Uniqueness Problems about Entire Functions with Their Difference Operator Sharing Sets

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1.Introduction and Main Results

Before proceeding, we spare the reader for a moment and assume some familiarity with the basics of Nevanlinna theory of meromorphic functions in \( \mathbb{C} \) such as the first and second main theorems and the usual notations such as the characteristic function \( T(r, f) \), the proximity function \( m(r, f) \), and the counting function \( N(r, f) \). \( S(r, f) \) denotes any quantity satisfying \( S(r, f) = o(T(r, f)) \) as \( r \to \infty \), except possibly on a set of finite logarithmic measure not necessarily the same at each occurrence, see e.g., [1–3].

Let \( f \) be a meromorphic functions on \( \mathbb{C} \). Here, the order \( \rho (f) \) is defined by

\[
\rho (f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r},
\]

(1)

and the exponent of convergence of zeros \( \lambda (f) \) is defined by

\[
\lambda (f) = \limsup_{r \to \infty} \frac{\log N(r, (1/f))}{\log r}.
\]

(2)

For a given \( a \in \mathbb{C} = \mathbb{C} \cup \{ \infty \} \), we say that two meromorphic functions \( f \) and \( g \) share \( a \) CM (counting multiplicities) when \( f \) and \( g \) have the same \( a \)-points. Let \( S \) be a finite set of some entire functions and \( f \) an entire function. Then, a set \( E_f (S) \) is defined as

\[
E_f (S) = \bigcup_{a \in S} \{ z | f (z) = a, \text{counting multiplicities} \}.
\]

(3)

Assume that \( g \) is another entire function. We say that \( f \) and \( g \) share a set \( S \), counting multiplicities (CM), provided that \( E_f (S) = E_g (S) \).

The uniqueness theory of meromorphic functions sharing sets generalizes that on sharing values and generally is more difficult. If meromorphic functions share a general set, it is not easy to determine these functions. In 1999, Li and Yang [4] deduced that if \( E_f (S) = E_g (S) \) with \( S \) contain two distinct constants, then \( f \) must have special forms. Fang and Zalcman [5] used the theory of normal family to solve the above problem by proving that there exists a finite set \( \tilde{S} \) containing three distinct elements such that if \( E_f (S) = E_g (S) \), then \( f = f' \).

Recently years, Nevanlinna characteristic of \( f(z+c) \), the value distribution theory for difference analogue, Nevanlinna theory of the difference operator, and the difference analogue of the lemma on the logarithmic derivative had been built, see e.g., [1–4, 6–14]. For meromorphic functions \( f(z) \), we define its shift by \( f_c (z) = f(z+c) \) and its difference operators by...
\[
\Delta_c f = f(z + c) - f(z), \\
\Delta_c^n f = \Delta_{c}^{-1} \left( \Delta_c f (z) \right), \quad n \in \mathbb{N}, n \geq 2. 
\] (4)

By Nevanlinna theory of the difference operator, a natural question to ask whether the derivative \( f' \) can be replaced by the difference operator \( \Delta_c f(z) = f(z + c) - f(z) \) in the above question?

In 2009, Liu [8] investigated the above question and proved the following result.

**Theorem 1.** Let \( a \) be a nonzero complex number and \( f \) be a transcendental entire function with finite order. If \( f \) and \( \Delta_c f \) share \( \{a, -a\} \) CM, then \( \Delta_c f(z) = f(z) \) for all \( z \in C \).

In 2012, from Theorem 1, considering the constant in set is replaced by the function, Li [9] proved the following theorem.

**Theorem 2.** If \( a \) and \( b \) are two distinct entire functions, then \( f \) is a nonconstant entire function whose \( \rho(f) \neq 1 \) and \( \lambda(f) < \rho(f) \) are two arbitrary complex numbers such that \( \omega_1 \neq \omega_2 \). Assume that \( \sigma \) is the order of \( f \), then for each \( c > 0 \), we have

\[
m(r, \frac{f(z + \omega_1)}{f(z + \omega_2)}) = O(r^{\sigma - 1 + c}).
\] (5)

**Lemma 2** (see [16]). Let \( g \) be a function transcendental and meromorphic in the plane with order less than 1. Set \( h > 0 \). Then, there exists an \( \varepsilon \)-set \( E \) such that

\[
g(z + \omega) \rightarrow 1, \quad \text{when } z \rightarrow \infty \text{ in } C \setminus E, \tag{6}
\]

uniformly in \( \omega \) for \( |\omega| < h \).

**Lemma 3** (see [3]). Suppose that \( f_1(z), f_2(z), \ldots, f_n(z) \) are meromorphic functions and \( g_1(z), g_2(z), \ldots, g_n(z) \) are entire functions satisfying the following conditions:

1. \( \sum_{j=1}^{n} f_j(z)e^{\alpha_j(z)} = 0 \).
2. \( g_1(z) - g_2(z) \) are not constants for \( 1 \leq j < k \leq n \).
3. \( f_j(z) \rightarrow 0, (j = 1, 2, \ldots, n) \).

**3. Proof of Theorems**

**Proof of Theorem 1.** Due to \( f \) and \( \Delta_c^2 f \) share \( \{a, b\} \) CM, so we set

\[
\frac{(\Delta_c^2 f - a)(\Delta_c^2 f - b)}{(f - a)(f - b)} = e_Q,
\] (7)

where \( Q \) is an entire function. And then it follows from (7) and \( \max\{\rho(a), \rho(b)\} < \rho(f) < \infty \) that \( Q \) is a polynomial.

Using Hadamard Factorization Theorem, we assume that \( f(z) = h(z)e^{P(z)} \), where \( h(\equiv 0) \) is an entire function and \( P \) is a polynomial which satisfies

\[
\lambda(f) = \rho(h) < \rho(f) = \deg(P). \tag{8}
\]

So,

\[
\Delta_c^2 f = f(z + 2c) - 2f(z + c) + f(z) = (h(z + 2c) - 2h(z + c) + h(z))e^{P(z)}. \tag{9}
\]
Put the forms of $f$ and $\Delta_2^zf$ into (7) to yield
\[
\begin{align*}
&\left\{ \left[ h(z + 2c)e^{P(z+2c)} - 2h(z + c)e^{P(z+c)} + h(z)e^{P(z)} \right] - a(z) \right\}, \\
&\left\{ \left[ h(z + 2c)e^{P(z+2c)} - 2h(z + c)e^{P(z+c)} + h(z)e^{P(z)} \right] - b(z) \right\} \\
&= \left( h(z)e^{P(z)} - a(z) \right) \left( h(z)e^{P(z)} - b(z) \right)e^z.
\end{align*}
\]

Take $w_1 = h(z + 2c)e^{P(z+2c)} - 2h(z + c)e^{P(z+c)} + h(z)e^{P(z)}$. We assume that $w_1 = 0$. Then,
\[
h(z + 2c)e^{P(z+2c)} - 2h(z + c)e^{P(z+c)} + h(z)e^{P(z)} = 0. \tag{11}
\]

By Lemma 3, if $p(z + 2c) - p(z)$, $p(z + c) - p(z)$, and $p(z + c) - p(z + c)$ are not constants, then $h(z) = 0$, a contradiction.

So, $p(z + 2c) - p(z) = a_1$, $p(z + c) - p(z) = b_1$, and $p(z + c) - p(z + c) = c_1$, (where $a_1, b_1, c_1$ are three constants).

We can get $p'(z + 2c) - p'(z) = 0$, $p'(z + c) - p'(z) = 0$, and $p'(z + c) - p'(z) = 0$.

Hence, $p(z)$ is a periodic function. We also know $p(z)$ is a polynomial. So, we get $p(z) = Az + B$ (where $A$ and $B$ are two constants, and $A \neq 0$). So, we obtain $f(z) = H(z)e^{Az}$. Here, $H(z)$ is an entire function and $\lambda(f) = \rho(H) < 1$.

If $w_1 \neq 0$ below, obviously, $w_1$ is a small function of $e^z$. Rewrite (10) as
\[
e^{O} = \frac{w_1 e^z}{h(z + 2c)} - \frac{(a_1/w_1)}{h(z + 2c)} \left[ \frac{e^z - (a/h)}{h(z + 2c)} \right] e^z. \tag{12}
\]

Note that $a \neq b$. Without loss of generality, we set $a \neq 0$. Suppose that $z_0$ is a zero of $e^z - (a/h)$, but not a zero of $w_1$. From (12), we may easily obtain that $z_0$ is a zero of $e^z - (a_1/w_1)$ or $e^z - (b/w)$. We denote by $N_1(r, e^z)$ the reduced counting function of those common zeros of $e^z - (a/h)$ and $e^z - (a_1/w_1)$. Similarly, we also denote $N_2(r, e^z)$ the reduced counting function of those common zeros of $e^z - (a/h)$ and $e^z - (b/w)$. Then,
\[
T(r, e^z) = N\left( r, \frac{1}{e^z - (a/h)} \right) + S(r, e^z) \tag{13}
\]

which implies that either $N_1(r, e^z) \neq S(r, e^z)$ or $N_2(r, e^z) \neq S(r, e^z)$. We distinguish the two cases as follows:

Case 1: $N_1(r, e^z) \neq S(r, e^z)$.

We may assume that $a_0$ is a common zero of $e^z - (a/h)$ and $e^z - (a_1/w_1)$. It is obvious that $a_0$ is a zero of $(a/h) - (a_1/w_1)$. If $(a/h) - (a_1/w_1) = 0$, then
\[
S(r, e^z) + N_1(r, e^z) \leq N\left( r, \frac{1}{(a/h) - (a_1/w_1)} \right) \tag{14}
\]
\[
T\left( r, \frac{a}{h} - \frac{a}{w_1} \right) = S(r, e^z),
\]

a contradiction. Hence,
\[
h = w_1. \tag{15}
\]

It deduces
\[
2\frac{h(z + c)}{h(z + 2c)} = e^{P(z+2c) - P(z+c)}. \tag{16}
\]

By Lemma 1, for any $\epsilon > 0$,
\[
m(r, e^{P(z+2c) - P(z+c)}) = m\left( r, \frac{h(z + c)}{h(z + 2c)} \right) + O(1) \tag{17}
\]
\[
= O\left( \frac{(2\epsilon)^{1+\epsilon}}{1+\epsilon} \right) + O(1).
\]

We also get $m(r, e^{P(z+2c) - P(z+c)}) = [A + o(1)]r^{\rho(f) - 1}$, where $A$ is a fixed positive constant.

If $\rho(f) > 1$, using $\rho(\hat{f}) > \rho(h)$ and the above estimates of $m(r, e^{P(z+2c) - P(z+c)})$, It easily gets a contradiction. So, $\rho(f) \leq 1$, this means that $e^{P(z+2c) - P(z+c)}$ is a nonzero constant $c_0$. Then, (16) changes to
\[
2\frac{h(z + c)}{h(z + 2c)} = c_0. \tag{18}
\]

Also, noting that $1 > \rho(f) > \rho(h)$. Then, by Lemma 2, we get that there exists an $\epsilon$-set $E$, as $z \notin E$ and $|z| \to \infty$ such that
\[
\frac{h(z + c)}{h(z + 2c)} \to 1. \tag{19}
\]

So, $c_0 = 2$ and $h(z + c) = h(z + 2c)$, and this also means that $h$ is a periodic function. If $h$ is a nonconstant function, then $\rho(h) \geq 1$, a contradiction. Therefore, $h$ is a constant. Noting that $\deg(p) = \rho(f) \leq 1$ and $f$ is a nonconstant entire function. Then, $\deg(p) = 1$. Thus, we may set $f = Ae^{cz}$, where $A$ and $\mu$ are two nonzero constants.

Using the assumption of Case 1, one has $f - a$ and $\Delta_2f - a$ as common zeros, which are not zeros of $a$. Suppose that $a_\varnothing$ is a common zero of $f - a$ and $\Delta_2f - a$
and not a zero of $a$. Then, $z_0$ is a zero of $\Delta^2_x f - f$.
Moreover,

$$f(z_0 + 2c) - 2f(z_0 + c) = 0,$$
(20)

this implies that $e^{\kappa x} = 2$. Finally, we deduce $\Delta_x f = f$, which is the desired result.

Case 2: $N_2(r, e^p) \neq S(r, e^p)$.

Suppose $b_0$ which is a common zero of $e^p - (a/h)$ and $e^p - (b/w_1)$. Then, it is obvious that $b_0$ is a zero of $(a/h) - (b/w_1)$. If $(a/h) - (b/w_1) \neq 0$, then

$$S(r, e^p) \neq N_3(r, e^p) \leq N \left( r, \frac{1}{(a/h) - (b/w_1)} \right) \leq T \left( r, \frac{a}{h} - \frac{b}{w_1} \right) = S(r, e^p),$$
(21)
a contradiction. Hence,

$$\frac{a}{h} - \frac{b}{w_1} = 0.$$  
(22)

If $b = 0$, then $(a/h) = 0$, a contradiction. Thus, $b = 0$.

We may set that $c_0$ is a zero of $e^{\kappa x} - (b/h)$, but not a zero of $w_1$. It follows from (12) that $c_0$ is a zero of $e^p - (a/w_1)$ or $e^p - (b/w_1)$. We take by $N_3(r, e^p)$ the reduced counting function of those common zeros of $e^p - (b/h)$ and $e^p - (a/w_1)$. Similarly, we denote by $N_4(r, e^p)$ the reduced counting function of those common zeros of $e^p - (b/h)$ and $e^p - (b/w_1)$. We obtain

$$T(r, e^p) = \overline{N} \left( r, \frac{1}{e^p - (b/h)} \right) + S(r, e^p) = N_3(r, e^p)$$
$$+ N_4(r, e^p) + S(r, e^p).$$

It implies that either $N_3(r, e^p) \neq S(r, e^p)$ or $N_4(r, e^p) \neq S(r, e^p)$. If $N_4(r, e^p) \neq S(r, e^p)$, likewise with Case 1, we deduce the desired result. Hence, we set that $N_3(r, e^p) \neq S(r, e^p)$ below. Similarly with Case 2, we also get that

$$\frac{b}{h} - \frac{a}{w_1} = 0.$$  
(24)

Combining (22) with (24), we deduce that

$$a^2 = b^2.$$  
(25)

Note that $a \neq b$. Thus, $a = -b$. Again using (24), we have $w_1 = -h$. We rewrite it as

$$h(z + 2c)e^{P(z+c) - P(z)} - 2h(z + c)e^{P(z+c) - P(z)} + 2h(z) = 0.$$  
(26)

Then,

$$h(z + 2c)e^{P(z+c) - P(z)} - 2h(z + c)e^{P(z+c) + 2h(z)e^{P(z)} = 0.$$  
(27)

By Lemma 3, if $p(z + 2c) - p(z), p(z + c) - p(z)$, and $p(z + 2c) - p(z + c)$ are not constants, then $h(z) = 0$; this is a contradiction.

So, $p(z + 2c) - p(z) = a_1$, $p(z + c) - p(z) = b_1$, and $p(z + 2c) - p(z + c) = c_1$ (where $a_1, b_1$, and $c_1$ are three constants).

We can get $p'(z + 2c) - p'(z) = 0$, $p'(z + c) - p'(z) = 0$.

Hence, $p(z)$ is a periodic function. We also know $p(z)$ is a polynomial. So, we get $p(z) = Az + B$ (where $A$ and $B$ are two constants, and $A \neq 0$). So, we obtain $f(z) = H(z)e^{Az}$. Here, $H(z)$ is an entire function and $\lambda(f) = \rho(H) < 1$.

Therefore, the proof of the main Theorem 3 is finished.  
\qed

Proof of Theorem 2. Note that $f$ and $\Delta^n_x f$ share \{a, b\} CM. So, we also set

$$\frac{(\Delta^n_x f - a)(\Delta^n_x f - b)}{(f - a)(f - b)} = e^Q,$$  
(28)

where $Q$ is an entire function. Furthermore, it deduces from (28) and $\max|p(a), p(b)| < \rho(f) < \infty$ that $Q$ is a polynomial.

Using Hadamard Factorization Theorem, we assume that $f(z) = H(z)e^{Az}$, where $h(\neq 0)$ is an entire function and $P$ is a polynomial satisfying

$$\lambda(f) = \rho(h) < \rho(f) = \deg(P).$$  
(29)

Then,

$$\Delta^n_x f = f(z + nc) + b_{n-1}f(z + (n - 1)c) + \cdots + b_1f(z + c) + b_0f(z)$$
$$= h(z + nc)e^{p(z+nc)} + b_{n-1}h(z + (n - 1)c)e^{p(z+(n-1)c)} + \cdots + b_0h(z)e^{p(z)},$$
(30)

where $b_0, b_1, \ldots, b_{n-1}$ are constants. Substituting the forms of $f$ and $\Delta^n_x f$ into (28) yields

$$\left\{ \begin{array}{l}
\left[ h(z + nc)e^{p(z+nc) - P(z)} + b_{n-1}h(z + (n - 1)c)e^{p(z+(n-1)c) - P(z)} + \cdots + b_0h(z)e^{P(z)} - a(z) \right], \\
\left[ h(z + nc)e^{p(z+nc) - P(z)} + b_{n-1}h(z + (n - 1)c)e^{p(z+(n-1)c) - P(z)} + \cdots + b_0h(z)e^{P(z)} - b(z) \right] \\
\end{array} \right\} = \left( h(z)e^{P(z) - a(z)} \right) \left( h(z)e^{P(z)} - b(z)e^{Q(z)} \right).$$  
(31)
Set \( w_1 = h(z + nc) e^{p(z + nc) - p(z)} + b_{n-1} h(z + (n - 1)c) e^{p(z + (n - 1)c) - p(z)} + \cdots + b_0 h(z) \). Suppose that \( w_1 = 0 \). Then,
\[
h(z + nc) e^{p(z + nc)} + b_{n-1} h(z + (n - 1)c) e^{p(z + (n - 1)c)} + \cdots + b_0 h(z) e^{p(z)} = 0.
\] (32)

By Lemma 3, if \( p(z + nc) - p(z), p(z + (n - 1)c) - p(z), \ldots, p(z + c) - p(z) \) are not constants, then \( h(z) = 0 \); a contradiction.

So, \( p(z + nc) - p(z) = a_n, p(z + (n - 1)c) - p(z) = a_{n-1}, \) and \( p(z + c) - p(z) = a_1 \), where \( a_n, a_{n-1}, \ldots, a_1 \) are three constants.

We can get \( p'(z + nc) - p'(z) = 0, p'(z + (n - 1)c) - p'(z) = 0, \) and \( p'(z + c) - p'(z) = 0 \).

Hence, \( p(z) \) is a periodic function. We also know \( p(z) \) is a polynomial. So, we get \( p(z) = Az + B \) (where \( A \) and \( B \) are two constants, and \( A \neq 0 \)). We obtain \( f(z) = H(z) e^{kz} \).

Here, \( H(z) \) is an entire function and \( \lambda(f) = \rho(H) < 1 \).

If \( w_1 \neq 0 \) below, obviously, \( w_1 \) is a small function of \( e^p \). Rewrite (31) as
\[
e^Q = w_1^2 [e^{p - (a/w_1)} - e^{p - (b/w_1)}]/h [e^{p - (a/h)} - e^{p - (b/h)}].
\] (33)

Due to \( a \neq b \), without loss of generality, we set \( a \neq 0 \). Let us assume that \( z_0 \) is a zero of \( e^p - (a/h) \), but not a zero of \( w_1 \). It deduces from (33) that \( z_0 \) is a zero of \( e^p - (a/w_1) \) or \( e^p - (b/w_1) \). We also take \( N_1(r, e^p) \) the reduced counting function of those common zeros of \( e^p - (a/h) \) and \( e^p - (a/w_1) \). Likewise, we denote by \( N_2(r, e^p) \) the reduced counting function of those common zeros of \( e^p \) and \( e^p - (b/w_1) \). Then,
\[
T(r, e^p) = N\left(r, \frac{1}{e^{p(z)} - (a/h)} \right) + S(r, e^p) = N_1(r, e^p) + N_2(r, e^p) + S(r, e^p),
\] (34)

which implies that either \( N_1(r, e^p) \neq S(r, e^p) \) or \( N_2(r, e^p) \neq S(r, e^p) \). We may distinguish the following two cases.

Case 1: \( N_1(r, e^p) \neq S(r, e^p) \).

We set \( a_0 \) a common zero of \( e^p - (a/h) \) and \( e^p - (a/w_1) \). Then, it is obvious that \( a_0 \) is a zero of \( (a/h) - (a/w_1) \). If \( (a/h) - (a/w_1) \neq 0 \), then
\[
S(r, e^p) \neq N_1(r, e^p) \leq N\left(r, \frac{1}{(a/h) - (a/w_1)} \right) \leq T\left(r, \frac{a}{h} - \frac{a}{w_1} \right) = S(r, e^p),
\] (35)
a contradiction. Hence,
\[
h = w_1.
\] (36)

It leads to
\[
h(z + nc) e^{p(z + nc)} + b_{n-1} h(z + (n - 1)c) e^{p(z + (n - 1)c)} + \cdots + (b_0 - 1)h(z) e^{p(z)} = 0.
\] (37)

By Lemma 3, if \( p(z + nc) - p(z), p(z + (n - 1)c) - p(z), \ldots, p(z + c) - p(z) \) are not constants, then \( h(z) = 0 \), a contradiction.

So, \( p(z + nc) - p(z) = a_n, p(z + (n - 1)c) - p(z) = a_{n-1}, \ldots, p(z + c) - p(z) = a_1 \), where \( a_n, a_{n-1}, a_1 \) are three constants.

We can get \( p'(z + nc) - p'(z) = 0, p'(z + (n - 1)c) - p'(z) = 0, \) and \( p'(z + c) - p'(z) = 0 \).

Here, \( p(z) \) is a periodic function. We also know \( p(z) \) is a polynomial. So, we get \( p(z) = Az + B \) (where \( A \) and \( B \) are two constants, and \( A \neq 0 \)). Finally, we get \( f(z) = H(z) e^{kz} \), where \( H(z) \) is an entire function and \( \lambda(f) = \rho(H) < 1 \).

Case 2: \( N_2(r, e^p) \neq S(r, e^p) \).

Suppose \( b_0 \) is a common zero of \( e^p - (a/h) \) and \( e^p - (b/w_1) \). Then, it is easy to see that \( b_0 \) is a zero of \( (a/h) - (b/w_1) \). If \( (a/h) - (b/w_1) \neq 0 \), then
\[
S(r, e^p) / N_2(r, e^p) \leq N\left(r, \frac{1}{(a/h) - (b/w_1)} \right) \leq T\left(r, \frac{a}{h} - \frac{b}{w_1} \right) = S(r, e^p),
\] (38)
a contradiction. Thus,
\[
\frac{a}{h} - \frac{b}{w_1} = 0.
\] (39)

If \( b = 0 \), then \( (a/h) = 0 \), a contradiction. Thus, \( b \neq 0 \).

We assume that \( c_0 \) is a zero of \( e^p - (b/w_1) \), but not a zero of \( w_1 \). It deduces from (33) that \( c_0 \) is a zero of \( e^p - (a/w_1) \) or \( e^p - (b/w_1) \). We take by \( N_3(r, e^p) \) the reduced counting function of those common zeros of \( e^p - (b/h) \) and \( e^p - (a/w_1) \). Similarly, we denote by \( N_4(r, e^p) \) the reduced counting function of those common zeros of \( e^p - (b/h) \) and \( e^p - (b/w_1) \). Then,
\[
T(r, e^p) = N\left(r, \frac{1}{e^{p} - (b/h)} \right) + S(r, e^p) = N_3(r, e^p) + N_4(r, e^p) + S(r, e^p),
\] (40)
and this implies that either $N_3(r, e^p) \neq S(r, e^p)$ or $N_4(r, e^p) \neq S(r, e^p)$. If $N_4(r, e^p) \neq S(r, e^p)$, Similarly, as the same way in Case 1, we get the desired result. So, we assume that $N_3(r, e^p) \neq S(r, e^p)$ as follows. Similarly, as the way in Case 2, we can get that

$$
\frac{b}{h} - \frac{a}{w_1} = 0.
$$

(41)

It follows from (39) and (41) that

$$a^2 = b^2.
$$

(42)

Note that $a \neq b$. Thus, $a = -b$. Again by (41), one has $w_1 = -h$. We also rewrite it as

$$h(z + nc)e^{p(z+nc)} + b_{n-1}h(z + (n-1)c)e^{p(z+(n-1)c)} + \cdots + (b_0 + 1)h(z)e^{p(z)} = 0.
$$

(43)

By Lemma 3, if $p(z + nc) - p(z)$, $p(z + (n-1)c) - p(z)$, $p(z + c) - p(z)$ are not constants, then $h(z) = 0$, a contradiction.

So, $p(z + nc) - p(z) = a_n$, $p(z + (n-1)c) - p(z) = a_{n-1}$, $p(z + c) - p(z) = a_1$ (where $a_n, a_{n-1}, \ldots, a_1$ are three constants).

We also get $p'(z + nc) - p'(z) = 0$, $p'(z + c) - p'(z) = 0$.

Hence, $p(z)$ is a periodic function. We also know $p(z)$ is a polynomial. So, we get $p(z) = Az + B$ (where $A$ and $B$ are two constants, and $A \neq 0$). Finally, we get $f(z) = H(z)e^{Az}$. Here, $H(z)$ is an entire function and $\lambda(f) = \rho(H) < 1$.

Proof of Theorem 4 is completed. $\square$

\section*{Data Availability}

No data were used to support this study.

\section*{Conflicts of Interest}

The authors declare that they have no conflicts of interest.

\section*{Authors’ Contributions}

All authors typed, read, and approved the final manuscript.

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