{xh} denotes the unique minimizer of the variational problem:

$$(P_1^{xh}): \inf_{F \in P_q(\mathbb{R}^d)} \{I(F) := E(F) + hW_c^h(F, G_1^{xh})\}. \quad (28)$$

$P_q(\mathbb{R}^d)$ is the set of all probability density on $\mathbb{R}^d$ having a finite q -moment, and $W_c^h(F, G_1^{xh})$ stands for the Kantorovich work defined as

$$W_c^h(F, G_1^{xh}) = \inf_{\gamma \in \Pi(F, G_1^{xh})} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{y-z}{h}\right) d\gamma(y, z) \right\},$$

$$E(F) := \frac{1}{\rho_1^h(x)} \int_{\mathbb{R}^d} G(\rho_1^h(x) F(v)) dv. \quad (29)$$

We obtain the terms f_k^h , for $k \geq 2$, by induction as follows:

- (i) By fixing v , we define $g_k^h(x, v) := f_{k-1}^h(x - hv, v)$.
- (ii) By fixing x , we define

$$f_k^h(x, v) := \rho_k^h(x) F_k^{xh}(v), \quad (30)$$

where

$$\rho_k^h(x) := \int_{\mathbb{R}^d} g_k^h(x, v) dv, \quad (31)$$

and F_k^{xh} is the unique minimizer of the variational problem.

$$(P_k^{xh}): \inf_{F \in P_q(\mathbb{R}^d)} \{I(F) := E(F) + hW_c^h(F, G_k^{xh})\}, \quad (32)$$

with

$$G_k^x(v) := \frac{g_k^h(x, v)}{\rho_k^h(x)}, \quad (33)$$

$$E(F) := \frac{1}{\rho_k^h(x)} \int_{\mathbb{R}^d} G(\rho_k^h(x) F(v)) dv.$$

Existence of F_1^{xh} and F_k^{xh} will be proved farther in Sections 2 and 3, respectively.

2.2. c -Wasserstein Metric. In this section, we define a Wasserstein metric corresponding to a cost function c , and we study its topology.

Definition 2. Assume that $c: \mathbb{R}^d \rightarrow [0, \infty)$ satisfies H_c . Let $\rho_1, \rho_2 \in P(\mathbb{R}^d)$ two probability measures on $\mathbb{R}^d$. We define the c -Wasserstein metric between ρ_1 and ρ_2 by

$$W_c(\rho_1, \rho_2) := \inf \left\{ \lambda > 0, \inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) d\gamma(x, y) \leq 1 \right\}. \quad (34)$$

Theorem 1. Assume that $c: \mathbb{R}^d \rightarrow [0, \infty)$ satisfies H_c . Then, W_c is a distance on the probability space $P(\mathbb{R}^d)$. Furthermore, if $(\rho_n)_n$ is a sequence in $P(\mathbb{R}^d)$ and $\rho \in P(\mathbb{R}^d)$, then $(\rho_n)_n$ converges to ρ in the metric space $(P(\mathbb{R}^d), W_c)$ if and only if $(\rho_n)_n$ converges narrowly to ρ in $P(\mathbb{R}^d)$.

Proof. Let $\rho_1, \rho_2 \in P(\mathbb{R}^d)$ be two probability measures on $\mathbb{R}^d$ such that $W_c(\rho_1, \rho_2) = 0$. Then, there exists a sequence $(\lambda_n)_n$ in $(0, \infty)$ which converges to 0 such that

$$\inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda_n}\right) d\gamma \leq 1, \quad \text{for all } n \in \mathbb{N}. \quad (35)$$

Denote by γ_n the solution of Kantorovich problem:

$$(K_n): \inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda_n}\right) d\gamma. \quad (36)$$

Then, we obtain

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda_n}\right) d\gamma_n \leq 1, \quad \text{for all } n \in \mathbb{N}. \quad (37)$$

Since λ_n converges to 0, then $|x-y/\lambda_n|$ tends to ∞ and when n goes to ∞ , for all $x, y \in \mathbb{R}^d$ such that $x \neq y$. Then, using the fact that c is coercive, we deduce that there exists $N_0 \in \mathbb{N}$ such that

$$c\left(\frac{x-y}{\lambda_n}\right) \geq \left|\frac{x-y}{\lambda_n}\right|, \quad \text{for all } n \geq N_0. \quad (38)$$

This with (37) implies that

$$\inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| d\gamma \leq \lambda_n, \quad \text{for all } n \geq N_0. \quad (39)$$

Thus, we conclude that

$$\inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| d\gamma = 0. \quad (40)$$

Let γ_0 be the solution of the Kantorovich problem:

$$(K_0): \inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| d\gamma. \quad (41)$$

Then, using (40), we obtain

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| d\gamma_0 = 0. \quad (42)$$

We deduce that $x = y\gamma_0$ a.e. So for all $\phi \in C_b(\mathbb{R}^d)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(x) \rho_1 dx &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x) d\gamma_0(x, y) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(y) d\gamma_0(x, y) = \int_{\mathbb{R}^d} \phi(y) \rho_2 dy. \end{aligned} \quad (43)$$

Consequently, $\rho_1 = \rho_2$.

Let us fix two probability measures ρ_1 and ρ_2 on $\mathbb{R}^d$. Since c is even, then

$$\inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) d\gamma = \inf_{\bar{\gamma} \in \Pi(\rho_2, \rho_1)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) d\bar{\gamma}, \quad (44)$$

for all $\lambda > 0$. We deduce from (44) that

$$\begin{aligned} W_c(\rho_1, \rho_2) &= \inf \left\{ \lambda > 0, \inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) d\gamma(x, y) \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0, \inf_{\bar{\gamma} \in \Pi(\rho_2, \rho_1)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) d\bar{\gamma}(x, y) \leq 1 \right\} \\ &= W_c(\rho_2, \rho_1). \end{aligned} \quad (45)$$

Let ρ_1, ρ_2, ρ_3 be three probability measures on $\mathbb{R}^d$. Define $\lambda_1 = W_c(\rho_1, \rho_2)$ and $\lambda_2 = W_c(\rho_2, \rho_3)$. Denote by $\gamma_1 \in \Pi(\rho_1, \rho_2)$ the solution of Kantorovich problem

$$(K_1): \inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda_1}\right) d\gamma, \quad (46)$$

and denote by $\gamma_2 \in \Pi(\rho_2, \rho_3)$ the solution of the Kantorovich problem:

$$(K_2): \inf_{\gamma \in \Pi(\rho_2, \rho_3)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda_2}\right) d\gamma. \quad (47)$$

Using the Gluing lemma [8], there exists a probability measure σ on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ such that

$$\begin{aligned} \sigma(A \times B \times \mathbb{R}^d) &= \gamma_1(A \times B), \\ \sigma(\mathbb{R}^d \times A \times B) &= \gamma_2(A \times B), \end{aligned} \quad (48)$$

for some Borel subsets A and B of $\mathbb{R}^d$. Let γ_3 be a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ defined by $\gamma_3(A \times B) = \sigma(A \times \mathbb{R}^d \times B)$. Then, $\gamma_3 \in \Pi(\rho_1, \rho_3)$, and we use the convexity of c to get that

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-z}{\lambda_1 + \lambda_2}\right) d\gamma_3 &= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-z}{\lambda_1 + \lambda_2}\right) d\sigma \\ &\leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda_1}\right) d\gamma_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{y-z}{\lambda_2}\right) d\gamma_2 \\ &\leq \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2} = 1. \end{aligned} \quad (49)$$

So, $\inf_{\gamma \in \Pi(\rho_1, \rho_3)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x-z/\lambda_1 + \lambda_2) d\gamma \leq 1$, and we conclude that

$$W_c(\rho_1, \rho_2) \leq \lambda_1 + \lambda_2 = W_c(\rho_1, \rho_2) + W_c(\rho_2, \rho_3). \quad (50)$$

Hence, W_c is a distance on $P(\mathbb{R}^d)$.

Let us now study the topology of W_c .

Let $(\rho_n)_n$ be a sequence on $P(\mathbb{R}^d)$ and $\rho \in P(\mathbb{R}^d)$ such that $W_c(\rho_n, \rho)$ converges to 0 when n tends to ∞ . Define $\lambda_n = W_c(\rho_n, \rho)$, since $(\lambda_n)_n$ converges to 0, then we use the fact that c is coercive to have

$$\inf_{\gamma \in \Pi(\rho_n, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| d\gamma \leq \lambda_n, \quad (51)$$

when $n \rightarrow \infty$. We deduce that

$$\limsup_{n \rightarrow \infty} \inf_{\gamma \in \Pi(\rho_n, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| d\gamma = 0. \quad (52)$$

Note that the 1-Wasserstein metric between ρ_n and ρ is

$$W_1(\rho_n, \rho) = \inf_{\gamma \in \Pi(\rho_n, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| d\gamma. \quad (53)$$

We deduce that $W_1(\rho_n, \rho)$ converges to 0 when n tends to ∞ . Since the 1-Wasserstein metric W_1 induces the narrow topology of $P(\mathbb{R}^d)$, we conclude that the sequence $(\rho_n)_n$ converges narrowly to ρ in $P(\mathbb{R}^d)$.

Assume now that the sequence $(\rho_n)_n$ converges narrowly to ρ in $P(\mathbb{R}^d)$. Fix $\lambda > 0$ and denote by γ_n^λ the solution of Kantorovich problem:

$$(K): \inf_{\gamma \in \Pi(\rho_n, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) d\gamma. \quad (54)$$

Since ρ_n converges narrowly to ρ , then γ_n^λ converges narrowly to some $\gamma^\lambda \in \Pi(\rho, \rho)$ and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) d\gamma_n^\lambda = \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) d\gamma^\lambda = 0. \quad (55)$$

So,

$$\lim_{n \rightarrow \infty} \inf_{\gamma \in \Pi(\rho_n, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) d\gamma = 0, \quad (56)$$

for all $\lambda > 0$. Hence, for all $\lambda > 0$, there exists $N_\lambda > 0$ such that

$$\inf_{\gamma \in \Pi(\rho_n, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) d\gamma \leq 1, \quad \text{for all } n \geq N_\lambda. \quad (57)$$

So, $W_c(\rho_n, \rho) \leq \lambda$ for all $n \geq N_\lambda$. Then,

$$\lim_{n \rightarrow \infty} W_c(\rho_n, \rho) = 0. \quad (58)$$

Consequently, $(\rho_n)_n$ converges narrowly to ρ in the metric space $(P(\Omega), W_c)$.

We establish now the existence of solution for the variational problem (P) defined by

$$(P): \inf_{\rho \in P_q(\mathbb{R}^d)} \left\{ \int_{\mathbb{R}^d} G(\rho) dx + h W_c^h(\rho_0, \rho) \right\}, \quad (59)$$

in the metric space $(P(\mathbb{R}^d), W_c)$, where

$$W_c^h(\rho_0, \rho) = \inf_{\gamma \in \Pi(\rho_0, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{h}\right) d\gamma. \quad (60)$$

$P_q(\mathbb{R}^d)$ is the set of all probability measures on $\mathbb{R}^d$ having q -finite moment, that is,

$$P_q(\mathbb{R}^d) = \left\{ \rho \in P(\mathbb{R}^d), \int_{\mathbb{R}^d} |x|^q \rho dx < \infty \right\}, \quad (61)$$

and $h > 0$ being a time step. $\square$

Lemma 1. Assume that ρ_0 , G , and c satisfy, respectively, H_{f_0} , H_G , and H_c . Then, the following is obtained:

- (i) The map $\rho \longrightarrow W_c^h(\rho_0, \rho)$ is lower semicontinuous in $(P(\mathbb{R}^d), W_c)$.
- (ii) The functional $\rho \longrightarrow \int_{\mathbb{R}^d} G(\rho) dx$ is lower semicontinuous in $L^1(\mathbb{R}^d)$.
- (iii) The set $\left\{ \rho \in P_M(\mathbb{R}^d), \int_{\mathbb{R}^d} G(\rho) dx \leq \int_{\mathbb{R}^d} G(\rho_0) dx \right\}$ is a closed subset of $L^1(\mathbb{R}^d)$,

$$\text{where } P_M(\mathbb{R}^d) = \left\{ \rho \in P(\mathbb{R}^d), \rho \leq M \right\}. \quad (62)$$

Proof

- (i) Let $\rho \in P(\mathbb{R}^d)$. Let $(\rho_n)_n$ be a sequence in $P(\mathbb{R}^d)$ such that $(\rho_n)_n$ converges to ρ in metric space

$(P(\mathbb{R}^d), W_c)$. Denote by γ_n^h the solution of the Kantorovich problem:

$$(K_n^h): \inf_{\gamma \in \Pi(\rho_n, \rho_0)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{h}\right) d\gamma. \quad (63)$$

We have

$$\inf_{\gamma \in \Pi(\rho_n, \rho_0)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{h}\right) d\gamma = \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{h}\right) d\gamma_n^h. \quad (64)$$

Since $(\rho_n)_n$ converges to ρ narrowly, then $(\gamma_n^h)_n$ converges to $\gamma^h \in \Pi(\rho_0, \rho)$ narrowly and

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{h}\right) d\gamma_n^h \geq \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{h}\right) d\gamma^h \geq W_c^h(\rho_0, \rho). \quad (65)$$

This implies that

$$\liminf_{n \rightarrow \infty} W_c^h(\rho_0, \rho_n) \geq W_c^h(\rho_0, \rho). \quad (66)$$

Thus, we obtain the proof of (i).

- (ii) Since G is convex and $G \in C^1((0, \infty))$,

$$\int_{\mathbb{R}^d} G(\rho_n) dx \geq \int_{\mathbb{R}^d} G(\rho) dx + \int_{\mathbb{R}^d} G'(\rho)(\rho_n - \rho) dx. \quad (67)$$

Hence, if $(\rho_n)_n$ converges to ρ weakly in $L^1(\mathbb{R}^d)$ with $\rho \in L^\infty(\mathbb{R}^d)$, we have

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} G'(\rho)(\rho_n - \rho) dx = 0, \quad (68)$$

and then

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} G(\rho_n) dx \geq \int_{\mathbb{R}^d} G(\rho) dx, \quad (69)$$

which complete the proof of (ii). The proof of (iii) is a consequence of (ii). $\square$

2.3. Existence Results for the Discrete Problem (P_1^{xh}) . First of all, we introduce here for unbounded domains analogous of the maximum principle stated for bounded domains in [1]. This maximum principle plays a central role in the searching of solution for the discrete problem (P_1^{xh}) . It is also used to further establish the convergence of our algorithm towards a weak solution of the kinetic equation (1).

Proposition 2 (maximum principle). Assume that the initial datum f_0 satisfies $N \leq f_0 \leq M$ with $0 < N$, c satisfies (H_c) , and G satisfies (H_G) .

Then, any solution F_1^{xh} of the variational problem

$$(P_1^{xh}): \inf_{F \in P_q(\mathbb{R}^d)} \{E(F) + hW_c^h(F, G_1^{xh})\}, \quad (70)$$

satisfies $a(x) \leq F_1^{xh} \leq b(x)$, with $a(x) = (N/\rho_1^h(x))$ and $b(x) = (M/\rho_1^h(x))$.

Proof. We define $E_x := \{y \in \mathbb{R}^d, F_1^{xh} > b(x)\}$. Assume by contradiction that E_x has a positive Lebesgue measure. Then, $\gamma_x(E_x^c \times E_x) > 0$, where γ_x is the minimizer of

$$(K): \inf_{\gamma \in \Pi(G_1^{xh}, F_1^{xh})} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{y-z}{h}\right) d\gamma(y, z) \right\}. \quad (71)$$

Otherwise,

$$\begin{aligned} b(x)|E_x| &< \int_{E_x} F_1^{xh} d\nu = \gamma_x(\mathbb{R}^d \times E_x) = \gamma_x(E_x \times E_x) \\ &\leq \gamma_x(E_x \times \mathbb{R}^d) = \int_{E_x} G_1^{xh} d\nu \leq |E_x|b(x). \end{aligned} \quad (72)$$

This yields a contradiction.

Define $\gamma_x = 1_{E_x^c \times E_x} \gamma_x$, i.e., $\gamma_x(F) = \gamma_x(F \cap E_x^c \times E_x)$, for all $F \subset \mathbb{R}^d$, and denote by $\nu_{0,x}$ and $\nu_{1,x}$ the marginals of γ_x . We have $\gamma_x \ll \gamma_x$ with $\gamma_x \in \Pi(G_1^{xh}, F_1^{xh})$ and then $\nu_{0,x} \ll G_1^{xh} d\nu$ and $\nu_{1,x} \ll F_1^{xh} d\nu$.

Let $\nu_{0,x}$ and $\nu_{1,x}$ denote, respectively, the density functions of $\nu_{0,x}$ and $\nu_{1,x}$. Because

$$\begin{aligned} \int_{E_x} \nu_{0,x} dy &= \nu_{0,x}(E_x) = \gamma_x((E_x \times \mathbb{R}^d) \cap E_x^c \times E_x) = 0, \\ \int_{E_x^c} \nu_{1,x} dy &= \nu_{1,x}(E_x^c) = \gamma_x((\mathbb{R}^d \times E_x^c) \cap E_x^c \times E_x) = 0, \end{aligned} \quad (73)$$

we have $\nu_{0,x} = 0$ on E_x and $\nu_{1,x} = 0$ on E_x^c .

For $\varepsilon > 0$ is small enough, define $F_1^{xh,\varepsilon} = F_1^{xh} + \varepsilon(\nu_{0,x} - \nu_{1,x})$ and γ_x^ε by

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} H(x, y) d\gamma_x^\varepsilon &= \int_{\mathbb{R}^d \times \mathbb{R}^d} H(x, y) d\gamma_x \\ &+ \varepsilon \int_{E_x^c \times E_x} [H(x, x) - H(x, y)] d\gamma_x, \end{aligned} \quad (74)$$

for all $H \in C_0(\mathbb{R}^d \times \mathbb{R}^d)$.

$F_1^{xh,\varepsilon} \in P_q(\mathbb{R}^d)$, $\gamma_x^\varepsilon \in \Pi(G_1^{xh}, F_1^{xh,\varepsilon})$, and we get

$$\begin{aligned} I(F_1^{xh,\varepsilon}) - I(F_1^{xh}) &= h[W_c^h(F_1^{xh,\varepsilon}, G_1^{xh}) \\ &- W_c^h(F_1^{xh}, G_1^{xh})] + \frac{1}{\rho_1^h(x)} \int_{\mathbb{R}^d} [G(\rho_1^h(x) F_1^{xh,\varepsilon}) - G(\rho_1^h(x) F_1^{xh})]. \end{aligned} \quad (75)$$

Taking into account some ideas from [1], we shall prove that $I(F_1^{xh,\varepsilon}) - I(F_1^{xh}) < 0$, which leads to a contradiction.

Indeed,

$$\begin{aligned} &W_c^h(F_1^{xh,\varepsilon}, G_1^{xh}) - W_c^h(F_1^{xh}, G_1^{xh}) \\ &= \inf_{\gamma \in \Pi(F_1^{xh,\varepsilon}, F_1^{xh})} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{y-z}{h}\right) d\gamma - \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{y-z}{h}\right) d\gamma_x \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{y-z}{h}\right) d\gamma_x^\varepsilon - \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{y-z}{h}\right) d\gamma_x \\ &= -\varepsilon \int_{E_x^c \times E_x} c\left(\frac{y-z}{h}\right) d\gamma_x. \end{aligned} \quad (76)$$

Since $(y, z) \mapsto c(y-z/h)$ is nonnegative on $E_x^c \times E_x$ and $\gamma_x(E_x^c \times E_x) > 0$, we have

$$W_c^h(F_1^{xh,\varepsilon}, G_1^{xh}) - W_c^h(F_1^{xh}, G_1^{xh}) = -\varepsilon \int_{E_x^c \times E_x} c\left(\frac{y-z}{h}\right) d\gamma_x < 0. \quad (77)$$

Also, we have

$$\begin{aligned} &\frac{1}{\rho_1^h(x)} \int_{\mathbb{R}^d} [G(\rho_1^h(x) F_1^{xh,\varepsilon}) - G(\rho_1^h(x) F_1^{xh})] d\nu \\ &= \frac{1}{\rho_1^h(x)} \int_{\mathbb{R}^d} [G(\rho_1^h(x) F_1^{xh} + \varepsilon \rho_1^h(x) (\nu_{0,x} - \nu_{1,x})) - G(\rho_1^h(x) F_1^{xh})] d\nu \\ &= \frac{1}{\rho_1^h(x)} \int_{E_x} [G((\rho_1^h(x) F_1^{xh} - \varepsilon \rho_1^h(x) \nu_{1,x})) - G(\rho_1^h(x) F_1^{xh})] d\nu \\ &\quad + \frac{1}{\rho_1^h(x)} \int_{E_x^c} [G(\rho_1^h(x) F_1^{xh} + \varepsilon \rho_1^h(x) \nu_{0,x}) - G(\rho_1^h(x) F_1^{xh})] d\nu. \end{aligned} \quad (78)$$

G is being convex and of class C^1 , then

$$G(\rho_1^h F_1^{xh}) - G(\rho_1^h F_1^{xh} - \varepsilon \rho_1^h \nu_{1,x}) \geq G'(\rho_1^h F_1^{xh} - \varepsilon \rho_1^h \nu_{1,x}) \varepsilon \rho_1^h \nu_{1,x}, \quad (79)$$

on E_x and

$$\begin{aligned} &G(\rho_1^h(x) F_1^{xh} + \varepsilon \rho_1^h(x) \nu_{0,x}) - G(\rho_1^h(x) F_1^{xh}) \\ &\leq \varepsilon \rho_1^h \nu_{0,x} G'(\rho_1^h F_1^{xh} + \varepsilon \rho_1^h \nu_{0,x}), \end{aligned} \quad (80)$$

on E_x^c .

Hence, we have

$$\begin{aligned} &\frac{1}{\rho_1^h(x)} \int_{\mathbb{R}^d} [G(\rho_1^h(x) F_1^{xh,\varepsilon}) - G(\rho_1^h(x) F_1^{xh})] d\nu \\ &\leq \varepsilon \left[\int_{E_x^c} [G'(\rho_1^h F_1^{xh} + \varepsilon \rho_1^h \nu_{0,x}) \nu_{0,x} d\nu \right. \\ &\quad \left. - \int_{E_x} [G'(\rho_1^h F_1^{xh} - \varepsilon \rho_1^h \nu_{1,x}) \nu_{1,x} d\nu] \right] \\ &= \varepsilon \int_{E_x^c \times E_x} [G'(\rho_1^h F_1^{xh} + \varepsilon \rho_1^h \nu_{0,x}) - G'(\rho_1^h F_1^{xh} - \varepsilon \rho_1^h \nu_{1,x})] d\gamma_x. \end{aligned} \quad (81)$$

Since G' is continuous, then

$$\lim_{\varepsilon \rightarrow 0} [G'(\rho_1^h F_1^{xh} + \varepsilon \rho_1^h \nu_{0,x}) - G'(\rho_1^h F_1^{xh} - \varepsilon \rho_1^h \nu_{1,x})] = 0. \quad (82)$$

Consequently, there exists $\delta > 0$ such that, for $0 < \varepsilon < \delta$, then

$$\begin{aligned} 0 &\leq G'(\rho_1^h F_1^{xh} + \varepsilon \rho_1^h \nu_{0,x}) - G'(\rho_1^h F_1^{xh} - \varepsilon \rho_1^h \nu_{1,x}) \\ &\leq \frac{1}{2\gamma_x(E_x^c \times E_x)} \int_{E_x^c \times E_x} c\left(\frac{y-z}{h}\right) d\gamma_x. \end{aligned} \quad (83)$$

So, we fix $\varepsilon > 0$ small, such that $0 < \varepsilon < \delta$, and we use (81) and (83) to obtain

$$\begin{aligned} &\frac{1}{\rho_1^h(x)} \int_{\mathbb{R}^d} [G(\rho_1^h(x) F_1^{xh,\varepsilon}) - G(\rho_1^h(x) F_1^{xh})] dv \\ &\leq \frac{\varepsilon}{2} \int_{E_x^c \times E_x} c\left(\frac{y-z}{h}\right) d\gamma_x. \end{aligned} \quad (84)$$

Now, fixing $\varepsilon < \delta$ and combining (77) and (84) yield

$$\begin{aligned} &I(F_1^{xh,\varepsilon}) - I(F^{xh}) \\ &< -\varepsilon \int_{E_x^c \times E_x} c\left(\frac{y-z}{h}\right) d\gamma_x + \frac{\varepsilon}{2} \int_{E_x^c \times E_x} c\left(\frac{y-z}{h}\right) d\gamma_x \\ &= -\frac{\varepsilon}{2} \int_{E_x^c \times E_x} c\left(\frac{y-z}{h}\right) d\gamma_x < 0. \end{aligned} \quad (85)$$

It is a contradiction since F_1^{xh} is a minimizer of I on $P_q(\mathbb{R}^d)$. Consequently, E_x is negligible and then $F_1^{xh} \leq b(x)$.

The proof of $F_1^{xh} \geq a(x)$ is analogous to that of $F_1^x \leq b(x)$. Thus, we conclude that $a(x) \leq F_1^{xh} \leq b(x)$. $\square$

Lemma 2. Let $f_0 \in P_q(\Omega \times \mathbb{R}^d)$ be a probability density on $\Omega \times \mathbb{R}^d$ such that $0 < N \leq f_0 \leq M$, $\int_{\Omega \times \mathbb{R}^d} G(f_0) dx dv < \infty$ and $\int_{\Omega \times \mathbb{R}^d} |v|^q f_0(x, v) dx dv < \infty$.

The variational problem (P_1^{xh}) defined in (32) admits a unique minimizer F_1^{xh} in $P_q(\mathbb{R}^d)$ and $(N/\rho_1^h(x)) \leq F_1^{xh} \leq (M/\rho_1^h(x))$.

Proof. Since $0 < N \leq f_0 \leq M$ and $G_1^{xh}(\nu) = f_0(x, \nu)/\rho_1^h(x)$, then $(N/\rho_1^h(x)) \leq G_1^{xh} \leq (M/\rho_1^h(x))$. Then, we use Proposition 2, and we obtain that any minimizer F_1^{xh} of the variational problem P_1^{xh} satisfies $a(x) \leq F_1^{xh} \leq b(x)$, with $a(x) = N/\rho_1^h(x)$ and $b(x) = M/\rho_1^h(x)$.

By using Proposition 2 and the fact that G is convex, we obtain that

$$E(F) \geq 1/\rho_1^h(x) \int_{\mathbb{R}^d} G(\rho_1^h(x) G_1^{xh}(\nu)) dv + G'(N) - G'(M), \quad (86)$$

for all probability density $F \in P_q(\mathbb{R}^d)$ such that $N \leq F \leq M$. We use now Lemma 1 to get that the functional

$F \mapsto 1/\rho^h(x) \int_{\mathbb{R}^d} G(\rho^h(x) F(\nu)) dv$ is lower semicontinuous on the Wasserstein space $(P_q(\mathbb{R}^d), W_c)$. We conclude then that the problem (P_1^{xh}) admits a solution F_1^{xh} . The strict convexity of G and c implies the strict convexity of the map $F \mapsto W_c^h(G_1^{xh}, F)$ and that of the maps $F \mapsto 1/\rho^h(x) \int_{\mathbb{R}^d} G(\rho^h(x) F(\nu)) dv$ and accordingly the uniqueness of the minimizer F_1^{xh} of (P_1^{xh}) . $\square$

3. Euler–Lagrange Equation for the Problem (P_k^{xh})

In this section, we prove that the sequence $(f_k^h)_k$ is a time discretization of the kinetic equation (1). In order to achieve it, we need the following lemma.

Lemma 3 (explicit expression for optimal maps). Assume that G satisfies H_G and c satisfies H_c . Then, the Monge problem

$$(M): \inf_{T_{\#} F_k^{xh} = G_k^{xh}} \int_{\mathbb{R}^d} c\left(\frac{v - T(v)}{h}\right) F_k^{xh} dv, \quad (87)$$

admits a unique solution T_k such that

$$T_k(v) = v + h \nabla c^*(\nabla_v G'(\rho_k^h F_k^{xh})), \quad (88)$$

where F_k^x is the unique minimizer of the variational problem.

$$(P_k^{xh}): \inf_{F \in P_q(\mathbb{R}^d)} \left\{ I(F) := \frac{1}{\rho_k^h} \int_{\mathbb{R}^d} G(\rho_k^h F) + h W_c^h(G_k^{xh}, F) \right\}. \quad (89)$$

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ be a test function and consider the diffeomorphism map $(T_\varepsilon)_{\varepsilon \in \mathbb{R}}$ in $C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ defined by

$$\begin{cases} \frac{\partial T_\varepsilon}{\partial \varepsilon} = \psi \circ T_\varepsilon, \\ T_0 = \text{id}. \end{cases} \quad (90)$$

Define the probability density $G_\varepsilon^x = T_{\varepsilon\#} F_k^x$ on $\mathbb{R}^d$. Since G satisfies (H_G) and T_ε is a diffeomorphism pushing F_k^{xh} forward to G_ε^x , we obtain the following Monge–Kantorovich-type energy inequality:

$$\begin{aligned} &\frac{1}{\rho_k^h(x)} \left[\int_{\mathbb{R}^d} G(\rho_k^h G_\varepsilon^x) dv - \int_{\mathbb{R}^d} G(\rho_k^h F_k^{xh}) dv \right] \\ &\geq \int_{\mathbb{R}^d} \langle T_\varepsilon(v) - v, \nabla_v (G'(\rho_k^h F_k^{xh})) \rangle F_k^{xh} dv. \end{aligned} \quad (91)$$

Recalling the definition of T_ε , we have

$$\left| \frac{T_\varepsilon(v) - v}{\varepsilon} \right| \leq \|\psi\|_\infty, \quad \text{for all } v \in \mathbb{R}^d \text{ and } \varepsilon > 0. \quad (92)$$

Dividing (91) by $\varepsilon > 0$ and using (92) and the dominated convergence theorem, we have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon \rho_k^h(x)} \left[\int_{\mathbb{R}^d} G(\rho_k^h G_\varepsilon^x) dv - \int_{\mathbb{R}^d} G(\rho_k^h F_k^{xh}) dv \right] \\
& \geq \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d} \left\langle \frac{T_\varepsilon(v) - v}{\varepsilon}, \nabla_v (G'(\rho_k^h F_k^{xh})) \right\rangle F_k^{xh} dv \quad (93) \\
& \geq \int_{\mathbb{R}^d} \langle \psi, \nabla_v (G'(\rho_k^h F_k^{xh})) \rangle F_k^{xh} dv.
\end{aligned}$$

Furthermore, since $T_\varepsilon^{-1} \# G_\varepsilon^x = F_k^{xh}$, then the Monge–Kantorovich-type energy inequality gives

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon \rho_k^h(x)} \left[\int_{\mathbb{R}^d} G(\rho_k^h F_k^{xh}) dv - \int_{\mathbb{R}^d} G(\rho_k^h G_\varepsilon^x) dv \right] \\
& \geq \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d} \left\langle \frac{T_\varepsilon^{-1}(v) - v}{\varepsilon}, \nabla_v (G'(\rho_k^h G_\varepsilon^x)) \right\rangle G_\varepsilon^x dv \quad (94) \\
& = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d} \left\langle \frac{v - T_\varepsilon(v)}{\varepsilon}, \nabla_v G'((\rho_k^h G_\varepsilon^x) \circ T_\varepsilon) \right\rangle F_k^x dv \\
& = - \int_{\mathbb{R}^d} \langle \psi, \nabla_v G'(\rho_k^h F_k^{xh}) \rangle F_k^{xh} dv.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon \rho_k^h(x)} \left[\int_{\mathbb{R}^d} G(\rho_k^h G_\varepsilon^x) dv - \int_{\mathbb{R}^d} G(\rho_k^h F_k^{xh}) dv \right] \\
& \leq \int_{\mathbb{R}^d} \langle \psi, \nabla_v G'(\rho_k^h F_k^{xh}) \rangle F_k^{xh} dv. \quad (95)
\end{aligned}$$

We combine now (93) and (95) to derive

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon \rho_k^h(x)} \left[\int_{\mathbb{R}^d} G(\rho_k^h G_\varepsilon^x) dv - \int_{\mathbb{R}^d} G(\rho_k^h F_k^{xh}) dv \right] \\
& = \int_{\mathbb{R}^d} \langle \psi, \nabla_v G'(\rho_k^h F_k^{xh}) \rangle F_k^{xh} dv. \quad (96)
\end{aligned}$$

Since $T_k \# F_k^{xh} = G_k^{xh}$ and $T_\varepsilon \# F_k^{xh} = G_\varepsilon^x$, then $T_\varepsilon \circ T_k^{-1} \# G_k^{xh} = G_\varepsilon^x$. So,

$$\begin{aligned}
W_c^h(G_k^{xh}, G_\varepsilon^x) &= \inf_{S \# G_k^{xh} = G_\varepsilon^x} \int_{\mathbb{R}^d} c\left(\frac{v - S(v)}{h}\right) G_k^{xh} dv \\
&\leq \int_{\mathbb{R}^d} c\left(\frac{v - T_\varepsilon \circ T_k^{-1}(v)}{h}\right) G_k^{xh} dv. \quad (97)
\end{aligned}$$

Thus, for $\varepsilon > 0$, we have

$$\begin{aligned}
& \frac{W_c^h(G_k^{xh}, G_\varepsilon^x) - W_c^h(G_k^{xh}, F_k^{xh})}{\varepsilon} \\
& \leq \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \left[c\left(\frac{v - T_\varepsilon \circ T_k^{-1}(v)}{h}\right) - c\left(\frac{v - T_k(v)}{h}\right) \right] G_k^{xh} dv. \quad (98)
\end{aligned}$$

We use the convexity of c and the fact that $T_k \# F_k^{xh} = G_k^{xh}$ to obtain

$$\begin{aligned}
& \frac{hW_c^h(G_k^{xh}, G_\varepsilon^x) - hW_c^h(G_k^{xh}, F_k^{xh})}{\varepsilon} \\
& \leq - \int_{\mathbb{R}^d} \left\langle \nabla c\left(\frac{T_k(v) - T_\varepsilon(v)}{h}\right), \frac{T_\varepsilon(v) - v}{\varepsilon} \right\rangle F_k^{xh} dv. \quad (99)
\end{aligned}$$

Now, from (92) and the dominated convergence theorem, we obtained

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \frac{hW_c^h(G_k^{xh}, G_\varepsilon^x) - hW_c^h(G_k^{xh}, F_k^{xh})}{\varepsilon} \\
& \leq - \int_{\mathbb{R}^d} \left\langle \nabla c\left(\frac{T_k(v) - v}{h}\right), \psi \right\rangle F_k^{xh} dv. \quad (100)
\end{aligned}$$

Since F_k^{xh} minimizes the functional

$$I(F) = \frac{1}{\rho_k^h(x)} \int_{\mathbb{R}^d} G(\rho_k^h F) dv + hW_c^h(G_k^{xh}, F), \quad (101)$$

on the probability space, the Euler–Lagrange equation yields

$$\lim_{\varepsilon \rightarrow 0^+} \frac{I(G_\varepsilon^x) - I(F_k^{xh})}{\varepsilon} = 0. \quad (102)$$

Consequently, by using (96) and (100), we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} \langle \psi, \nabla_v G'(\rho_k^h F_k^{xh}) \rangle F_k^{xh} dv \\
& - \int_{\mathbb{R}^d} \left\langle \nabla c\left(\frac{T_k(v) - v}{h}\right), \psi \right\rangle F_k^{xh} dv \geq 0. \quad (103)
\end{aligned}$$

Next, by replacing ψ by $-\psi$ in (96), we obtain the desired equation:

$$\int_{\mathbb{R}^d} \langle \psi, \nabla_v G'(\rho_k^h F_k^{xh}) \rangle F_k^{xh} dv - \int_{\mathbb{R}^d} \left\langle \nabla c\left(\frac{T_k(v) - v}{h}\right), \psi \right\rangle F_k^{xh} dv = 0. \quad (104)$$

Thus, we conclude that

$$\nabla c\left(\frac{T_k(v) - v}{h}\right) = \nabla_v G'(\rho_k^h F_k^{xh}) a.e - F_k^{xh}. \quad (105)$$

Note that ∇c is invertible and $(\nabla c)^{-1} = \nabla c^*$. Then, we obtain the explicit expression of the optimal map T_k :

$$T_k(v) = v + h \nabla c^*(\nabla_v (G'(\rho_k^h F_k^{xh}))) - a.e F_k^{xh}. \quad (106)$$

The proof of this lemma is complete.

We are now ready to show that the sequence $(f_k^h)_k$ satisfies the time discretization (17) of the kinetic equation (1).

Let $\psi \in C_c^\infty(\Omega \times \mathbb{R}^d)$, then

$$\begin{aligned}
& \int_{\Omega \times \mathbb{R}^d} (f_k^h(x, v) - f_{k-1}^h(x, v)) \psi(x, v) dx dv \\
& = \int_{\Omega \times \mathbb{R}^d} (f_k^h(x, v) - g_{k-1}^h(x, v)) \psi(x, v) dx dv \quad (107) \\
& + \int_{\Omega \times \mathbb{R}^d} (g_k^h(x, v) - f_{k-1}^h(x, v)) \psi(x, v) dx dv.
\end{aligned}$$

By using $T_k \# F_k^{xh} = G_k^{xh}$, $f_k^h = \rho_k^h F_k^{xh}$, and $g_k^h = \rho_k^h G_k^{xh}$, we have

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^d} (f_k^h(x, v) - g_{k-1}^h(x, v)) \psi(x, v) dx dv &= \int_{\Omega \times \mathbb{R}^d} (\psi(x, v) - \psi(x, T_k(x))) f_k^h(x, v) dv, \\ \int_{\Omega \times \mathbb{R}^d} (g_k^h(x, v) - f_{k-1}^h(x, v)) \psi(x, v) dx dv &= \int_{\Omega \times \mathbb{R}^d} (\psi(x + hv, v) - \psi(x, v)) f_{k-1}^h dx dv. \end{aligned} \quad (108)$$

Next, we use the Taylor formula and the expression of the optimal T_k to obtain

$$\begin{aligned} &\int_{\Omega \times \mathbb{R}^d} (f_k^h(x, v) - f_{k-1}^h(x, v)) \psi(x, v) dx dv \\ &= h \int_{\Omega \times \mathbb{R}^d} \langle v, \nabla_x \psi(x, v) \rangle f_{k-1}^h(x, v) dx dv \\ &\quad - h \int_{\Omega \times \mathbb{R}^d} \langle \nabla c^*(\nabla_v(G'(f_k^h))) \rangle f_k^h dx dv \\ &\quad + h^2 \int_{\Omega \times \mathbb{R}^d} \langle v^\tau, \nabla_x^2 \psi(x + h\theta v, v) \rangle f_{k-1}^h dx dv \\ &\quad + h^2 \int_{\Omega \times \mathbb{R}^d} \langle \sigma_k^\tau, \nabla_v^2 \psi(x, v + \alpha h \sigma_k) \rangle f_k^h dx dv. \end{aligned} \quad (109)$$

Here, $\sigma_k = \nabla c^*(\nabla_v(G'(f_k^h)))$ and $\theta, \alpha \in]0, 1[$. Now define

$$\begin{aligned} A_k^h[\psi] &:= \frac{h^2}{2} \int_{\Omega \times \mathbb{R}^d} \langle (\sigma_k)^\tau, \nabla^2 \psi(x, v + h\theta \sigma_k) \rangle \cdot (\sigma_k) \rangle f_k^h(x, v) dx dv \\ &\quad + \frac{h^2}{2} \int_{\Omega \times \mathbb{R}^d} \langle v^\tau, \nabla_x^2 \psi(x + \alpha h v, v) \rangle f_{k-1}^h dx dv. \end{aligned} \quad (110)$$

If we show that $A_k^h[\psi]$ tends to 0 as h goes to 0, then we are done.

Indeed, from the maximum principle, $N \leq f_k^h \leq M$, and then

$$\begin{aligned} |A_k^h[\psi]| &\leq \frac{h^2}{2} \|\nabla_v^2 \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)} \int_{\Omega \times \mathbb{R}^d} |\nabla c^*(\nabla_v(G'(f_k^h)))|^2 f_k^h dx dv \\ &\quad + \frac{h^2}{2} \|\nabla_x^2 \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)} \|f_0\|_{L^\infty(\Omega \times \mathbb{R}^d)} \int_{K_\psi} |v|^2 dx dv, \end{aligned} \quad (111)$$

where K_ψ is a compact subset of $\Omega \times \mathbb{R}^d$ such that $\text{supp} \psi \subset K_\psi$.

Since F_k^{xh} minimizes the general functional energy $I(F) = 1/\rho_k^h \int_{\mathbb{R}^d} G(\rho_k^h F) dv + h W_c^h(F, G_k^{xh})$ on the metric space $(P_q(\mathbb{R}^d), W_c)$, then

$$\frac{1}{\rho_k^h} \int_{\mathbb{R}^d} G(\rho_k^h F_k^{xh}) dv + h W_c^h(F_k^{xh}, G_k^{xh}) \leq \frac{1}{\rho_k^h} \int_{\mathbb{R}^d} G(\rho_k^h G_k^{xh}) dv. \quad (112)$$

We use in (112) the expression of the optimal maps T_k and the definition of the f_k^h , and then, we have

$$\begin{aligned} &\int_{\Omega \times \mathbb{R}^d} G(f_{k-1}^h) dx dv - \int_{\Omega \times \mathbb{R}^d} G(f_k^h) dx dv \\ &\geq h \int_{\Omega \times \mathbb{R}^d} c(\nabla c^*(\nabla_v(G'(f_k^h)))) f_k^h dx dv. \end{aligned} \quad (113)$$

Because $G \in C^1((0, \infty))$ is convex and $N \leq f_k^h \leq M$, we have

$$\int_{\Omega \times \mathbb{R}^d} G(f_{k-1}^h) dx dv - \int_{\Omega \times \mathbb{R}^d} G(f_k^h) dx dv \leq G'(M) - G'(N). \quad (114)$$

By using (114) and the fact that $c(x) \geq A_1 |x|^q$, then (113) becomes

$$h \int_{\Omega \times \mathbb{R}^d} |\nabla c^*(\nabla_v(G'(f_k^h)))|^q f_k^h dx dv \leq \frac{G'(M) - G'(N)}{A_1}. \quad (115)$$

Recalling (115) and (111), we obtain

$$\begin{aligned} |A_k^h(\psi)| &\leq h \|\nabla_x^2 \psi\|_\infty |K_\psi|^2 M \\ &\quad + h \|\nabla_v^2 \psi\|_\infty 2^{q-1} \frac{G'(M) - G'(N)}{A_1} + h 2^{q-1}, \quad \text{if } q \geq 2, \end{aligned} \quad (116)$$

$$\begin{aligned} |A_k^h(\psi)| &\leq h \|\nabla_x^2 \psi\|_\infty |K_\psi|^2 M + h^{q-1} \|\nabla_v^2 \psi\|_\infty \\ &\quad \cdot \frac{G'(M) - G'(N)}{A_1}, \quad \text{if } q < 2. \end{aligned} \quad (117)$$

We combine (116) and (117) to conclude that $A_k^h(\psi)$ tends to 0, when h goes to 0. Accordingly, we conclude that the sequence $(f_k^h)_k$ resulted from a time discretization of the kinetic equation (1).

Recalling Section 2.1, we define an approximate solution f^h over $[0, \infty[\times \Omega \times \mathbb{R}^d$ of the kinetic equation (1) as follows:

$$\begin{cases} f^h(t, x, v) = f_k^h(x - (t - t_k)v, v), & \text{if } t \in [t_k, t_{k+1}), \\ f^h(0, x, v) = f_0(x, v). \end{cases} \quad (118)$$

In the next section, we establish the convergence of the sequence $(f^h)_h$ to a weak solution f of (1). $\square$

4. Convergence Results

4.1. Weak Convergence of $(f^h)_h$. Let us consider the sequence $(f^h)_h$ as defined in (118).

Lemma 4. Assume that f_0 satisfies (H_{f_0}) . Then, for all $k \geq 1$, we have

$$\int_{\Omega \times \mathbb{R}^d} |f_k^h(x, v)|^p dx dv \leq \int_{\Omega \times \mathbb{R}^d} |f_0(x, v)|^p dx dv. \quad (119)$$

Proof. Taking $H(t) = t^p$, $p > 1$, in Corollary 1, we obtain

$$\int_{\mathbb{R}^d} |G_k^{xh}|^p dv - \int_{\mathbb{R}^d} |F_k^{xh}|^p dv \geq \int_{\mathbb{R}^d} \left\langle T_k(v) - v, \nabla_v \left(H'(F_k^{xh}) \right) \right\rangle F_k^{xh} dv, \quad (120)$$

where T_k is c -optimal map that pushes F_k^{xh} forward to G_k^{xh} . We use expression of T_k and we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} |G_k^{xh}|^p dv - \int_{\mathbb{R}^d} |F_k^{xh}|^p dv \\ & \geq \int_{\mathbb{R}^d} \left\langle T_k(v) - v, \nabla_v \left(H'(F_k^{xh}) \right) \right\rangle F_k^{xh} dv \\ & = p(p-1)h \int_{\mathbb{R}^d} \left\langle \nabla c^*(\nabla_v G'(f_k^h)), \nabla_v (F_k^{xh}) \right\rangle (F_k^{xh})^{p-1} dv. \end{aligned} \quad (121)$$

Since $\nabla_v G'(f_k^h) = G''(f_k^h) \nabla_v f_k^h$ and $f_k^h = \rho_k^h F_k^{xh}$, (121) becomes

$$\begin{aligned} & \int_{\mathbb{R}^d} |G_k^{xh}|^p dv - \int_{\mathbb{R}^d} |F_k^{xh}|^p dv \\ & \geq \frac{hp(p-1)}{\rho_k^h(x)} \int_{\mathbb{R}^d} \frac{(F_k^{xh})^{p-1}}{G''(f_k^h)} \left\langle \nabla c^*(\nabla_v G'(f_k^h)), \nabla_v G'(f_k^h) \right\rangle dv. \end{aligned} \quad (122)$$

From the convexity of c and c^* , we have

$$a \cdot b = c(a) + c^*(b), \quad \text{if } a, b \in \mathbb{R}^d, \text{ with } a = \nabla c^*(b). \quad (123)$$

Thus, using (123) with $a = \nabla c^*(\nabla_v(G'(f_k^h)))$ and $b = \nabla_v G'(f_k^h)$, (122) becomes

$$\int_{\mathbb{R}^d} |G_k^{xh}|^p dv - \int_{\mathbb{R}^d} |F_k^{xh}|^p dv \geq \frac{hp(p-1)}{\rho_k^h(x)} \int_{\mathbb{R}^d} \frac{(F_k^{xh})^{p-1}}{G''(f_k^h)} c(\nabla c^*(\nabla_v G'(f_k^h))) dv + \frac{hp(p-1)}{\rho_k^h(x)} \int_{\mathbb{R}^d} \frac{(F_k^{xh})^{p-1}}{G''(f_k^h)} c^*(\nabla_v G'(f_k^h)) dv. \quad (124)$$

Since $c \geq 0$ and $c(0) = 0$, then $c^* \geq 0$; hence,

$$\int_{\mathbb{R}^d} |G_k^{xh}|^p dv - \int_{\mathbb{R}^d} |F_k^{xh}|^p dv \geq 0. \quad (125)$$

Multiplying (125) by $|\rho_k^h|^p$, we obtain after integration

$$\int_{\Omega \times \mathbb{R}^d} |f_k^h|^p dx dv \leq \int_{\Omega \times \mathbb{R}^d} |g_k^h|^p dx dv = \int_{\Omega \times \mathbb{R}^d} |f_{k-1}^h|^p dx dv, \quad (126)$$

and then by an iteration process on k , we get the proof of Lemma 4. $\square$

Lemma 5. Assume that $f_0 \in L^p(\Omega \times \mathbb{R}^d) \cap P_q(\Omega \times \mathbb{R}^d)$ with $\int_{\Omega \times \mathbb{R}^d} G(f_0) dx dv < \infty$ and $0 < N \leq f_0 \leq M$. For $0 < T < \infty$, we have

$$\int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^h(t, x, v)|^p dt dx dv \leq T \int_{\Omega \times \mathbb{R}^d} |f_0(x, v)|^p dx dv. \quad (127)$$

Proof. Let $0 < T < \infty$ and let $h > 0$ be a step such that $T/h \in \mathbb{N}^*$.

We use Lemma 3 and the definition of f^h in (118) to obtain

$$\int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^h(t, x, v)|^p dt dx dv = \sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^d} |f_k^h|^p dx dv \leq T \int_{\Omega \times \mathbb{R}^d} |f_0(x, v)|^p dx dv < \infty. \quad (128)$$

4.2. Weak Convergence of the Linear Term. Here, we study the weak convergence of the linear term.

Proposition 3. Assume that f_0 satisfy (H_{f_0}) . Then, there exists a function f in $L^p([0, \infty) \times \Omega \times \mathbb{R}^d) \cap L^\infty([0, T] \times \Omega \times \mathbb{R}^d)$ until a subsequence $(f^h)_h$ converges to f weakly in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ and weakly- $*$ in $L^\infty([0, T] \times \Omega \times \mathbb{R}^d)$.

Moreover,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_{[0,T] \times \Omega \times \mathbb{R}^d} [\partial_t \psi + v \cdot \nabla_x \psi] f^{h_k} dt dx dv \\ & = \int_{[0,T] \times \Omega \times \mathbb{R}^d} [\partial_t \psi + v \cdot \nabla_x \psi] f dt dx dv, \end{aligned} \quad (129)$$

$\square$

for every test function $\psi \in C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)$, with $\text{supp} \psi(., x, v) \subset [-T, T]$.

Proof. Since $f_0 \in L^p(\Omega \times \mathbb{R}^d) \cap L^\infty(\Omega \times \mathbb{R}^d)$ (see Proposition 2 and Lemma 4), then

$$\begin{aligned} \|f^h\|_{L^\infty} &\leq \|f_0\|_{L^\infty(\Omega \times \mathbb{R}^d)}, \\ \|f^h\|_{L^p} &\leq T^{1/p} \|f_0\|_{L^p(\Omega \times \mathbb{R}^d)}. \end{aligned} \quad (130)$$

Hence, the sequence $(f^h)_h$ is bounded in $L^p([0, T] \times \Omega \times \mathbb{R}^d) \cap L^\infty([0, T] \times \Omega \times \mathbb{R}^d)$, and then, there is subsequence of $(f^h)_h$ still denoted $(f^h)_h$ that converges to a nonnegative function f weakly in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ and weakly-* in $L^\infty[0, T] \times \Omega \times \mathbb{R}^d$.

Since $\partial_t \psi + v \cdot \nabla_x \psi \in L^q([0, T] \times \Omega \times \mathbb{R}^d)$, for every $\psi \in C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)$ such that $\text{supp} \psi(., x, v) \subset [-T, T]$, we finally obtain Proposition 3. $\square$

4.3. Weak Convergence of the Nonlinear Term. In this section, we establish the weak convergence of the nonlinear term.

First, we prove that the sequence $\nabla_v G'(f^h)$ is bounded in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$, and that $\nabla c^*(\nabla_v(G'(f^h)))$ is bounded in $L^q([0, T] \times \Omega \times \mathbb{R}^d)$ where $0 < T < \infty$ fixed.

Lemma 6. Assume that f_0 , c , and G satisfy, respectively, (H_{f_0}) , (H_c) , and (H_G) . Then, for all $0 < T < \infty$ fixed,

$$\begin{aligned} &\int_{[0, T] \times \Omega \times \mathbb{R}^d} \left| \nabla c^*(\nabla_v(G'(f^h))) \right|^q dt dx dv \\ &\leq \frac{T}{A_1 N} [G'(M) - G'(N)], \\ &\int_{[0, T] \times \Omega \times \mathbb{R}^d} \left| \nabla_v(G'(f^h)) \right|^p dt dx dv \\ &\leq [A_2 T + G'(M) - G'(N)] \frac{pq^{p-1} A_2^{p-1}}{N}. \end{aligned} \quad (131)$$

Proof. Using (115), we have

$$\begin{aligned} &\int_{\Omega \times \mathbb{R}^d} \left| \nabla c^*(\nabla_v(G'(f_k^h))) \right|^q f_k^h dx dv \\ &\leq \frac{G'(M) - G'(N)}{A_1}. \end{aligned} \quad (132)$$

But $f^h \geq N$, and then

$$\begin{aligned} &\int_{[0, T] \times \Omega \times \mathbb{R}^d} \left| \nabla c^*(\nabla_v(G'(f^h))) \right|^q dt dx dv \\ &\leq [G'(M) - G'(N)]' \frac{T}{NA_1}. \end{aligned} \quad (133)$$

Recalling (123), with $a = \nabla c^*(\nabla_v(G'(f^h)))$ and $b = \nabla_v G'(f^h)$, we have

$$\begin{aligned} &\nabla c^*(\nabla_v(G'(f^h))) \cdot \nabla_v G'(f^h) \\ &- c(\nabla c^*(\nabla_v(G'(f^h)))) = c^*(\nabla_v G'(f^h)), \end{aligned} \quad (134)$$

with $c \geq 0$.

Then,

$$\begin{aligned} &\int_{[0, T] \times \Omega \times \mathbb{R}^d} c^*(\nabla_v(G'(f^h))) f^h dt dx dv \\ &\leq \int_{[0, T] \times \Omega \times \mathbb{R}^d} \nabla c^*(\nabla_v(G'(f^h))) \nabla_v G'(f^h) dt dx dv \\ &= \sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^d} \nabla c^*(\nabla_v(G'(f_k^h))) \nabla_v G'(f_k^h) dx dv \\ &\leq \sum_{k=1}^{T/h} \left[\int_{\Omega \times \mathbb{R}^d} G(f_{k-1}^h) dx dv - \int_{\Omega \times \mathbb{R}^d} G(f_k^h) dx dv \right]. \end{aligned} \quad (135)$$

Moreover,

$$\begin{aligned} &\sum_{k=1}^{T/h} \left[\int_{\Omega \times \mathbb{R}^d} G(f_{k-1}^h) dx dv - \int_{\Omega \times \mathbb{R}^d} G(f_k^h) dx dv \right] \\ &= \int_{\Omega \times \mathbb{R}^d} G(f_0) dx dv - \int_{\Omega \times \mathbb{R}^d} G(f_{T/h}) dx dv \\ &\leq G'(M) - G'(N), \end{aligned} \quad (136)$$

and then,

$$\int_{[0, T] \times \Omega \times \mathbb{R}^d} c^*(\nabla_v(G'(f^h))) f^h dt dx dv \leq G'(M) - G'(N). \quad (137)$$

Because $c(y) \leq A_2 |y|^q + A_2$, we have $x \cdot y - c(y) \geq x \cdot y - A_2 |y|^q - A_2$ for all $x, y \in \mathbb{R}^d$ and then

$$c^*(x) := \sup_{y \in \mathbb{R}^d} \{x \cdot y - c(y)\} \geq -A_2 + \frac{|x|^p}{pq^{p-1} A_2^{p-1}}. \quad (138)$$

Consequently,

$$\begin{aligned} &-A_2 T + \frac{1}{pq^{p-1} A_2^{p-1}} \int_{[0, T] \times \Omega \times \mathbb{R}^d} \left| \nabla_v(G'(f^h)) \right|^p f^h dt dx dv \\ &\leq G'(M) - G'(N). \end{aligned} \quad (139)$$

Using $f^h \geq N$, we obtain

$$\begin{aligned} &\int_{[0, T] \times \Omega \times \mathbb{R}^d} \left| \nabla_v(G'(f^h)) \right|^p dt dx dv \\ &\leq [TA_2 + G'(M) - G'(N)] \frac{pq^{p-1} A_2^{p-1}}{N}. \end{aligned} \quad (140)$$

$\square$

4.4. Strong Convergence of $(f^h)_h$. In order to prove the strong convergence result, we need to establish the following compactness results for $(f^h)_h$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$.

Lemma 7 (velocity-compactness). Assume that f_0 satisfies (H_{f_0}) . Fix $0 < T < \infty$ and $0 \neq \eta \in \mathbb{R}^d$ such that $|\eta|$ is small. Then,

$$\int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^h(t, x, v + \eta) - f^h(t, x, v)|^p dt dx dv \leq |\eta|^p K, \quad (141)$$

where $K := \inf_{t \in [N, M]} 1/|G''(t)|^p [TA_2 + G'(M) - G'(N)]$ ($p q^{p-1} A_2^{p-1}/N$) is a constant.

Proof. We use $\nabla_v G'(f^h) = G''(f^h) \nabla_v f^h$, $N \leq f^h \leq M$, and (140) to produce that

$$\int_{[0,T] \times \Omega \times \mathbb{R}^d} |\nabla_v f^h|^p dt dx dv \leq K. \quad (142)$$

Then, f^h and $\nabla_v f^h \in L^p([0, T] \times \Omega \times \mathbb{R}^d)$ and approximating $f^h(t, x, \cdot)$ by $C_c^\infty(\mathbb{R}^d)$ functions in $W^{1,p}(\mathbb{R}^d)$, we obtain

$$\begin{aligned} & \int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^h(t, x, v + \eta) - f^h(t, x, v)|^p dt dx dv \\ & \leq |\eta|^p \int_{[0,T] \times \Omega \times \mathbb{R}^d} |\nabla_v f^h|^p dt dx dv. \end{aligned} \quad (143)$$

This implies that

$$\int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^h(t, x, v + \eta) - f^h(t, x, v)|^p dt dx dv \leq |\eta|^p K. \quad (144)$$

□

Lemma 8 (position-compactness). Assume that f_0 satisfies (H_{f_0}) . We fix $0 < T < \infty$ and $0 \neq \eta \in \mathbb{R}$ such that $|\eta|$ is small. Then,

$$\int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^h(t, x + h\eta v, v) - f^h(t, x, v)|^p dt dx dv \leq |\eta|^p T K_1, \quad (145)$$

where $K_1 = 2^p \|f_0\|_{L^p(\Omega \times \mathbb{R}^d)}^p$.

Proof. By fixing v , (17) becomes

$$f_k^h - f_{k-1}^h + h v \cdot \nabla_x f_k^h = o(h), \quad (146)$$

where $o(h)$ tends to 0 when h goes to 0. We use (146), and the fact that $(f_k^h)_k$ is bounded in $L^p(\Omega \times \mathbb{R}^d)$ to deduce that $(h v \cdot \nabla_x f_k^h)_k$ is bounded in $L^p(\Omega \times \mathbb{R}^d)$. Then, by using Lemma 4, we have

$$\|h v \cdot \nabla_x f_k^h\|_{L^p(\Omega \times \mathbb{R}^d)} \leq 2 \|f_0\|_{L^p(\Omega \times \mathbb{R}^d)}. \quad (147)$$

Thus, $f^h(t, \cdot, v) \in L^p(\Omega)$, and approximating $f^h(t, \cdot, v)$ by $C_c^\infty(\mathbb{R}^d)$ functions, we use the fact that $(h v \cdot \nabla_x f_k^h)_k$ is bounded in $L^p(\Omega \times \mathbb{R}^d)$ to deduce that

$$\begin{aligned} & \int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^h(t, x + h\eta v, v) - f^h(t, x, v)|^p dt dx dv \\ & \leq |\eta|^p \int_{[0,T] \times \Omega \times \mathbb{R}^d} |h v \cdot \nabla_x f^h(t, x, v)|^p dt dx dv \\ & \leq |\eta|^p 2^p T \left(\int_{\Omega \times \mathbb{R}^d} |f_0(x, v)|^p dx dv \right). \end{aligned} \quad (148)$$

□

Lemma 9. Assume that (H_{f_0}) holds. Let $\tau > 0$ be quite small such that $[t, t + \tau] \subset [0, T]$, $\forall t \in [0, T]$, where $0 < T < \infty$ is fixed. We have

$$\begin{aligned} & \int_{[0,T] \times \Omega \times \mathbb{R}^d} [G'(f^h(t + \tau)) - G'(f^h(t))] [f^h(t + \tau) - f^h(t)] dt dx dv \\ & \leq \tau K_2(\Omega, T, N, M, f_0, G', q, p), \end{aligned} \quad (149)$$

where $K_2(\Omega, T, N, M, f_0, G', q, p)$ is a positive constant depending on $\Omega, T, N, M, f_0, G', q, p$.

Proof. Choose $h > 0$ such that $T/h \in \mathbb{N}^*$ and $\tau/h = N \in \mathbb{N}^*$. Define

$$\begin{aligned} A = & \int_{[0,T] \times \Omega \times \mathbb{R}^d} [G'(f^h(t + \tau)) - G'(f^h(t))] \\ & \cdot [f^h(t + \tau) - f^h(t)] dt dx dv. \end{aligned} \quad (150)$$

Using the definition of f^h in (118), we obtain

$$A = \sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^d} [\psi_{h,k}(x, v)] [f_{k-1+N}^h(x, v) - f_{k-1}^h(x, v)] dx dv, \quad (151)$$

where $\psi_{h,k}(x, v) = [G'(f_{k-1+N}^h(x, v)) - G'(f_{k-1}^h(x, v))]$.

Note that

$$f_{k-1+N}^h(x, v) - f_{k-1}^h(x, v) = \sum_{i=0}^{N-1} [f_{k-1+N-i}^h(x, v) - f_{k-2+N-i}^h(x, v)]. \quad (152)$$

Using (152), we have

$$\begin{aligned} A = & \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h \int_{\Omega \times \mathbb{R}^d} [\psi_{h,k}(x, v)] [f_{k-1+N-i}^h(x, v) - f_{k-2+N-i}^h(x, v)] dx dv \\ = & C + D, \end{aligned} \quad (153)$$

where

$$\begin{aligned} C = & \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h \int_{\Omega \times \mathbb{R}^d} [\psi_{h,k}(x, v)] [f_{k-1+N-i}^h(x, v) - g_{k-1+N-i}^h(x, v)] dx dv, \\ D = & \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h \int_{\Omega \times \mathbb{R}^d} [\psi_{h,k}(x, v)] [g_{k-1+N-i}^h(x, v) - f_{k-2+N-i}^h(x, v)] dx dv. \end{aligned} \quad (154)$$

Using $S_{k,N}^{h,i}$ as a c -optimal map that pushes $F_{k-1+N-i}^{xh}$ forward to $G_{k-1+N-i}^{xh}$, we have

$$C = \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h \int_{\Omega \times \mathbb{R}^d} [\psi_{h,k}(x, v) - \psi_{h,k}(x, S_{k,N}^{h,i}(v))] f_{k-1+N-i}^h(x, v) dx dv. \quad (155)$$

On the contrary, $\nabla_v G'(f_{k-1+N-i}^h) \in L^p(\Omega \times \mathbb{R}^d)$ (see Lemma 6), and from assumptions on G , $G'(s)/s$ is continuous on $\mathbb{R}^*$, and then there exists a constant K such that $G'(s) \leq K \cdot s$ for $0 < N \leq s \leq M$. Since $0 < N \leq f_{k-1+N-i}^h \leq M$ (see Lemma 2) and $f_{k-1+N-i}^h \in L^p(\Omega \times \mathbb{R}^d)$, then $G'(f_{k-1+N-i}^h) \in L^p(\Omega \times \mathbb{R}^d)$. So $G'(f_{k-1+N-i}^h(x, \cdot)) \in W^{1,p}(\mathbb{R}^d)$.

Approximating $G'(f_{k-1+N-i}^h(x, \cdot))$ by $C_c^\infty(\mathbb{R}^d)$ functions and using the dominated convergence theorem, we have

$$C = \sum_{i=1}^{N-1} \sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^d} f_{k-1+N-i}^h(x, v) \langle v - S_{k,N}^{h,i}(x, v), \nabla_v \psi_{h,k}(x, v + \theta(v - S_{k,N}^{h,i}(x, v))) \rangle dx dv, \quad \text{for } \theta \in]0, 1[. \quad (156)$$

Note $v - S_{k,N}^{h,i} = -h \nabla c^*(\nabla_v G'(f_{k-1+N-i}^h)) \in L^q(\Omega \times \mathbb{R}^d)$

that
and

$\nabla_v \psi_{h,k} = \nabla_v G'(f_{k-1+N}^h) - \nabla_v G'(f_{k-1}^h) \in L^p(\Omega \times \mathbb{R}^d)$ (see Lemma 6).

So, by using Hölder's inequality, we have that

$$\begin{aligned} C &\leq \sum_{i=1}^{N-1} \sum_{k=1}^{T/h} h \left(\int_{\Omega \times \mathbb{R}^d} |v - S_{k,N}^{h,i}(v)|^q f_{k-1+N-i}^h(x, v) dx dv \right)^{1/q} \left(\int_{\Omega \times \mathbb{R}^d} |\nabla_v \psi_{h,k}|^p \right)^{1/p} \\ &\leq \sum_{i=1}^{N-1} \sum_{k=1}^{T/h} \frac{h}{A_1^{1/q}} \left(\int_{\Omega} \rho_{k-1+N-i}^h(x) W_c^h(F_{k-1+N-i}^{xh}, G_{k-1+N-i}^{xh}) dx \right)^{1/q} \left(\int_{\Omega \times \mathbb{R}^d} |\nabla_v \psi_{h,k}|^p \right)^{1/p}. \end{aligned} \quad (157)$$

Using the algorithm defined in Section 2.1 and Lemma 6, we have

Thus,

$$\begin{aligned} &\int_{\Omega} \rho_{k-1+N-i}^h(x) W_c^h(F_{k-1+N-i}^{xh}, G_{k-1+N-i}^{xh}) dx \\ &\leq h^{q-1} \left[\int_{\Omega \times \mathbb{R}^d} G(f_{k-2+N-i}^h(x, v)) dx dv \right. \\ &\quad \left. - \int_{\Omega \times \mathbb{R}^d} G(f_{k-1+N-i}^h(x, v)) dx dv \right]. \end{aligned} \quad (158)$$

$$\begin{aligned} C &\leq \sum_{i=1}^{N-1} \sum_{k=1}^{T/h} \frac{h}{A_1^{1/q}} \left[\int_{\Omega \times \mathbb{R}^d} G(f_{k-2+N-i}^h(x, v)) dx dv - \int_{\Omega \times \mathbb{R}^d} G(f_{k-1+N-i}^h(x, v)) dx dv \right]^{1/q} \times \\ &\quad h^{(q-1)/q} \left(\int_{\Omega \times \mathbb{R}^d} |\nabla_v \psi_{h,k}|^p \right)^{1/p}. \end{aligned} \quad (159)$$

Applying the Hölder's inequality in the previous relation yields

$$\begin{aligned} C &\leq \frac{1}{A_1^{1/q}} h^{1-(1/p)} h^{(q-1)/q} \left\{ \sum_{k=1}^{T/h} \left[\int_{\Omega \times \mathbb{R}^d} G(f_{k-2+N-i}^h(x, v)) dx dv - \int_{\Omega \times \mathbb{R}^d} G(f_{k-1+N-i}^h(x, v)) dx dv \right] \right\}^{1/q} \times \\ &\quad \sum_{i=1}^{N-1} \left(\sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^d} |\nabla_v \psi_{h,k}|^p \right)^{1/p}. \end{aligned} \quad (160)$$

We notice that

$$\begin{aligned}
& \sum_{k=1}^{T/h} \left[\int_{\Omega \times \mathbb{R}^d} G(f_{k-2+N-i}^h(x, v)) dx dv \right. \\
& \quad \left. - \int_{\Omega \times \mathbb{R}^d} G(f_{k-1+N-i}^h(x, v)) dx dv \right] \\
&= \int_{\Omega \times \mathbb{R}^d} G(f_{-1+N-i}^h(x, v)) dx dv \\
& \quad - \int_{\Omega \times \mathbb{R}^d} G(f_{(T/h)-1+N-i}^h(x, v)) dx dv \leq [G'(M) - G'(N)],
\end{aligned} \tag{161}$$

Next, we use Lemma 6, and we have

$$\begin{aligned}
& \left(\sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^d} |\nabla_v \psi_{h,k}|^p \right)^{1/p} = \left(\sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^d} \left| \nabla_v G'(f_{k-1+N}^h) - \nabla_v G'(f_{k-1}^h) \right|^p \right)^{1/p} \\
& \leq 2^{(p-1)/p} \left(\sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^d} |\nabla_v G'(f_{k-1+N}^h)|^p + \sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^d} |\nabla_v G'(f_{k-1}^h)|^p \right)^{1/p} \\
& \leq 2^{(p-1)/p} (2M_0(T, \Omega))^{1/p} = 2(M_0(T, \Omega))^{1/p}.
\end{aligned} \tag{162}$$

We combine (160), (161), and (162) to get

$$\begin{aligned}
C & \leq \sum_{i=1}^{N-1} 2h^{1-(1/p)} h^{(q-1)/q} [G'(M) - G'(N)]^{1/q} (M_0(T, \Omega))^{1/p} \\
& \leq Nh^{2-((1/p)+(1/q))} [G'(M) - G'(N)]^{1/q} (M_0(T, \Omega))^{1/p}.
\end{aligned} \tag{163}$$

Writing $Nh = \tau$, we then have

$$C \leq [G'(M) - G'(N)]^{1/q} (M_0(T, \Omega))^{1/p} \tau. \tag{164}$$

On the contrary,

$$\begin{aligned}
D &= \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h \int_{\Omega \times \mathbb{R}^d} [\psi_{h,k}(x, v)] \\
& \quad [f_{k-2+N-i}^h(x, v) - g_{k-1+N-i}^h(x, v)] dx dv \\
&= \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h \int_{\Omega \times \mathbb{R}^d} [\psi_{h,k}(x, v)] \\
& \quad [f_{k-2+N-i}^h(x, v) - f_{k-2+N-i}^h(x - hv, v)] dx dv.
\end{aligned} \tag{165}$$

Since $G'(f_{k-1+N-i}^h(\cdot, v)) \in W^{1,p}(\Omega)$ and then $\psi_{h,k}(\cdot, v) \in W^{1,p}(\Omega)$, we approximate $\psi_{h,k}(\cdot, v)$ by $C_c^\infty(\Omega)$ functions and we get

$$D = - \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h^2 \int_{\Omega \times \mathbb{R}^d} v \cdot \nabla_x \psi_{h,k}(x, v) f_{k-2+N-i}^h(x, v) dx dv. \tag{166}$$

Note that $v \cdot \nabla_x (G'(f_k^h)) = G''(f_k^h) v \cdot \nabla_x f_k^h$, and according to Lemma 8, $v \cdot \nabla_x f_k^h \in L^p(\Omega \times \mathbb{R}^d)$. So, since G'' is continuous and $N \leq f_k^h \leq M$, we have $(v \cdot \nabla_x G'(f_k^h))_{h,k} \in L^p(\Omega \times \mathbb{R}^d)$.

Consequently,

$$\begin{aligned}
D &= - \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h^2 \int_{\Omega \times \mathbb{R}^d} v \cdot \nabla_x \psi_{h,k}(x, v) f_{k-2+N-i}^h(x, v) dx dv \\
&= - \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h^2 \int_{\Omega \times \mathbb{R}^d} v \cdot \nabla_x G'(f_{k-1+N}^h)(x, v) f_{k-2+N-i}^h(x, v) dx dv \\
& \quad + \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h^2 \int_{\Omega \times \mathbb{R}^d} v \cdot \nabla_x G'(f_{k-1}^h)(x, v) f_{k-2+N-i}^h(x, v) dx dv \\
&\leq \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h^2 \left(\int_{\Omega \times \mathbb{R}^d} |v \cdot \nabla_x G'(f_{k-1}^h)(x, v)|^p dx dv \right)^{1/p} \|f_0\|_{L^\infty} \\
& \quad + \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h^2 \left(\int_{\Omega \times \mathbb{R}^d} |v \cdot \nabla_x G'(f_{k-1+N}^h)(x, v)|^p dx dv \right)^{1/p} \|f_0\|_{L^\infty}.
\end{aligned} \tag{167}$$

Next, we use the fact that $v \cdot \nabla_x G'(f_k^h) = G''(f_k^h) v \cdot \nabla_x f_k^h$ and the boundedness in $L^p(\Omega \times \mathbb{R}^d)$, of $(v \cdot \nabla_x f_k^h)_{h,k}$ (see Lemma 8), to get

$$\begin{aligned}
D &\leq 2 \|f_0\|_{L^\infty(\Omega \times \mathbb{R}^d)} N \frac{T}{h} \sup_{s \in [N, M]} G''(s) K_1(\Omega) h^2 \\
&= 2 \|f_0\|_{L^\infty(\Omega \times \mathbb{R}^d)} \sup_{s \in [N, M]} G''(s) K_1(\Omega) T \tau.
\end{aligned} \tag{168}$$

Thus,

$$A = C + D \leq K_2(A_1, \Omega, T, N, M, f_0, G', q, p) \tau, \tag{169}$$

where $K_2(A_1, \Omega, T, N, M, f_0, G', q, p) := 2 \|f_0\|_{L^\infty(\Omega \times \mathbb{R}^d)} \sup_{s \in [N, M]} G''(s) K_1(\Omega) T + [G'(M) - G'(N)]^{1/q} (M_0(T, \Omega))^{1/p}$.

Let us process now to the following estimate. $\square$

Lemma 10. Assume that f_0 satisfies (H_{f_0}) . Then, for every $0 < T < \infty$ fixed and for all $\tau > 0$ small, we have

$$\int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^h(t+\tau, x, v) - f^h(t, x, v)|^p dt dx dv \leq C(A_1, \Omega, T, N, M, f_0, G', q, p)\tau, \quad (170)$$

where $C(A_1, \Omega, T, N, M, f_0, G', q, p)$ is positive constant.

Proof. Since $G \in C^2$, we use Taylor's formula to have that

$$\begin{aligned} G'(f^h(t+\tau, x, v) - G'(f^h(t, x, v))) \\ = G''(f_\theta^h)(f^h(t+\tau, x, v) - f^h(t, x, v)), \end{aligned} \quad (171)$$

where $\theta \in]0, 1[$ and $f_\theta^h := \theta f^h(t) + (1-\theta)f^h(t+\tau)$.

Since $0 < N \leq f_0 \leq M$, we have $0 < N \leq f^h \leq M$ (see Lemma 2). Thus, using (171) and the fact that G'' is continuous and nonnegative, we have

$$\begin{aligned} \int_{[0,T] \times \Omega \times \mathbb{R}^d} [G'(f^h(t+\tau)) - G'((f^h))][f^h(t+\tau) - f^h(t)] dt dx dv \\ \geq \inf_{t \in [N, M]} G''(t) \int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^h(t+\tau, x, v) - f^h(t, x, v)|^2 dt dx dv. \end{aligned} \quad (172)$$

Define $\beta = \inf_{t \in [N, M]} G''(t) > 0$. We use Lemma 9 and obtain

$$\begin{aligned} \int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^h(t+\tau, x, v) - f^h(t, x, v)|^2 dt dx dv \\ \leq \frac{1}{\beta} K_2(A_1, \Omega, T, N, M, f_0, G', q, p)\tau. \end{aligned} \quad (173)$$

Using $N \leq f^h \leq M$, we have

$$\begin{aligned} \int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^h(t+\tau, x, v) - f^h(t, x, v)|^p dt dx dv \\ \leq \frac{1}{\beta} C(A_1, \Omega, T, N, M, f_0, G', q, p)\tau, \end{aligned} \quad (174)$$

where $C(\Omega, T, N, M, f_0, G', q, p)$ is positive constant.

We will use Lemmas 7, 8, and 10, to prove that the sequence $(f^h)_h$ is precompact in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$. $\square$

Lemma 11. Assume that f_0 , c , and G satisfy, respectively, (H_{f_0}) , (H_c) , and (H_G) . Then, $(f^h)_h$ is precompact in $L^p([0, T] \times \Omega \times B_R)$, where B_R is the open ball centered at origin of radius R .

Proof. Define $K = [0, T] \times \Omega \times B_R$. Then, $K = \bigcup_{\delta > 0} K^\delta$, with $K^\delta = \{y \in K, \text{dist}(y, K^c) > \delta\}$. $K^\delta \subset K$, $\overline{K^\delta} \subset K$, and $\overline{K^\delta}$ is compact and $\text{dist}(K^\delta, K^c) > \delta/2$. So, there exists $\delta_1, \delta_2, \dots, \delta_l > 0$ such that $\overline{K^\delta} \subset \bigcup_{i=1}^l K^{\delta_i}$.

Define $\delta_0 = \min_{i=1,2,\dots,l} \delta_i$ and we have $K^{\delta_0} \subset K$ and $\text{dist}(K^{\delta_0}, K^c) > \delta_0/2$.

Let $\tau > 0$, $\eta_1, \eta_2 \in \mathbb{R}^d$, such that $0 < \tau < \delta_{0,\varepsilon}$, $|\eta_1| < \delta_{0,\varepsilon}$, and $|\eta_2| < \delta_{0,\varepsilon}$, where $\delta_{0,\varepsilon} = \min\{\delta_0/2, \varepsilon\}$.

We use Lemmas 7, 8, and 10 to have

$$\int_{K^\delta} |f^h(t+\tau, x, v) - f^h(t, x, v)|^p dt dx dv < c_1 \varepsilon,$$

$$\int_{K^\delta} |f^h(t, x + \eta_1, v) - f^h(t, x, v)|^p dt dx dv < \varepsilon^p c_2, \quad (175)$$

$$\int_{K^\delta} |f^h(t, x, v + \eta_2) - f^h(t, x, v)|^p dt dx dv < c_3 \varepsilon^p.$$

c_1, c_2 , and c_3 are defined in Lemmas 7, 8, and 10. So, by using the Riesz–Fréchet–Kolmogorov's theorem, we deduce that the sequence $(f^h)_h$ is precompact in $L^p(K^\delta)$. Therefore, $\lim_{\delta \rightarrow 0} |K \setminus K^\delta| = 0$, and then for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $|K \setminus K^{\delta_\varepsilon}| < \varepsilon \|f_0\|_{L^\infty}$, and since $(f^h)_h$ is bounded in $L^\infty([0, T] \times \Omega \times B_R)$, we have $\|f^h\|_{L^p(K \setminus K^{\delta_\varepsilon})}^p \leq \|f_0\|_{L^\infty} |K \setminus K^{\delta_\varepsilon}| < \varepsilon$.

We use corollary IV.26 of [9], to conclude that the sequence $(f^h)_h$ is precompact in $L^p([0, T] \times \Omega \times B_R)$, which implies that $(f^h)_h$ converges strongly to some function f_R in $L^p([0, T] \times \Omega \times B_R)$.

Now, let us prove that the sequence $(f^h)_h$ is precompact in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$. $\square$

Lemma 12. Assume that f_0 , c , and G satisfy, respectively, (H_{f_0}) , (H_c) , and (H_G) . Then, the sequence $(f^h)_h$ is precompact in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$.

Proof. Since $f_0 \in L^p([0, T] \times \Omega \times \mathbb{R}^d)$, Lemma 3 implies that $(f^h)_h$ is bounded in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$. Then, $(f^h)_h$ converges weakly to some function f in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ (up to a subsequence).

Let B_n be an open ball of radius n centered at origin, with $n \in \mathbb{N}^*$. Then, $B_n \subset B_{n+1}$ and $\bigcup_{n \in \mathbb{N}^*} B_n = \mathbb{R}^d$.

Using previous results, the sequence $(f^h)_h$ is precompact in $L^p([0, T] \times \Omega \times B_n)$. A subsequence $(f^{h_n})_n$ converges strongly to some function f_n in $L^p([0, T] \times \Omega \times B_n)$. Let us denote by (f^{h_n}) the subsequence of $(f^h)_h$ which converges strongly to f_n in $L^p([0, T] \times \Omega \times B_n)$. Fix ϕ in $L^q([0, T] \times \Omega \times \mathbb{R}^d)$. We have

$$\begin{aligned} \int_{[0,T] \times \Omega \times \mathbb{R}^d} (f^{h_n} - f_n) \phi dt dx dv &= \int_{[0,T] \times \Omega \times B_n} (f^{h_n} - f_n) \phi dt dx dv \\ &+ \int_{[0,T] \times \Omega \times B_n^c} (f^{h_n} - f_n) \phi dt dx dv, \end{aligned} \quad (176)$$

with

$$\left| \int_{[0,T] \times \Omega \times B_n^c} (f^{h_n} - f_n) \phi dt dx dv \right| \leq \|\phi\|_{L^q([0,T] \times \Omega \times \mathbb{R}^d)} 2M |B_n^c|^{1/p}. \quad (177)$$

$|B_n^c|$ tends to 0, when n goes to ∞ ; for $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

$$\left| \int_{[0,T] \times \Omega \times B_n^c} (f^{h_n} - f_n) \phi dt dx dv \right| \leq \frac{\varepsilon}{2}, \quad (178)$$

for all $n \geq N_\varepsilon$. Since f^{h_n} converges strongly to f_n in $L^p([0, T] \times \Omega \times B_n)$, we deduce that

$$\left| \int_{[0,T] \times \Omega \times \mathbb{R}^d} (f^{h_n} - f_n) \phi dt dx dv \right| \leq \varepsilon, \quad (179)$$

when h_n goes to 0 and $n \geq N_\varepsilon$. We conclude that (f^{h_n}) converges weakly to f_n in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$, for $n \geq N_\varepsilon$. So $f_n = f$ for $n \geq N_\varepsilon$.

Now, let prove that $\|f^{h_n} - f\|_{L^p([0,T] \times \Omega \times \mathbb{R}^d)}$ tends to 0 as $h_n \rightarrow 0$ and $n \geq N_\varepsilon$.

Note that

$$\begin{aligned} & \|f^{h_n} - f\|_{L^p([0,T] \times \Omega \times \mathbb{R}^d)}^p \\ &= \int_{[0,T] \times \Omega \times B_n} |f^{h_n} - f|^p dt dx dv + \int_{[0,T] \times \Omega \times B_n^c} |f^{h_n} - f|^p dt dx dv \\ &\leq \int_{[0,T] \times \Omega \times B_n} |f^{h_n} - f|^p dt dx dv + 2^p |f_0|_{L^\infty}^p T |\Omega| |B_n^c|. \end{aligned} \quad (180)$$

Let us recall that

$$\lim_{h_n \rightarrow 0} \int_{[0,T] \times \Omega \times B_n} |f^{h_n} - f|^p dt dx dv = 0, \quad \forall n \geq N_\varepsilon. \quad (181)$$

Since $|B_n^c| < (\varepsilon/2)$, for $n \geq N_\varepsilon$, as

$$\lim_{h_n \rightarrow 0} \int_{[0,T] \times \Omega \times B_{N_\varepsilon}^c} |f^{h_n} - f|^p dt dx dv = 0, \quad (182)$$

we can find $n_\varepsilon \in \mathbb{N}$ such that for every $n \geq N_\varepsilon$, we have

$$\int_{[0,T] \times \Omega \times B_{N_\varepsilon}^c} |f^{h_n} - f|^p dt dx dv < \frac{\varepsilon}{2}. \quad (183)$$

Then, for all $n \geq N_\varepsilon$,

$$\int_{[0,T] \times \Omega \times \mathbb{R}^d} |f^{h_n} - f|^p dt dx dv < \varepsilon. \quad (184)$$

Consequently, (f^{h_n}) converges strongly to f in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$.

Using the previous lemma, $(f^h)_h$ converges strongly to f in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ (up to a subsequence). G' is being continuous, then $(G'(f^h))_h$ converges strongly to $G'(f)$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$. We conclude that the sequence $\nabla_v G'(f^h)_h$ converges to $\nabla_v G'(f)$ in $[C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)]'$. Let us recall that $(\nabla_v G'(f^h))_h$ is bounded in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$, so $(\nabla_v G'(f^h))_h$ converges weakly to $\nabla_v G'(f)$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$.

On the contrary, $(\nabla_c^*(\nabla_v G'(f^h)))_h$ is bounded in $L^q([0, T] \times \Omega \times \mathbb{R}^d)$. Then, a subsequence $(\nabla_c^*(\nabla_v G'(f^h)))_h$ converges weakly to σ in $L^q([0, T] \times \Omega \times \mathbb{R}^d)$.

We derive from the proof of Theorem 3.10 in [1] that the sequence $\text{div}_v(f^h \nabla_c^*(\nabla_v G'(f^h)))$ converges weakly to $\text{div}_v(f \nabla_c^*(\nabla_v G'(f)))$ in $[C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)]'$. $\square$

Theorem 2. Assume that f_0 , c , and G satisfy, respectively, (H_{f_0}) , (H_c) , and (H_G) . Let $0 < T < \infty$ and $u \in C_c^\infty(\mathbb{R})$ be such that $u \geq 0$ and $\text{supp } u \subset [-T, T]$. Then,

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f^h \nabla_c^*(\nabla_v G'(f^h)), \nabla_v G'(f^h) \rangle u(t) dt dx dv \\ &= \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f \sigma, \nabla_v G'(f) \rangle u(t) dt dx dv. \end{aligned} \quad (185)$$

Moreover, $(\text{div}_v(f^h \nabla_c^*(\nabla_v G'(f^h))))_h$ converges weakly to $\text{div}_v(f \sigma)$ in $[C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)]'$ and $\text{div}_v(f \sigma) = \text{div}_v(f \nabla_c^*(\nabla_v G'(f)))$ in the weak sense.

Proof. Let us recall the following:

- (i) $(f^h)_h$ converges strongly to f in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$
- (ii) $(\nabla_v G'(f^h))_h$ converges weakly to $\nabla_v G'(f)$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$
- (iii) $(\sigma^h = \nabla_c^*(\nabla_v G'(f^h)))_h$ converges weakly to σ in $L^q([0, T] \times \Omega \times \mathbb{R}^d)$

Then, we have

$$\begin{aligned} & \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f^h \nabla_c^*(\nabla_v G'(f^h)), \nabla_v G'(f) \rangle u(t) dt dx dv \\ &= \int_{[0,T] \times \Omega \times \mathbb{R}^d} (f^h - f) \langle \nabla_c^*(\nabla_v G'(f^h)), \nabla_v G'(f) \rangle u(t) dt dx dv \\ &\quad + \int_{[0,T] \times \Omega \times \mathbb{R}^d} f \langle \nabla_c^*(\nabla_v G'(f^h)), \nabla_v G'(f) \rangle u(t) dt dx dv. \end{aligned} \quad (186)$$

Since $\nabla_v(G'(f)) \in L^p([0, T] \times \Omega \times \mathbb{R}^d)$ and $(\nabla_c^*(\nabla_v G'(f^h)))_h$ is bounded in $L^q([0, T] \times \Omega \times \mathbb{R}^d)$, we use (i) and (iii) to have

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f^h \sigma^h, \nabla_v G'(f) \rangle u(t) dt dx dv \\ &= \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f \sigma, \nabla_v G'(f) \rangle u(t) dt dx dv. \end{aligned} \quad (187)$$

Also, we have

$$\begin{aligned} & \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f^h \nabla_c^*(\nabla_v G'(f)), \nabla_v G'(f^h) - \nabla_v G'(f) \rangle u(t) dt dx dv \\ &= \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle (f^h - f) \nabla_c^*(\nabla_v G'(f)), \nabla_v G'(f^h) - \nabla_v G'(f) \rangle u(t) dt dx dv \\ &\quad + \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f \nabla_c^*(\nabla_v G'(f)), \nabla_v G'(f^h) - \nabla_v G'(f) \rangle u(t) dt dx dv. \end{aligned} \quad (188)$$

Using (i), (ii), and the fact that $\nabla_c^*(\nabla_v G'(f)) \in L^q([0, T] \times \Omega \times \mathbb{R}^d)$ and that $(\nabla_v G'(f^h))_h$ is bounded in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$, we get

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f^h \nabla_c^*(\nabla_v G'(f)), \nabla_v G'(f^h) - \nabla_v G'(f) \rangle \\ & \quad u(t) dt dx dv = 0. \end{aligned} \quad (189)$$

To conclude with Theorem 2, we need to establish the following limit, whose proof is derived from the three following lemmas:

$$\begin{aligned}
& \lim_{h \rightarrow 0} \int_{[0, \infty] \times \Omega \times \mathbb{R}^d} \langle f^h \nabla c^*(\nabla_v G'(f^h)), \nabla_v G'(f^h) \rangle u(t) dt dx dv \\
&= \int_{[0, \infty] \times \Omega \times \mathbb{R}^d} \langle f \sigma, \nabla_v G'(f) \rangle u(t) dt dx dv.
\end{aligned} \tag{190}$$

□

Lemma 13

$$\begin{aligned}
& \liminf_{h \rightarrow 0} \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f^h \sigma^h, \nabla_v G'(f^h) \rangle u(t) dt dx dv \\
&\geq \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f \sigma, \nabla_v G'(f) \rangle u(t) dt dx dv.
\end{aligned} \tag{191}$$

Proof. Since c^* is convex and $f^h \geq 0, u \geq 0$, then

$$\begin{aligned}
& f^h \langle \nabla c^*(\nabla_v G'(f^h)) - \nabla c^*(\nabla_v G'(f)), \nabla_v G'(f^h) \\
&\quad - \nabla_v G'(f) \rangle u(t) \geq 0.
\end{aligned} \tag{192}$$

So,

$$\begin{aligned}
& \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f^h \nabla c^*(\nabla_v G'(f^h)), \nabla_v G'(f^h) \rangle u(t) dt dx dv \\
&\geq \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f^h \nabla c^*(\nabla_v G'(f^h)), \nabla_v G'(f) \rangle u(t) dt dx dv \\
&\quad + \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f^h \nabla c^*(\nabla_v G'(f)), \nabla_v G'(f^h) - \nabla_v G'(f) \rangle \\
&\quad u(t) dt dx dv.
\end{aligned} \tag{193}$$

Passing to the limit, we obtain

$$\begin{aligned}
& \liminf_{h \rightarrow 0} \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f^h \nabla c^*(\nabla_v G'(f^h)), \nabla_v G'(f^h) \rangle u(t) dt dx dv \\
&\geq \liminf_{h \rightarrow 0} \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f^h \nabla c^*(\nabla_v G'(f^h)), \nabla_v G'(f) \rangle u(t) dt dx dv \\
&\quad + \liminf_{h \rightarrow 0} \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f^h \nabla c^*(\nabla_v G'(f)), \nabla_v G'(f^h) \\
&\quad - \nabla_v G'(f) \rangle u(t) dt dx dv.
\end{aligned} \tag{194}$$

Taking into account (187) and (189) in (194) enables us to conclude the proof of Lemma 13. □

Lemma 14

$$\begin{aligned}
& \limsup_{h \rightarrow 0} \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f^h \sigma^h, \nabla_v G'(f^h) \rangle u(t) dt dx dv \\
&\leq \int_{\Omega \times \mathbb{R}^d} [f_0 G'(f_0) - G^*(G'(f_0))] u(0) dx dv \\
&\quad + \int_{[0, T] \times \Omega \times \mathbb{R}^d} [f G'(f) - G^*(G'(f))] u'(t) dt dx dv.
\end{aligned} \tag{195}$$

Proof. Using the proof of Lemma 6, we have

$$\begin{aligned}
& \int_{\Omega \times \mathbb{R}^d} G(f_{k-1}^h) dx dv - \int_{\Omega \times \mathbb{R}^d} G(f_k^h) dx dv \\
&\geq \int_{\Omega \times \mathbb{R}^d} \langle T_k(v) - v, \nabla_v G'(f_k^h) \rangle f_k^h dx dv,
\end{aligned} \tag{196}$$

where

$$T_k(v) = v + h \nabla c^*(\nabla_v G'(f_k^h)), \tag{197}$$

is a c -optimal map that pushes F_k^{xh} forward G_k^{xh} . Thus,

$$\begin{aligned}
& \int_{\Omega \times \mathbb{R}^d} \frac{G(f_k^h) - G(f_{k-1}^h)}{h} dx dv \\
&\leq - \int_{\Omega \times \mathbb{R}^d} \langle \nabla c^*(\nabla_v G'(f_k^h)), \nabla_v G'(f_k^h) \rangle f_k^h dx dv.
\end{aligned} \tag{198}$$

Since $u \geq 0$, then

$$\begin{aligned}
& \sum_{k=1}^{T/h} \int_{t_{k-1}}^{t_k} \int_{\Omega \times \mathbb{R}^d} \frac{G(f_k^h) - G(f_{k-1}^h)}{h} u(t) dx dv \\
&\leq - \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle \nabla c^*(\nabla_v G'(f^h)), \nabla_v G'(f^h) \rangle f^h u(t) dt dx dv.
\end{aligned} \tag{199}$$

Also,

$$\begin{aligned}
& \sum_{k=1}^{T/h} \int_{t_{k-1}}^{t_k} \int_{\Omega \times \mathbb{R}^d} \frac{G(f_k^h) - G(f_{k-1}^h)}{h} u(t) dx dv \\
&= - \int_{[0, T] \times \Omega \times \mathbb{R}^d} G(f^h) \partial_t^h u(t) dt dx dv \\
&\quad - \frac{1}{h} \int_0^h \int_{\Omega \times \mathbb{R}^d} G(f_0) u(t) dt dx dv,
\end{aligned} \tag{200}$$

where $\partial_t^h u(t) = (u(t+h) - u(t))/h$.

Let us prove that

$$\begin{aligned}
& \lim_{h \rightarrow 0} \int_{[0, T] \times \Omega \times \mathbb{R}^d} G(f^h) \partial_t^h u(t) dt dx dv \\
&= \int_{[0, T] \times \Omega \times \mathbb{R}^d} G(f) u'(t) dt dx dv.
\end{aligned} \tag{201}$$

Indeed,

$$\begin{aligned}
& \left| \int_{[0, T] \times \Omega \times \mathbb{R}^d} G(f^h) \partial_t^h u(t) - G(f) u'(t) dt dx dv \right| \\
&\leq \int_{[0, T] \times \Omega \times \mathbb{R}^d} |G(f^h) - G(f)| |u'(t)| dt dx dv \\
&\quad + \int_{[0, T] \times \Omega \times \mathbb{R}^d} |\partial_t^h u(t) - u'(t)| |G(f^h)| dt dx dv.
\end{aligned} \tag{202}$$

Using (i), the continuity of G , the boundedness of $(f^h)_h$ in $L^\infty([0, \infty] \times \Omega \times \mathbb{R}^d)$, and the fact that $(\partial_t^h u)_h$ converges uniformly to u' on the compact $[0, T]$, we obtain

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} G(f^h) \partial_t^h u(t) dt dx dv \\ &= \int_{[0,T] \times \Omega \times \mathbb{R}^d} G(f) u'(t) dt dx dv. \end{aligned} \quad (203)$$

Next, we combine (199), (200), and (203) to get

$$\begin{aligned} & \int_{[0,T] \times \Omega \times \mathbb{R}^d} G(f) u'(t) dt dx dv + \int_{\Omega \times \mathbb{R}^d} G(f_0) u(0) dx dv \geq \\ & \limsup_{h \rightarrow 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*(\nabla_v G'(f^h)), \nabla_v G'(f^h) \rangle u(t) dt dx dv. \end{aligned} \quad (204)$$

Since $G \in C^1$ is strictly convex, $G^*(G'(s)) = sG'(s) - G(s)$, $\forall s > 0$. Consequently, replacing $G(f) = fG'(f) - G^*(G'(f))$ and $G(f_0) = f_0G'(f_0) - G^*(G'(f_0))$ in (5), we obtain Lemma 14. $\square$

Lemma 15

$$\begin{aligned} & \int_{\Omega \times \mathbb{R}^d} [f_0 G'(f_0) - G^*(G'(f_0))] u(0) dx dv \\ &+ \int_{[0,T] \times \Omega \times \mathbb{R}^d} [f G'(f) - G^*(G'(f))] u'(t) dt dx dv \\ &\leq \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f \sigma, \nabla_v G'(f) \rangle u(t) dt dx dv. \end{aligned} \quad (205)$$

Proof. Let us write $\psi(t, x, v) = G'(f)u(t)$ for $(t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^d$. we have $\psi(t, \cdot, \cdot) \in W^{1,p}(\Omega \times B_R)$.

Approximating $\psi(t, \cdot, \cdot)$ by $C_c^\infty(\Omega \times \mathbb{R}^d)$ functions and using (17), we have

$$\begin{aligned} \frac{A_k^h[\psi]}{h} &= \int_{\Omega \times B_R} \frac{f_k^h - g_k^h}{h} \psi(t, x + (t - t_k)v, v) dx dv \\ &+ \int_{\Omega \times B_R} f_k^h \langle \nabla c^*(\nabla G'(f_k^h)), \nabla_v \psi(t, x + (t - t_k)v, v) \rangle dx dv, \end{aligned} \quad (206)$$

where $A_k^h[\psi]$ is defined in (17) and tends to 0 as $h \rightarrow 0$. By integrating (206) over $[0, T]$, one obtains

$$\begin{aligned} \sum_{i=1}^{T/h} \int_{t_i}^{t_{i+1}} \frac{A_i^h[\psi]}{h} &= \sum_{i=1}^{T/h} \int_{t_i}^{t_{i+1}} \int_{\Omega \times B_R} \frac{f_i^h - g_i^h}{h} \psi(t, x + (t - t_i)v, v) dt dx dv \\ &+ \sum_{i=1}^{T/h} \int_{t_i}^{t_{i+1}} \int_{\Omega \times B_R} f_i^h \langle \nabla c^*(\nabla_v G'(f_i^h)), \nabla_v \psi(t, x + (t - t_i)v, v) \rangle dt dx dv. \end{aligned} \quad (207)$$

Next, from the definition of f^h , we derive that

$$\begin{aligned} & \sum_{k=1}^{T/h} \int_{t_k}^{t_{k+1}} \int_{\Omega \times B_R} \frac{f_k^h - g_k^h}{h} \psi(t, x + (t - t_k)v, v) dt dx dv \\ &= - \int_{[0,T] \times \Omega \times B_R} f^h \left[\frac{\psi(t + h, x, v) - \psi(t, x, v)}{h} \right] dt dx dv \\ &\quad - \frac{1}{h} \int_0^h \int_{\Omega \times B_R} f^h \psi(t, x, v) dt dx dv, \end{aligned} \quad (208)$$

$$\begin{aligned} & \sum_{i=1}^{T/h} \int_{t_i}^{t_{i+1}} \int_{\Omega \times B_R} f_i^h \langle \nabla c^*(\nabla_v G'(f_i^h)), \nabla_v \psi(t, x + (t - t_i)v, v) \rangle dt dx dv \\ &= \int_{[0,T] \times \Omega \times B_R} f^h \langle \nabla c^*(\nabla_v G'(f^h)), \nabla_v \psi(t, x, v) \rangle dt dx dv \\ &\quad - \int_0^h \int_{\Omega \times B_R} f_0 \langle \nabla c^*(\nabla_v G'(f_0)), \nabla_v \psi(t, x, v) \rangle dt dx dv. \end{aligned} \quad (209)$$

We combine (208) and (209) to obtain

$$\begin{aligned} B^h[\psi] &= - \int_{[0,T] \times \Omega \times B_R} f^h \left[\frac{G'(f(t+h))u(t+h) - G'(f(t))u(t)}{h} \right] dt dx dv \\ &\quad - \frac{1}{h} \int_0^h \int_{\Omega \times B_R} f_0 G'(f) u(t) dt dx dv \\ &\quad + \int_{[0,T] \times \Omega \times B_R} f^h \langle \nabla c^*(\nabla_v G'(f^h)), \nabla_v \psi \rangle dt dx dv \\ &\quad - \int_0^h \int_{\Omega \times B_R} f_0 \langle \nabla c^*(\nabla_v G'(f_0)), \nabla_v \psi \rangle dt dx dv, \end{aligned} \quad (210)$$

where

$$B^h[\psi] = \sum_{i=1}^{T/h} \int_{t_i}^{t_{i+1}} \frac{A_i^h[\psi]}{h} dt. \quad (211)$$

Let us show that $B^h[\psi]$ tends to 0 as $h \rightarrow 0$. Assume that $\psi \in C_c^2(\mathbb{R} \times \Omega \times \mathbb{R}^d)$.

(i) If $q \geq 2$, we use (116) and we have

$$\begin{aligned} |A_k^h[\psi]| &\leq \frac{h}{2} \|\nabla_v^2 \psi\|_{L^\infty} \left[h \int_{\Omega \times \mathbb{R}^d} \|\nabla_v(G'(f_k^h))\|^p f_k^h dx dv + h \right] \\ &\quad + \frac{h^2}{2} \|\nabla_x^2 \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)} \|f_0\|_{L^\infty(\Omega \times \mathbb{R}^d)} \int_{\Omega \times K_\psi} |v|^2 dx dv. \end{aligned} \quad (212)$$

Recalling the fact that $c(x) \geq A_1 |x|^q$, we have

$$h \int_{\Omega \times \mathbb{R}^d} \|\nabla_v(G'(f_k^h))\|^p f_k^h dx dv \geq \frac{h}{A_1} \int_{\Omega} \rho_k^h(x) W_c^h(F_k^{xh}, G_k^{xh}) dx. \quad (213)$$

From the statements in Section 2.1, we get

$$\int_{\Omega} \rho_k^h(x) W_c^h(F_k^{xh}, G_k^{xh}) dx \leq \frac{1}{A_1} \left[\int_{\Omega \times \mathbb{R}^d} G(f_{k-1}^h) - \int_{\Omega \times \mathbb{R}^d} G(f_k^h) \right] dx dv, \quad (214)$$

and accordingly,

$$\begin{aligned} |A_k^h[\psi]| &\leq \frac{h}{2A_1} \|\nabla_v^2 \psi\|_{L^\infty} \left[\int_{\Omega \times \mathbb{R}^d} G(f_{k-1}^h) dx dv - \int_{\Omega \times \mathbb{R}^d} G(f_k^h) dx dv + h \right] \\ &+ \frac{h^2}{2} \|\nabla_x^2 \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)} \|f_0\|_{L^\infty(\Omega \times \mathbb{R}^d)} \int_{\Omega \times K_\psi} |v|^2 dx dv. \end{aligned} \quad (215)$$

So

$$\begin{aligned} |B^h[\psi]| &\leq \frac{1}{2A_1} \|\nabla_v^2 \psi\|_{L^\infty} \left[\sum_{k=1}^{T/h} \int_{t_k}^{t_{k+1}} \left(\int_{\Omega \times \mathbb{R}^d} G(f_{k-1}^h) dx dv \right. \right. \\ &\quad \left. \left. - \int_{\Omega \times \mathbb{R}^d} G(f_k^h) dx dv \right) + hT \right] \\ &+ \frac{h}{2} T \|\nabla_x^2 \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)} \|f_0\|_{L^\infty(\Omega \times \mathbb{R}^d)} \int_{\Omega \times K_\psi} |v|^2 dx dv, \end{aligned} \quad (216)$$

and then,

$$\begin{aligned} |B^h[\psi]| &\leq \frac{1}{2} \|\nabla_v^2 \psi\|_{L^\infty} \left[h \left(\int_{\Omega \times \mathbb{R}^d} G(f_0) dx dv \right. \right. \\ &\quad \left. \left. - \int_{\Omega \times \mathbb{R}^d} G(f_{T/h}^h) dx dv \right) + hT \right] \\ &+ \frac{h}{2} T \|\nabla_x^2 \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)} \|f_0\|_{L^\infty(\Omega \times \mathbb{R}^d)} \int_{\Omega \times K_\psi} |v|^2 dx dv. \end{aligned} \quad (217)$$

From the proof of Lemma 6, we have

$$\int_{\Omega \times \mathbb{R}^d} G(f_0) dx dv - \int_{\Omega \times \mathbb{R}^d} G(f_{T/h}^h) dx dv \leq G'(M) - G'(N). \quad (218)$$

Then, we obtain

$$\begin{aligned} |B^h[\psi]| &\leq \frac{1}{2} \|\nabla_v^2 \psi\|_{L^\infty} [h(G'(M) - G'(N)) + hT] \\ &+ \frac{h}{2} T \|\nabla_x^2 \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)} \|f_0\|_{L^\infty(\Omega \times \mathbb{R}^d)} \int_{\Omega \times K_\psi} |v|^2 dx dv. \end{aligned} \quad (219)$$

So, $B^h[\psi]$ tends to 0 as $h \rightarrow 0$.

(ii) If $q < 2$, we have

$$\int_{\Omega \times \mathbb{R}^d} \left(h |\nabla_v G'(f_k^h)|^p \right)^{2/q} f_k^h dx dv \leq \int_{\Omega \times \mathbb{R}^d} h |\nabla_v G'(f_k^h)|^p f_k^h dx dv, \quad (220)$$

and then,

$$\begin{aligned} |A_k^h[\psi]| &\leq 2 \frac{h^{2-(2/q)}}{2} \|\nabla_v^2 \psi\|_{L^\infty} \int_{\Omega \times \mathbb{R}^d} h |\nabla_v G'(f_k^h)|^p f_k^h dx dv \\ &+ \frac{h^2}{2} \|\nabla_x^2 \psi\|_{L^\infty} \|f_0\|_{L^\infty} \int_{\Omega \times K_\psi} |v|^2 dx dv. \end{aligned} \quad (221)$$

From the statements in Section 2.1, we obtain

$$\begin{aligned} |A_k^h[\psi]| &\leq \frac{h^{2-2/q}}{2} \|\nabla_v^2 \psi\|_{L^\infty} \left[\int_{\Omega \times \mathbb{R}^d} G(f_{k-1}^h) dx dv \right. \\ &\quad \left. - \int_{\Omega \times \mathbb{R}^d} G(f_k^h) dx dv \right] \\ &+ \frac{h^2}{2} \|\nabla_x^2 \psi\|_{L^\infty} \|f_0\|_{L^\infty} \int_{\Omega \times K_\psi} |v|^2 dx dv, \end{aligned} \quad (222)$$

and then,

$$\begin{aligned} |B^h[\psi]| &\leq \frac{h^{2-(2/q)}}{2} \|\nabla_v^2 \psi\|_{L^\infty} [G'(M) - G'(N)] \\ &+ \frac{h}{2} T \|\nabla_x^2 \psi\|_{L^\infty} \|f_0\|_{L^\infty} \int_{\Omega \times K_\psi} |v|^2 dx dv. \end{aligned} \quad (223)$$

Thus, $B^h[\psi]$ tends to 0 as $h \rightarrow 0$.

By the change of variable $t = t + h$, we have

$$\begin{aligned} &\int_{[0,T] \times \Omega \times B_R} f^h \left[\frac{G'(f(t+h))u(t+h) - G'(f(t))u(t)}{h} \right] dt dx dv \\ &= X_1^h + X_2^h + X_3^h, \end{aligned} \quad (224)$$

where

$$\begin{aligned}
X_1^h &= \int_{[0,T] \times \Omega \times B_R} f^h(t-h) \left[\frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv, \\
X_2^h &= \int_{]-h,0[\times \Omega \times B_R} f^h(t-h) \left[\frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv, \\
X_3^h &= - \int_{]T-h,T[\times \Omega \times B_R} f^h(t-h) \left[\frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv.
\end{aligned} \tag{225}$$

Recalling the definition of f^h , we get

$$X_2^h = \int_{]-h,0[\times \Omega \times B_R} f_0 G'(f_0) \left[\frac{u(t) - u(t-h)}{h} \right] dt dx dv. \tag{226}$$

Thus,

$$\lim_{h \rightarrow 0} X_2^h = 0. \tag{227}$$

Besides,

and since $\text{supp } p(u) \subset [-T, T]$,

$$\lim_{h \rightarrow 0} |X_3^h| \leq M |G'(M)| [u(T) + u(T)] = 0. \tag{229}$$

We also have

$$\begin{aligned}
X_1^h &= \int_{[0,T] \times \Omega \times B_R} (f^h(t-h) - f) \left[\frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv \\
&+ \int_{[0,T] \times \Omega \times B_R} f \left[\frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv.
\end{aligned} \tag{230}$$

But

$$\left| \int_{[0,T] \times \Omega \times B_R} (f^h(t-h) - f) \left[\frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv \right| \leq \tag{231}$$

$$\begin{aligned}
&\|f^h(t-h) - f\|_{L^\infty} \int_{[0,T] \times \Omega \times B_R} \left| \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right|^p dt dx dv, \\
&\lim_{h \rightarrow 0} \|f^h(t-h) - f\|_{L^\infty} = 0,
\end{aligned} \tag{232}$$

and then by the dominated convergence theorem, we have

and using (232) and (233), we obtain that

$$\begin{aligned}
&\lim_{h \rightarrow 0} \int_{[0,T] \times \Omega \times B_R} \left| \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right|^p dt dx dv \\
&= \int_{[0,T] \times \Omega \times B_R} |\partial_t (G'(f(t))u(t))|^p < \infty,
\end{aligned} \tag{233}$$

$$\lim_{h \rightarrow 0} \int_{[0,T] \times \Omega \times B_R} (f^h(t-h) - f) \left[\frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv = 0. \tag{234}$$

Next, the equality

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{[0,T] \times \Omega \times B_R} f^h \left[\frac{G'(f(t+h))u(t+h) - G'(f(t))u(t)}{h} \right] dt dx dv \\ &= \lim_{h \rightarrow 0} \int_{[0,T] \times \Omega \times B_R} f \left[\frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv, \end{aligned} \quad (235)$$

follows by recalling (229), (227), and (234), and accordingly,

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_0^T \int_{\Omega \times B_R} f \left[\frac{G'(f)u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv \\ &+ \int_{\Omega \times B_R} f_0 G'(f_0)u(0) dx dv \\ &= \int_0^T \int_{\Omega \times B_R} \langle f\sigma, \nabla_v G'(f) \rangle u(t) dt dx dv. \end{aligned} \quad (236)$$

We notice that

$$\begin{aligned} & f \left[\frac{G'(f)u(t) - G'(f(t-h))u(t-h)}{h} \right] \\ &= f G'(f) \partial_t^{-h} u(t) + \frac{1}{h} f u(t-h) [G'(f) - G'(f(t-h))], \end{aligned} \quad (237)$$

where $\partial_t^{-h} u(t) = u(t) - u(t-h)/h$, and since $G \in C^1$ is convex, then

$$G^*(G'(f)) - G^*(G'(f(t-h))) \leq f(G'(f) - G'(f(t-h))). \quad (238)$$

Accordingly,

$$\begin{aligned} & f \left[\frac{G'(f)u(t) - G'(f(t-h))u(t-h)}{h} \right] \\ &\geq f G'(f) \partial_t^{-h} u(t) + \frac{1}{h} u(t-h) \end{aligned} \quad (239)$$

$$[G^*(G'(f)) - G^*(G'(f(t-h)))],$$

and after integration, we get

$$\begin{aligned} & \int_0^T \int_{\Omega \times B_R} f \left[\frac{G'(f)u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv \\ &\geq \int_0^T \int_{\Omega \times B_R} f G'(f) \partial_t^{-h} u(t) dt dx dv \\ &+ \frac{1}{h} \int_0^T \int_{\Omega \times B_R} u(t-h) [G^*(G'(f)) \\ &- G^*(G'(f(t-h)))] dt dx dv. \end{aligned} \quad (240)$$

Since $\text{supp}(u) \subset [-T, T]$, $u(t) = 0$ for $t-h \in [T-h, T]$, and then, we obtain

$$\begin{aligned} & \int_0^T \int_{\Omega \times B_R} f \left[\frac{G'(f)u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv \geq \\ & \int_0^T \int_{\Omega \times B_R} [f G'(f) - G^*(G'(f))] \partial_t^{-h} u(t) dt dx dv + \\ & - \frac{1}{h} \int_0^T \int_{\Omega \times B_R} G^*(G'(f(t-h))) u(t-h) dt dx dv. \end{aligned} \quad (241)$$

Tending h to 0 in the relation above enables us to have

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_0^T \int_{\Omega \times B_R} f \left[\frac{G'(f)u(t) - G'(f(t-h))u(t-h)}{h} \right] dt dx dv \\ &\geq \int_0^T \int_{\Omega \times B_R} [f G'(f) - G^*(G'(f))] u'(t) dt dx dv \\ &- \int_{\Omega \times B_R} G^*(G'(f_0)) u(0) dx dv. \end{aligned} \quad (242)$$

Moreover, expressions (236) and (242) give

$$\begin{aligned} & \int_{\Omega \times B_R} [f_0 G'(f_0) - G^*(G'(f_0))] u(0) dx dv \\ &+ \int_{[0,T] \times \Omega \times B_R} [f G'(f) - G^*(G'(f))] u'(t) dt dx dv \\ &\leq \int_{[0,T] \times \Omega \times B_R} \langle f\sigma, \nabla_v G'(f) \rangle u(t) dt dx dv. \end{aligned} \quad (243)$$

So, tending R to ∞ in (243), we obtain Lemma 15.

Now, we derive (185) by combining Lemmas 13–15.

So, let us show that the sequence $(\text{div}_v(f^h \sigma^h))_h$ converges weakly to $\text{div}_v(f\sigma)$ in $[C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)]'$ and $\text{div}_v(f\sigma) = \text{div}_v(f \nabla c^*(\nabla_v G'(f)))$ in the weak sense.

So, fix $T > 0$ and let $\psi \in C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)$ such that $\text{supp} \psi(\cdot, x, v) \subset [-T, T]$.

We have

$$\lim_{h \rightarrow 0} \int_0^T \int_{\Omega \times \mathbb{R}^d} \langle f^h \nabla c^*(\nabla G'(f^h)) - f\sigma, \nabla_v \psi \rangle dt dx dv = 0. \quad (244)$$

Indeed,

$$\begin{aligned} & \int_0^T \int_{\Omega \times \mathbb{R}^d} \langle f^h \nabla c^*(\nabla_v G'(f^h)) - f\sigma, \nabla_v \psi \rangle dt dx dv \\ &= \int_0^T \int_{\Omega \times \mathbb{R}^d} f \langle \nabla c^*(\nabla G'(f^h)) - \sigma, \nabla_v \psi \rangle dt dx dv \\ &+ \int_0^T \int_{\Omega \times \mathbb{R}^d} (f^h - f) \langle \nabla c^*(\nabla_v G'(f^h)), \nabla_v \psi \rangle dt dx dv. \end{aligned} \quad (245)$$

We use (i), (iii), and the fact that $\nabla_v \psi \in L^p([0, T] \times \Omega \times \mathbb{R}^d)$ and $(\nabla c^*(\nabla_v G'(f^h)))_h$ is bounded in $L^q([0, T] \times \Omega \times \mathbb{R}^d)$, and we tend h to 0 in (245) to obtain (244).

Let us show that $\operatorname{div}_v(f\sigma) = \operatorname{div}_v(f\nabla c^*(\nabla_v G')(f))$ in the weak sense. So, let $\varepsilon > 0$ and $\psi \in C_c^\infty(\Omega \times \mathbb{R}^d)$ be a test function and $u \in C_c^\infty(\mathbb{R})$, $u \geq 0$, with $\operatorname{supp}(u) \subset [-T, T]$. Define $\omega_\varepsilon(t, x, v) = G'(f) - \varepsilon\psi$.

Since c^* is convex and $f^h \geq 0$, we have

$$\int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*((\nabla_v G')(f^h)) - \nabla c^*(\nabla \omega_\varepsilon), \nabla_v G'(f^h) - \nabla \omega_\varepsilon \rangle u(t) \geq 0. \quad (246)$$

Splitting (246) as follows,

$$\begin{aligned} & \int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*((\nabla_v G')(f^h)) - \nabla c^*(\nabla \omega_\varepsilon), \nabla_v G'(f^h) - \nabla \omega_\varepsilon \rangle u(t) \\ &= \int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*((\nabla_v G')(f^h)) - \nabla c^*(\nabla \omega_\varepsilon), \nabla_v G'(f^h) - \nabla_v G'(f) \rangle u(t) \\ &+ \int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*((\nabla_v G')(f^h)) - \nabla c^*(\nabla \omega_\varepsilon), \nabla_v G'(f) - \nabla \omega_\varepsilon \rangle u(t), \end{aligned} \quad (247)$$

we derive from (185) that

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*((\nabla_v G')(f^h)) \\ & - \nabla c^*(\nabla \omega_\varepsilon), \nabla_v G'(f^h) - \nabla_v G'(f) \rangle u(t) = 0. \end{aligned} \quad (248)$$

Consequently,

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*((\nabla_v G')(f^h)) - \nabla c^*(\nabla \omega_\varepsilon), \nabla_v G'(f^h) - \nabla \omega_\varepsilon \rangle u(t) \\ &= \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f\sigma - f\nabla c^*(\nabla \omega_\varepsilon), \nabla_v G'(f) - \nabla \omega_\varepsilon \rangle u(t) \geq 0. \end{aligned} \quad (249)$$

We divide (249) by ε , and we tend ε to 0 and have

$$\int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f\sigma - f\nabla c^*(\nabla_v G'(f)), \nabla_v \psi u(t) \rangle \leq 0 \quad (250)$$

Replacing ψ by $-\psi$ in the previous relationship, we obtain

$$\int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f\sigma - f\nabla c^*(\nabla_v G'(f)), \nabla_v \psi \rangle u(t) = 0. \quad (251)$$

We conclude that $(\operatorname{div}_v(f^h \nabla c^*(\nabla_v G'(f^h))))$ converges weakly to $\operatorname{div}_v(f \nabla c^*(\nabla_v G'(f)))$ in $[C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)]'$, and then, the proof of Theorem 2 is complete. $\square$

4.5. Existence of Solutions. Now, we combine the weak convergence of the sequence $(f^h)_h$ and that of the nonlinear term $\operatorname{div}_v(f^h \nabla c^*(\nabla_v G'(f^h)))$ to prove that the kinetic equation (1) admits a weak solution f in the sense of Definition 1.

Theorem 3. Let f_0 be a density of probability on $\Omega \times \mathbb{R}^d$ such that $f_0 + (1/f_0) \in L^\infty(\Omega \times \mathbb{R}^d)$, $\int_{\Omega \times \mathbb{R}^d} |v|^q f_0 dx dv < \infty$, and $\int_{\Omega \times \mathbb{R}^d} G(f_0) dx dv < \infty$ with G and c satisfying, respectively, H_G and H_c .

Then, for any test function $\psi \in C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)$ such that $\operatorname{supp} \psi(\cdot, x, v) \subset [-T, T]$, we have

$$\begin{aligned} & \int_{[0,T] \times \Omega \times \mathbb{R}^d} f(\partial_t \psi + v \cdot \nabla_x \psi - \nabla_v \psi \cdot \nabla c^*(\nabla_v G'(f))) dt dx dv \\ &= - \int_{\Omega \times \mathbb{R}^d} f_0 \psi(0) dx dv, \end{aligned} \quad (252)$$

where f is the limit function obtained in Proposition 3.

Proof. Fix $T > 0$ and let $\psi \in C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)$ be a test function such that $\operatorname{supp} \psi(\cdot, x, v) \subset [-T, T]$. Integrating (17) over $[0, T] \times \Omega \times \mathbb{R}^d$, we obtain

$$\begin{aligned} & \sum_{k=1}^{T/h} \int_{t_{k-1}}^{t_k} \int_{\Omega \times \mathbb{R}^d} \frac{f_k^h - f_{k-1}^h}{h} \psi(t, x + (t - t_{k-1})v, v) dt dx dv \\ &+ \sum_{k=1}^{T/h} \int_{t_{k-1}}^{t_k} \int_{\Omega \times \mathbb{R}^d} f_{k-1}^h v \cdot \nabla_x \psi(t, x + (t - t_{k-1})v, v) dt dx dv \\ &+ \sum_{k=1}^{T/h} \int_{t_{k-1}}^{t_k} \int_{\Omega \times \mathbb{R}^d} f_k^h \nabla_v \psi(t, x + (t - t_{k-1})v, v) \cdot \nabla c^*(\nabla_v G'(f_k^h)) = B^h[\psi], \end{aligned} \quad (253)$$

where $B^h[\psi]$ tends to 0 as $h \rightarrow 0$.

We observe that

$$\begin{aligned}
& \sum_{k=1}^{T/h} \int_{t_{k-1}}^{t_k} \int_{\Omega \times \mathbb{R}^d} \frac{f_k^h - f_{k-1}^h}{h} \psi(t, x + (t - t_{k-1})v, v) dt dx dv \\
&= \int_{[0, T] \times \Omega \times \mathbb{R}^d} f^h \left[\frac{\psi(t - h, x, v) - \psi(t, x, v)}{h} \right] dt dx dv \\
&\quad - \frac{1}{h} \int_0^h \int_{\Omega \times \mathbb{R}^d} f_0(x - tv, v) \psi(t - h, x, v) dt dx dv,
\end{aligned} \tag{254}$$

$$\begin{aligned}
& \sum_{k=1}^{T/h} \int_{t_{k-1}}^{t_k} \int_{\Omega \times \mathbb{R}^d} f_{k-1}^h v \cdot \nabla_x \psi(t, x + (t - t_{k-1})v, v) dt dx dv \\
&= \int_{[0, T] \times \Omega \times \mathbb{R}^d} f^h v \cdot \nabla_x \psi dt dx dv.
\end{aligned} \tag{255}$$

Hence,

$$\begin{aligned}
& \sum_{k=1}^{T/h} \int_{t_{k-1}}^{t_k} \int_{\Omega \times \mathbb{R}^d} f_k^h \nabla_v \psi(t, x + (t - t_{k-1})v, v) \\
&\quad \cdot \nabla_c^* (\nabla_v G'(f_k^h)) dt dx dv \\
&= \int_{[0, T] \times \Omega \times \mathbb{R}^d} f^h \nabla_v \psi \cdot \nabla_c^* (\nabla_v G'(f^h)) dt dx dv \\
&\quad - \int_{[0, h] \times \Omega \times \mathbb{R}^d} f_0(x - tv, v) \nabla_v \psi(t, x, v) \\
&\quad \cdot \nabla_c^* (\nabla_v G'(f_0(x - tv, v))) dt dx dv.
\end{aligned} \tag{256}$$

We combine (254), (255), and (256) to obtain

$$\begin{aligned}
B^h[\psi] &= - \int_{[0, T] \times \Omega \times \mathbb{R}^d} f^h \partial_t^{-h} \psi(t, x, v) dt dx dv \\
&\quad - \frac{1}{h} \int_0^h \int_{\Omega \times \mathbb{R}^d} f_0(x - tv, v) \psi(t - h, x, v) dt dx dv \\
&\quad + \int_{[0, T] \times \Omega \times \mathbb{R}^d} f^h \nabla_v \psi \cdot \nabla_c^* (\nabla_v G'(f^h)) dt dx dv \\
&\quad - \int_{[0, T] \times \Omega \times \mathbb{R}^d} f^h v \cdot \nabla_x \psi dt dx dv \\
&\quad - \int_{[0, h] \times \Omega \times \mathbb{R}^d} f_0(x - tv, v) \nabla_v \psi(t, x, v) \\
&\quad \cdot \nabla_c^* (\nabla_v G'(f_0(x - tv, v))) dt dx dv.
\end{aligned} \tag{257}$$

Here,

$$\partial_t^{-h} \psi(t, x, v) = \frac{\psi(t, x, v) - \psi(t - h, x, v)}{h}. \tag{258}$$

Finally, we use the weak convergence of $(f^h)_h$ to f , the weak convergence of $f^h \nabla_c^* (\nabla_v G'(f^h))$ to $f \nabla_c^* (\nabla_v G'(f))$, and the uniform convergence of $\partial_t^h \psi(\cdot, x, v)$ to $\partial_t \psi(\cdot, x, v)$ on the compact set $[0, T]$ when h tend to 0, to derive the following equality:

$$\begin{aligned}
& - \int_{[0, T] \times \Omega \times \mathbb{R}^d} f \partial_t \psi dt dx dv - \int_{\Omega \times \mathbb{R}^d} f_0 \psi(0) dx dv \\
& - \int_{[0, T] \times \Omega \times \mathbb{R}^d} f v \cdot \nabla_x \psi dt dx dv + \int_{[0, T] \times \Omega \times \mathbb{R}^d} \nabla_v \psi \\
& \quad \cdot \nabla_c^* (\nabla_v G'(f)) f dt dx dv = 0,
\end{aligned} \tag{259}$$

i.e.,

$$\begin{aligned}
& \int_{[0, T] \times \Omega \times \mathbb{R}^d} [\partial_t \psi + v \cdot \nabla_x \psi - \nabla_v \psi \cdot \nabla_c^* (\nabla_v G'(f))] f dt dx dv \\
&= - \int_{\Omega \times \mathbb{R}^d} f_0 \psi(0) dx dv.
\end{aligned} \tag{260}$$

In conclusion, we prove that f solves (1) in the weak sense. $\square$

5. Numerical Implementation

In this section, we use the iterative operator-splitting methods as in [10] to construct an approximate weak solution of the kinetic equation (1) as stated in Section 2.1, and we establish the rate of convergence of our algorithm to the exact solution of the kinetic equation. The scheme consisting after fixing a time step $h > 0$ and denoting $t_k = hk$, with $k \in \mathbb{N}^*$, in defining the approximate solution $u^i = u^i(t, x, v)$, $(t, x, v) \in [t_k, t_{k+1}) \times \Omega \times \mathbb{R}^d$ of (1) is as follows:

$$\begin{cases} \frac{\partial u^i}{\partial t} + v \cdot \nabla_x u^i = \operatorname{div}_v \{ u^{i-1} \nabla_c^* [\nabla_v (G'(u^{i-1}))] \}, & \text{in } [t_k, t_{k+1}) \times \Omega \times \mathbb{R}^d, \\ \frac{\partial u^{i+1}}{\partial t} + v \cdot \nabla_x u^i = \operatorname{div}_v \{ u^{i+1} \nabla_c^* [\nabla_v (G'(u^{i+1}))] \}, & \text{in } [t_k, t_{k+1}) \times \Omega \times \mathbb{R}^d, \end{cases} \tag{261}$$

where $i = 1; 3; 5; \dots; 2j+1$, with $j \in \mathbb{N}^*$ and $u^i(t_k, x, v) = f_k^h(x, v)$, $u^{i+1}(t_k, x, v) = f_k^h(x, v)$; f_k^h denotes the approximate solution of (1) at $t = t_k$ defined in Section 2.1.

Remark 2. By fixing u^{i-1} , we obtain

$$\begin{aligned} u^i(t, x, v) &= f_k^h(x - (t - t_k)v, v) \\ &+ \int_{t_k}^t \operatorname{div}_v \left\{ u^{i-1}(s, x - (s - t_k)v, v) \nabla c^* \right. \\ &\quad \left. (\nabla_v G'(u^{i-1}(s, x - (s - t_k)v, v))) \right\} ds. \end{aligned} \quad (262)$$

By fixing u^i , we obtain u^{i+1} as follows:

$$u^{i+1}(t, x, v) = \chi^{i+1}(t, x) U^{x, i+1}(t, v), \quad (263)$$

where

$$\begin{aligned} \chi^{i+1}(t, x) &= \int_{\mathbb{R}^d} u^i(t - h, x - hv, v) dv, \\ U^{x, i+1} &\in \operatorname{Argmin} \left\{ \frac{1}{\chi^{i+1}(t, x)} \int_{\mathbb{R}^d} \right. \\ &\quad \left. G(\chi^{i+1}(t, x) U) dv + h W_c^h(U, V^{x, i+1}) \right\}, \end{aligned} \quad (264)$$

with

$$V^{x, i+1}(t, v) = \frac{u^i(t - h, x - hv, v)}{\chi^{i+1}(t, x)}. \quad (265)$$

Note that, for $t = t_k$ and for $i = 1; 3; \dots; 2j+1$, $j \in \mathbb{N}^*$, we obtain $u^i(t_k, x, v) = f_k^h(x, v)$, $u^{i+1}(t_k, x, v) = f_k^h(x, v)$, $\chi^{i+1}(t_k, x) = \rho_k^h(x)$, $V^{x, i+1}(t_k, v) = G_k^{xh}(v)$, and $U^{x, i+1}(t_k, v) = F_k^{xh}(v)$, where ρ_k^h , G_k^{xh} , F_k^{xh} , and f_k^h are defined as in the descent algorithm (see Section 2.1).

5.1. Error Analysis. An estimate of the rate of convergence of u^i to the exact solution of (1) is given as follows.

Theorem 4. Assume that f_0 , c , and G satisfy, respectively, (H_{f_0}) , (H_c) , and (H_G) , and $\operatorname{supp} f_0(x, \cdot) \subset B_R$ for all $x \in \Omega$; $\nabla c^*(0) = 0$. Then, we obtain the following error estimation:

$$\|e^j(t)\|_{\infty, L^q(\Omega \times B_R)} \leq (Kh)^{j-1} \|e^0\|_{\infty, L^q(\Omega \times B_R)}, \quad (266)$$

where $e^j(t, x, v) = f(t, x, v) - u^j(t, x, v)$, $j \in \mathbb{N}^*$ is the error between the exact solution f and the approximate solution u^j and $q = p/(p-1) \leq p$ and $e^0 = f - f_k^h$, and K is a constant depending on N , M , R , and G' .

Proof. We use the error function $e^i = f - u^i$ and the linearized equation of (1) via iterative splitting methods. Then, we have for $i = 1; 3; \dots; 2j+1$:

$$\begin{aligned} \frac{\partial e^i}{\partial t} + (v \cdot \nabla_x e^i) &= \operatorname{div}_v(u^{i-1} \nabla c^*(\nabla_v G'(u^{i-1}))) e^{i-1} \\ &+ \operatorname{div}_v(u^{i-1} D^2 c^*(\nabla_v G'(u^{i-1})) \circ D_v^2 G'(u^{i-1})) e^{i-1} + o(\|e^{i-1}\|). \end{aligned} \quad (267)$$

Since $\operatorname{supp} f_0(x, \cdot) \subset B_R$ and $N \leq f_0 \leq M$, then Lemma 16 gives that $\operatorname{supp} u^i(t, x, \cdot) \subset B_R$ and $D^2 c^*(\nabla_v G'(u^{i-1})) \circ D_v^2 G'(u^{i-1}) \in L^\infty([t_k, t_{k+1}] \times \Omega \times B_R)$ and

$$\|D^2 c^*(\nabla_v G'(u^{i-1})) \circ D_v^2 G'(u^{i-1})\|_{L^\infty([t_k, t_{k+1}] \times \Omega \times B_R)} \leq 1 + \frac{M}{N}. \quad (268)$$

We conclude that $\operatorname{div}_v(u^{i-1} \nabla c^*(\nabla_v G'(u^{i-1}))) \in L^q(\Omega \times B_R)$ and $\operatorname{div}_v(u^{i-1} D^2 c^*(\nabla_v G'(u^{i-1})) \circ D_v^2 G'(u^{i-1})) \in L^q(\Omega \times B_R)$.

Thus,

$$\|\operatorname{div}_v(u^{i-1} \nabla c^*(\nabla_v G'(u^{i-1})))\|_{\infty, L^q(\Omega \times B_R)} \leq K_1, \quad (269)$$

$$\|\operatorname{div}_v(u^{i-1} D^2 c^*(\nabla_v G'(u^{i-1})) \circ D_v^2 G'(u^{i-1}))\|_{\infty, L^q(\Omega \times B_R)} \leq K_2, \quad (270)$$

$$\|\operatorname{div}_v(u^{i-1} D^2 c^*(\nabla_v G'(u^{i-1})) \circ D_v^2 G'(u^{i-1}))\|_{\infty, L^q(\Omega \times B_R)} \leq K_3, \quad (271)$$

where K_1 , K_2 , and K_3 are constant depending of N , M , R , and G' . By using (267), we have

$$\begin{aligned} e^i(t, x + tv, v) &= \int_{t_k}^t \operatorname{div}_v(u^{i-1} \nabla c^*(\nabla_v G'(u^{i-1}))) e^{i-1} ds \\ &+ \int_{t_k}^t \operatorname{div}_v(u^{i-1} D^2 c^*(\nabla_v G'(u^{i-1})) \circ D_v^2 G'(u^{i-1})) e^{i-1} ds \\ &+ \int_{t_k}^t o(\|e^{i-1}\|) ds. \end{aligned} \quad (272)$$

We now use (270) and (271) to obtain

$$\|e^i\|_{\infty, L^q(\Omega \times B_R)} \leq K_4 h \|e^{i-1}\|_{\infty, L^q(\Omega \times B_R)}, \quad (273)$$

with $K_4 = K_2 + K_3 + 1$. As in (272), we have

$$\begin{aligned} e^{i+1}(t, x, v) &= - \int_{t_k}^t (v \cdot \nabla_x u^i) e_i ds + \int_{t_k}^t \operatorname{div}_v(u^{i+1} \nabla c^*(\nabla_v G'(u^{i+1}))) e^{i+1} ds \\ &+ \int_{t_k}^t \operatorname{div}_v(u^{i+1} D^2 c^*(\nabla_v G'(u^{i+1})) \circ D_v^2 G'(u^{i+1})) e^{i+1} ds + \int_{t_k}^t o(\|e^{i+1}\|) ds. \end{aligned} \quad (274)$$

Next, from (269), (270), and (271), we derive

$$\|e^{i+1}\|_{\infty, L^q(\Omega \times B_R)} \leq h^2 K_1 K_4 \|e^{i-1}\|_{\infty, L^q(\Omega \times B_R)} + K_4 h \|e^{i+1}\|_{\infty, L^q(\Omega \times B_R)}. \quad (275)$$

Accordingly,

$$\|e^{i+1}\|_{\infty, L^q(\Omega \times B_R)} \leq Kh \|e^{i-1}\|_{\infty, L^q(\Omega \times B_R)}, \quad (276)$$

where K depends on K_4 and K_1 . Finally, we use (273) and (276) to conclude that

$$\|e^j\|_{\infty, L^q(\Omega \times B_R)} \leq (Kh)^{j-1} \|e^0\|_{\infty, L^q(\Omega \times B_R)}, \quad (277)$$

for $j \in \mathbb{N}^*$ □

5.2. Numerical Example. In this section, we solve (1), when $d = 1$, $G(t) = t \ln t$, and $c^*(t) = t^2/2$, i.e., the following Boltzmann equation:

$$\frac{\partial f}{\partial t} + v \cdot \partial_x f = \partial_v^2 f, \quad f = f(t, x, v), \quad (t, x, v) \in [0, \infty) \times \Omega \times (0, \infty). \quad (278)$$

We suppose $\Omega := (0, 1)/\mathbb{Z} = \{x = \bar{x} + n, \bar{x} \in (0, 1), n \text{ browsing } \mathbb{Z}\}$.

The initial datum $f(0, x, v) = f_0(x, v)$ is a probability density on $\Omega \times (0, \infty)$.

Let us show that the function f defined by

$$\begin{cases} f(t, x, v) = \bar{x} + \frac{v^3}{6} + \frac{11}{12\sqrt{\pi(4t+1)}} e^{-(v^2/4t+1)}, & \text{if } (t, \bar{x}, v) \in [0, +\infty) \times (0, 1) \times (0, 1), \\ f(t, x, v) = \frac{11}{12\sqrt{\pi(4t+1)}} e^{-(v^2/4t+1)}, & \text{if } (t, \bar{x}, v) \in [0, +\infty) \times (0, 1) \times [1, +\infty), \end{cases} \quad (279)$$

is one solution of the kinetic equation (278). Indeed,

$$\frac{\partial f(t, x, v)}{\partial t} + v \cdot \partial_x f = \partial_v^2 f, \quad \text{in } [0, +\infty) \times \Omega \times (0, +\infty), \quad (280)$$

with the initial datum

$$\begin{cases} f_0(x, v) = \bar{x} + \frac{v^3}{6} + \frac{11}{12\sqrt{\pi}} e^{-v^2}, & \text{if } (\bar{x}, v) \in (0, 1) \times (0, 1), \\ f_0(x, v) = \frac{11}{12\sqrt{\pi}} e^{-v^2}, & \text{if } (x, v) \in (0, 1) \times [1, +\infty). \end{cases} \quad (281)$$

Moreover,

$$\begin{aligned} \frac{\partial f}{\partial t} &= \left[\frac{11v^2}{3(4t+1)^2 \sqrt{\pi(4t+1)}} - \frac{11}{6(4t+1) \sqrt{\pi(4t+1)}} \right] e^{-(v^2/4t+1)}, \\ v \cdot \partial_x f &= \begin{cases} v, & \text{if } (t, \bar{x}, v) \in [0, +\infty) \times (0, 1) \times (0, 1), \\ 0, & \text{if } (t, \bar{x}, v) \in [0, +\infty) \times (0, 1) \times [1, +\infty), \end{cases} \\ \partial_v^2 f &= \begin{cases} v + \left[\frac{11v^2}{3(4t+1)^2 \sqrt{\pi(4t+1)}} - \frac{11}{6(4t+1) \sqrt{\pi(4t+1)}} \right] e^{-(v^2/4t+1)}, & \text{if } (t, \bar{x}, v) \in [0, +\infty) \times (0, 1) \times (0, 1), \\ \left[\frac{11v^2}{3(4t+1)^2 \sqrt{\pi(4t+1)}} - \frac{11}{6(4t+1) \sqrt{\pi(4t+1)}} \right] e^{-(v^2/4t+1)}, & \text{if } (t, \bar{x}, v) \in [0, +\infty) \times (0, 1) \times [1, +\infty). \end{cases} \end{aligned} \quad (282)$$

FIGURE 1: Numerical solution $f_1^{1/10}(x, v)$ for $h = 1/10$.

Hence, f satisfies the kinetic equation (280).

Suppose $t \in [0, +\infty[$ fixed, one has

$$\begin{aligned} \int_{\Omega \times (0, +\infty)} f(t, x, v) dx dv &= \int_{(0,1) \times (0,1)} \left[\bar{x} + \frac{v^3}{6} + \frac{11}{12\sqrt{\pi(4t+1)}} e^{-(v^2/4t+1)} \right] d\bar{x} dv \\ &+ \int_{(0,1) \times [1, +\infty)} \left[\frac{11}{12\sqrt{\pi(4t+1)}} e^{-(v^2/4t+1)} \right] d\bar{x} dv \\ &= \int_{(0,1)} \bar{x} d\bar{x} + \int_{(0,1)} \frac{v^3}{6} dv + \int_{(0,+\infty)} \left[\frac{11}{12\sqrt{\pi(4t+1)}} e^{-(v^2/4t+1)} \right] dv, \end{aligned} \quad (283)$$

with

$$\begin{aligned} \int_{(0,1)} \bar{x} d\bar{x} &= \frac{1}{2}, \\ \int_{(0,1)} \frac{v^3}{6} dv &= \frac{1}{24}, \\ \int_{(0,+\infty)} \left[\frac{11}{12\sqrt{\pi(4t+1)}} e^{-(v^2/4t+1)} \right] dv &= \frac{11}{24}. \end{aligned} \quad (284)$$

Consequently, $\int_{\Omega \times (0, +\infty)} f(t, x, v) dx dv = 1$ for all $t \geq 0$.

5.3. The Figures. Here, we perform the first-order approximation f_1 of the solution of (278) using the algorithm in (261). We limit our self to the first-order approximation just for the sake of simplicity and to avoid very complex and cumbersome calculations. Numerical computation carried out for $h \in \{1/10, 1/50, 1/100\}$ and the graphical representations of both the analytical solution f and the approximate solution f_1 with Scilab software shows a coherence between the approximate solution and the analytical solution.

Moreover, an error tables obtained for some values of the time t in the interval $[h, 2h]$ for different values of the step h are presented below.

By a simple computation, the expression of f_1^h appears as follows:

$$\begin{cases} f_1^h(x, v) = \bar{x} + \frac{v^3}{6} + \frac{11hv^2}{3\sqrt{\pi}} e^{-v^2} + \frac{11(1-2h)}{12\sqrt{\pi}} e^{-v^2}, & \text{is } (\bar{x}, v) \in (0, 1) \times (0, 1), \\ f_1^h(x, v) = \frac{11hv^2}{3\sqrt{\pi}} e^{-v^2} + \frac{11(1-2h)}{12\sqrt{\pi}} e^{-v^2}, & \text{is } (\bar{x}, v) \in (0, 1) \times [1, +\infty). \end{cases} \quad (285)$$

For some values of h , we draw the figures of both the analytical solution and the numerical solution (approximate solution). We notice that Figure 1 represents an approximation of Figure 2 when $h = (1/10)$, Figure 3 represents an

approximation of Figure 4 when $h = (1/50)$, and Figure 5 is an approximation of Figure 6 when $h = (1/100)$.

Errors progression is in terms of the L^∞ and the L^2 norms between the numerical solution f_1^h and the analytical

FIGURE 2: Analytical solution $f(3/20, x, v)$ for $h = (1/10)$ and $t = 3/20 \in [1/10, 2/10]$.

FIGURE 3: Numerical solution $f_1^{1/50}(x, v)$ for $h = 1/50$.

FIGURE 4: Analytical solution $f(3/100, x, v)$ for $h = 1/50$ and $t = 3/100 \in [1/50, 1/25]$.

FIGURE 5: Numerical solution $f_1^{1/100}(x, v)$ for $h = 1/100$. FIGURE 6: Analytical solution $f(3/200, x, v)$ for $h = 1/100$ and $t = 3/200 \in [1/100, 1/50]$.

TABLE 1: Errors' progression.

h	1/10	1/50	1/100	1/500	1/1000
δt	3/20	3/100	3/200	3/1000	3/2000
$\ f - f_1^h\ _{L^2}$	0.03422791	0.03422791	0.002478	0.001420191	0.00027920
$\ f - f_1^h\ _{\max}$	0.0145619	0.078035	0.0045067	0.0010067	0.0008798

solution f . One can actually notice the convergence of our method in the L^2 and in L^∞ norms when h is decreasing to 0, which is explained as follows (Table 1).

Appendix

Here, we give the regularity result necessary in the proof of Theorem 4.

Lemma 16. Assume that f_0 , c , and G satisfy, respectively, (H_{f_0}) , (H_c) , and (H_G) , $\text{supp } f_0(x, \cdot) \subset B_R$, for all $x \in \Omega$ and $\nabla c^*(0) = 0$. Then, $\text{supp } f_k^h(x, \cdot) \subset B_R$, for all $x \in \Omega$ and for all $k \in \mathbb{N}^*$.

Moreover, $D^2 c^*(\nabla_v G'(f_k^h)) \circ D_v^2 G'(f_k^h) \in L^\infty(\Omega \times B_R)$ and

$$\|h D^2 c^*(\nabla_v G'(f_k^h)) \circ D_v^2 G'(f_k^h)\|_{L^\infty(\Omega \times B_R)} \leq 1 + \frac{M}{N}. \quad (\text{A.1})$$

Proof. Since $\text{supp } f_0(x, \cdot) \subset B_R$, then $f_0(x, B_R^c) = 0$, where B_R^c is the complement of B_R . In Section 2.1, we have $f_1(x, v) = \rho_1^h(x) F_1^x(v)$, where

$$\rho_1^h(x) = \int_{\mathbb{R}^d} f_0(x - hv, v) dv, \quad (\text{A.2})$$

and F_1^x is the unique solution of the variational problem.

$$(P_1^x): \inf_{F \in P_q(\mathbb{R}^d)} \left\{ \frac{1}{\rho_1^h(x)} \int_{\mathbb{R}^d} G(F \rho_1^h(x)) dv + h W_c^h(F, G_1^x) \right\}, \quad (\text{A.3})$$

with

$$G_1^x(v) = \frac{f_0(x - hv, v)}{\rho_1^h(x)}, \quad (\text{A.4})$$

$$W_c^h(F, G_1^x) = \inf_{T_{\#} F = G_1^x} \int_{\mathbb{R}^d} c\left(\frac{v - T(v)}{h}\right) F dv.$$

Since $f_0(x, B_R^c) = 0$, then $G_1^x(B_R^c) = 0$.

The c -optimal map T_1 that pushes F_1^x forward to G_1^x satisfies then

$$F_1^x(T_1^{-1}(B_R^c)) = G_1^x(B_R^c) = 0. \quad (\text{A.5})$$

We now use the explicit expression of T_1 (see Lemma 3):

$$T_1(v) = v + h \nabla c^*(\nabla_v G'(f_1)), \quad (\text{A.6})$$

along with (A.5) and $\nabla c^*(0) = 0$ to get

$$T_1(T_1^{-1}(B_R^c)) = T_1^{-1}(B_R^c). \quad (\text{A.7})$$

This implies that $T_1^{-1}(B_R^c) = B_R^c$, and then, $F_1^x(B_R^c) = 0$. Thus, we deduce that $\text{supp } F_1^x \subset B_R$.

Since $f_1(x, v) = \rho_1^h(x) F_1^x(v)$, we obtain $\text{supp } f_1(x, \cdot) \subset B_R$, for all $x \in \Omega$. Finally, we obtain by induction $\text{supp } f_k^h(x, \cdot) \subset B_R$, for all $x \in \Omega$.

Since F_k^{xh} and G_k^{xh} have compact support and that c is strictly convex, then the c -optimal maps T_k that pushes F_k^{xh} forward to G_k^{xh} is differentiable, and we have

$$\nabla_v T_k(v) = i d + h D^2 c^*(\nabla_v G'(f_k^h)) \circ D_v^2 G'(f_k^h). \quad (\text{A.8})$$

Hence, $\nabla_v T_k(v)$ satisfies the Jacobian equation:

$$\rho_k^h F_k^{xh} = \det(\nabla_v T_k(v)) \rho_k^h G_k^{xh}. \quad (\text{A.9})$$

Since $N \leq f_0 \leq M$, the maximum principle gives $N \leq f_k^h \leq M$. Then, we deduce that

$$\frac{N}{M} \leq \det(\nabla_v T_k(v)) \leq \frac{M}{N}. \quad (\text{A.10})$$

Note that $\nabla_v T_k$ is diagonalizable and has positive eigenvalues (see [1]). Then, we deduce that $\nabla_v T_k(v) \in L^\infty(\Omega \times B_R)$ and $\|\nabla_v T_k(v)\|_{L^\infty(\Omega \times B_R)} \leq (M/N)$. And by using (A.8), we obtain

$$\|h D^2 c^*(\nabla_v G'(f_k^h)) \circ D_v^2 G'(f_k^h)\|_{L^\infty(\Omega \times B_R)} \leq 1 + \frac{M}{N}. \quad (\text{A.11})$$

□

Data Availability

There are no data underlying the findings in this paper to be shared.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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