Research Article

On Fully Degenerate Daehee Numbers and Polynomials of the Second Kind

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In a study, Carlitz introduced the degenerate exponential function and applied that function to Bernoulli and Eulerian numbers and degenerate special functions have been studied by many researchers. In this paper, we define the fully degenerate Daehee polynomials of the second kind which are different from other degenerate Daehee polynomials and derive some new and interesting identities and properties of those polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p -adic integers, the field of p -adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively.

Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then, the p -adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{n=0}^{p^N-1} f(x), \quad (1)$$

(see [1–3]).

From (1), we have

$$\begin{aligned} I_0(f_n) - I_0(f) &= \sum_{l=0}^{n-1} f'(l), \quad \text{where } f_n(x) \\ &= f(x+n), f'(l) = \left. \frac{df(x)}{dx} \right|_{x=l}. \end{aligned} \quad (2)$$

In particular, if $n = 1$, then

$$I_0(f_1) - I_0(f) = f'(0). \quad (3)$$

The Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \geq 0), \quad (4)$$

and the Stirling numbers of the second kind are given by

$$x^n = \sum_{l=0}^n S_2(n, l) (x)_l, \quad (5)$$

where $(x)_0 = 1$ and $(x)_n = x(x-1) \cdots (x-n+1)$ ($n \geq 1$) (see [4, 5]).

From (4) and (5), we can derive the following equations:

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}, \quad (6)$$

$$(\log(x+1))^n = n! \sum_{l=n}^{\infty} S_1(l, n) \frac{x^l}{l!}, \quad (n \geq 0), \quad (7)$$

(see [4–6]).

In addition,

$$\log(1+t) = \sum_{n=1}^{\infty} (-1)^{n+1} (n-1)! \frac{t^n}{n!}, \quad (8)$$

[4, 5].

The *Bernoulli polynomials of order r* are defined by the following generating function:

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (9)$$

(see [7–9]).

Carlitz's degenerate Bernoulli polynomials of order r are defined by the generating function to be

$$\sum_{n=0}^{\infty} \beta_n^{(r)}(x|\lambda) \frac{t^n}{n!} = \left(\frac{t}{(1+\lambda t)^{1/\lambda} - 1}\right)^r (1+\lambda t)^{x/\lambda}, \quad (10)$$

where $\lambda \in \mathbb{R}$ (see [10]). By (10), we know that

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \sum_{n=0}^{\infty} \beta_n^{(r)}(x|\lambda) \frac{t^n}{n!} &= \lim_{\lambda \rightarrow 0} \left(\frac{t}{(1+\lambda t)^{1/\lambda} - 1}\right)^r (1+\lambda t)^{x/\lambda} \\ &= \left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \end{aligned} \quad (11)$$

and thus, we obtain

$$\lim_{\lambda \rightarrow 0} \beta_n^{(r)}(x|\lambda) = B_n^{(r)}(x). \quad (12)$$

In [11], the degenerate Bernoulli polynomials are defined as

$$\left(\frac{\log(1+\lambda t)^{1/\lambda}}{(1+\lambda t)^{1/\lambda} - 1}\right)^r (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} b_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}, \quad (13)$$

which are different from Carlitz's degenerate numbers and polynomials.

By (13), we know that

$$\lim_{\lambda \rightarrow 0} b_{n,\lambda}^{(r)}(x) = B_n^{(r)}(x). \quad (14)$$

Note that by (3),

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{x+x_1+\cdots+x_r/\lambda} d\mu_0(x_1) \cdots d\mu_0(x_r) \\ &= \left(\frac{(1/\lambda)\log(1+\lambda t)}{(1+\lambda t)^{1/\lambda} - 1}\right)^r (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} b_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}, \end{aligned} \quad (15)$$

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{x+x_1+\cdots+x_r/\lambda} d\mu_0 \cdots d\mu_0(x_r) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \left(\frac{x+x_1+\cdots+x_r}{n}\right) \lambda^n t^n d\mu_0(x_1) \cdots d\mu_0(x_r) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x+x_1+\cdots+x_r)_{n,\lambda} d\mu_0(x_1) \cdots d\mu_0(x_r) \frac{t^n}{n!}. \end{aligned} \quad (16)$$

By (15) and (16), we know that

$$b_{n,\lambda}^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x+x_1+\cdots+x_r)_{n,\lambda} d\mu_0(x_1) \cdots d\mu_0(x_r). \quad (17)$$

The *higher-order Daehee polynomials* are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}, \quad (18)$$

(see [12–15]).

In particular, if $r = 1$, then $D_n^{(1)}(x) = D_n(x)$ is called the *Daehee polynomials*.

By replacing t as $\log(1+t)$ in (18), we have

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{1}{n!} (\log(1+t))^n \\ &= \left(\sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{1}{n!}\right) \left(n! \sum_{l=0}^{\infty} S_1(l, n) \frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n B_m^{(r)}(x) S_1(n, m)\right) \frac{t^n}{n!}, \end{aligned} \quad (19)$$

and so, we obtain

$$D_n^{(r)}(x) = \sum_{m=0}^n B_m^{(r)}(x) S_1(n, m). \quad (20)$$

Note that by (3), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!} &= \left(\frac{\log(1+t)}{t} \right)^r (1+t)^x \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x+x_1+\cdots+x_r} d\mu_0(x_1) \cdots d\mu_0(x_r) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x+x_1+\cdots+x_r}{n} t^n d\mu_0(x_1) \cdots d\mu_0(x_r) \\
&= \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x+x_1+\cdots+x_r)_n d\mu_0(x_1) \cdots d\mu_0(x_r) \right) \frac{t^n}{n!},
\end{aligned} \tag{21}$$

and thus, we know that

$$D_n^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x+x_1+\cdots+x_r)_n d\mu_0(x_1) \cdots d\mu_0(x_r). \tag{22}$$

Carlitz introduced the degenerate Bernoulli polynomials in [10] and the degeneration of special functions have been studied (see [6, 16–29]).

In particular, the degenerate Stirling numbers of the second kind with a generating function are defined as

$$\frac{1}{m!} \left((1+\lambda t)^{1/\lambda} - 1 \right)^m = \sum_{n=m}^{\infty} S_{2,\lambda}(n, m) \frac{t^n}{n!}, \tag{23}$$

where m is a given nonnegative integer in [6, 10, 22, 30].

After introducing Daehee numbers and polynomials [31], it plays an important role of developing various generalized polynomials, and interesting properties are obtained (see [8, 15, 21, 22, 28, 30–35]).

In this paper, we define the new degenerate Daehee polynomials and numbers which are called the degenerate Daehee polynomials of the second kind and investigate identities and properties of new polynomials.

2. Fully Degenerate Daehee Polynomials of the Second Kind

Let us assume that $\lambda \in \mathbb{R}$. By (3), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} (1+\lambda \log(1+t))^{x+y/\lambda} d\mu_0(y) \\
&= \frac{(1/\lambda) \log(1+\lambda \log(1+t))}{(1+\lambda \log(1+t))^{1/\lambda} - 1} (1+\lambda \log(1+t))^{x/\lambda}.
\end{aligned} \tag{24}$$

By (24), we define the *degenerate Daehee polynomials of the second kind* by the generating function to be

$$\begin{aligned}
&\frac{(1/\lambda) \log(1+\lambda \log(1+t))}{(1+\lambda \log(1+t))^{1/\lambda} - 1} (1+\lambda \log(1+t))^{x/\lambda} \\
&= \sum_{n=0}^{\infty} D_n(x|\lambda) \frac{t^n}{n!}.
\end{aligned} \tag{25}$$

In the special case, $x=0$, and $D_n(\lambda) = D_n(0|\lambda)$ are called the *degenerate Daehee numbers of the second kind*.

Note that

$$\begin{aligned}
\lim_{\lambda \rightarrow 0} \sum_{n=0}^{\infty} D_n(x|\lambda) \frac{t^n}{n!} &= \lim_{\lambda \rightarrow 0} \frac{(1/\lambda) \log(1+\lambda \log(1+t))}{(1+\lambda \log(1+t))^{1/\lambda} - 1} \\
&\quad \cdot (1+\lambda \log(1+t))^{x/\lambda} \\
&= \frac{\log(1+t)}{t} (1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!},
\end{aligned} \tag{26}$$

and thus, we know that

$$\lim_{\lambda \rightarrow 0} D_n(x|\lambda) = D_n(x), \quad (n \geq 0). \tag{27}$$

From (7) and (24), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} (1+\lambda \log(1+t))^{x+y/\lambda} d\mu_0(y) \\
&= \sum_{l=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x+y}{l} \lambda^l (\log(1+t))^l d\mu_0(y) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n S_1(n, l) \lambda^l l! \int_{\mathbb{Z}_p} \binom{x+y}{l} d\mu_0(y) \right) \frac{t^n}{n!}.
\end{aligned} \tag{28}$$

By (25) and (28), we have

$$D_{n,\lambda} = \sum_{k=0}^n S_1(n, l) \lambda^l l! \int_{\mathbb{Z}_p} \binom{x+y}{l} d\mu_0(y), \quad (n \geq 0). \tag{29}$$

Since

$$\binom{\frac{x+y}{\lambda}}{l} = \frac{((x+y)/\lambda)((x+y)/\lambda - 1)((x+y)/\lambda - 2) \dots ((x+y)/\lambda - l + 1)}{l!}$$

$$= \frac{(x+y)(x+y-\lambda)(x+y-2\lambda) \dots (x+y-(l-1)\lambda)}{\lambda^l l!},$$
(30)

by (29) and (30), we have

$$D_n(\lambda) = \sum_{k=0}^n S_1(n, l) \int_{\mathbb{Z}_p} (x+y)_{l,\lambda} d\mu_0(y), \quad (n \geq 0), \quad (31)$$

where $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda) \dots (x-(n-1)\lambda)$.

Thus, by (29) and (30), we have the following theorem which is Witt's type formula about degenerate Daehee polynomials of the second kind.

Theorem 1. For each $n \geq 0$, we have

$$D_n(\lambda) = \sum_{k=0}^n S_1(n, l) \lambda^l l! \int_{\mathbb{Z}_p} \binom{\frac{x+y}{\lambda}}{l} d\mu_0(y)$$

$$= \sum_{k=0}^n S_1(n, l) \int_{\mathbb{Z}_p} (x+y)_{l,\lambda} d\mu_0(y). \quad (32)$$

By replacing t as $e^t - 1$ in (25), we obtain the following:

$$\sum_{n=0}^{\infty} D_n(x|\lambda) \frac{(e^t - 1)^n}{n!} = \sum_{n=0}^{\infty} D_n(x|\lambda) \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n D_m(x|\lambda) S_2(m, n) \right) \frac{t^n}{n!}. \quad (33)$$

On the other hand,

$$\sum_{n=0}^{\infty} D_n(x|\lambda) \frac{(e^t - 1)^n}{n!} = \frac{(1/\lambda) \log(1 + \lambda t)}{(1 + \lambda t)^{1/\lambda} - 1} (1 + \lambda t)^{x/\lambda}$$

$$= \sum_{n=0}^{\infty} B_n(x|\lambda) \frac{t^n}{n!}, \quad (34)$$

where $B_n(x|\lambda)$ is the degenerate Bernoulli polynomials of the second kind of order $r \in \mathbb{Z}$ which are defined by the generating function to be

$$\left(\frac{(1/\lambda) \log(1 + \lambda t)}{(1 + \lambda t)^{1/\lambda} - 1} \right)^r (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} B_n^{(r)}(x|\lambda) \frac{t^n}{n!} \quad (35)$$

(see [20, 25]).

In particular, if $r = 1$, $B_n^{(1)}(x|\lambda) = B_n(x|\lambda)$ is called the degenerate Bernoulli polynomials of the second kind.

For positive integer d with $d \equiv 1 \pmod{2}$, if we put $f(x) = (1 + \lambda \log(1 + t))^{x/\lambda}$, then, by (2), we obtain

$$\int_{\mathbb{Z}_p} (1 + \lambda \log(1 + t))^{x+d/\lambda} d\mu_0(y)$$

$$- \int_{\mathbb{Z}_p} (1 + \lambda \log(1 + t))^{x/\lambda} d\mu_0(y)$$

$$= \sum_{l=0}^{d-1} \frac{1}{\lambda} (1 + \lambda \log(1 + t))^{x/\lambda} \log(1 + \lambda \log(1 + t)). \quad (36)$$

By (36), we have

$$\int_{\mathbb{Z}_p} (1 + \lambda \log(1 + t))^{x/\lambda} d\mu_0(x)$$

$$= \frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{d/\lambda} - 1} \sum_{l=0}^{d-1} (1 + \lambda \log(1 + t))^{l/\lambda}$$

$$= d \sum_{l=0}^{d-1} \frac{(1/\lambda) \log(1 + (\lambda/d) d \log(1 + t))}{(1 + (\lambda/d) d \log(1 + t))^{d/\lambda} - 1}$$

$$\cdot \left(1 + \frac{\lambda}{d} d \log(1 + t) \right)^{(d/\lambda)(l/d)}. \quad (37)$$

Note that by (6),

$$\frac{(1/\lambda) \log(1 + (\lambda/d) d \log(1 + t))}{(1 + (\lambda/d) d \log(1 + t))^{d/\lambda} - 1} (1 + (\lambda/d) d \log(1 + t))^{(d/\lambda)(l/d)}$$

$$= \sum_{n=0}^{\infty} B_n \left(\frac{l}{d} \middle| \frac{\lambda}{d} \right) \frac{(d \log(1 + t))^n}{n!}$$

$$= \sum_{n=0}^{\infty} B_n \left(\frac{l}{d} \middle| \frac{\lambda}{d} \right) \frac{d^n}{n!} \sum_{m=n}^{\infty} S_1(m, n) \frac{x^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n B_m \left(\frac{l}{d} \middle| \frac{\lambda}{d} \right) d^m S_1(n, m) \right) \frac{t^n}{n!}. \quad (38)$$

By (24), (37), and (38), we have

$$\sum_{n=0}^{\infty} D_n(\lambda) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + \lambda \log(1 + t))^{x/\lambda} d\mu_0(x)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \sum_{l=0}^{d-1} B_m \left(\frac{l}{d} \middle| \frac{\lambda}{d} \right) d^{m+1} S_1(n, m) \right) \frac{t^n}{n!}. \quad (39)$$

Hence, by (33), (34), and (39), we obtain the following theorem which shows the relationship between degenerate Dahee polynomials of the second kind and degenerate Bernoulli polynomials of the second kind.

Theorem 2. For nonnegative integer n and $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have

$$\begin{aligned} B_n(x|\lambda)(x) &= \sum_{m=0}^n D_{m,\lambda}(x) S_2(n, m), \\ D_{n,\lambda} &= \sum_{m=0}^n \sum_{l=0}^{d-1} B_m\left(\frac{l}{d} \middle| \frac{\lambda}{d}\right) d^{m+1} S_1(n, m). \end{aligned} \quad (40)$$

By (25), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} D_n(x|\lambda) \frac{t^n}{n!} &= \left(\frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{1/\lambda} - 1} \right) \\ &\quad \cdot (1 + \lambda \log(1 + t))^{x/\lambda} \\ &= \left(\sum_{n=0}^{\infty} D_n(\lambda) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \left(\frac{x}{\lambda} \right) \lambda^n (\log(1 + t))^n \right) \\ &= \left(\sum_{n=0}^{\infty} D_n(\lambda) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \left(\frac{x}{\lambda} \right) \lambda^n n! \sum_{l=n}^{\infty} S_1(l, n) \frac{x^l}{l!} \right) \\ &= \left(\sum_{n=0}^{\infty} D_n(\lambda) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{l=0}^n (x)_{l,\lambda} S_1(n, l) \frac{x^l}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{l=0}^k \binom{n}{k} (x)_{l,\lambda} S_1(k, l) D_{n-k}(\lambda) \right) \frac{t^n}{n!}. \end{aligned} \quad (41)$$

By comparing the coefficients on both sides of (41), we obtain the following theorem.

Theorem 3. For nonnegative integer n , we have

$$D_n(x|\lambda) = \sum_{k=0}^n \sum_{l=0}^k \binom{n}{k} (x)_{l,\lambda} S_1(k, l) D_{n-k}(\lambda). \quad (42)$$

Note that if we put $f(x) = (1 + \lambda \log(1 + t))^{x/\lambda}$, then

$$\begin{aligned} f'(0) &= \frac{1}{\lambda} \log(1 + \lambda \log(1 + t)) \\ &= \frac{1}{\lambda} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\lambda^n}{n} (\log(1 + t))^n \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\lambda^{n-1}}{n} n! \sum_{l=n}^{\infty} S_1(l, n) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=1}^n (-1)^{m+1} \lambda^{m-1} (m-1)! S_1(n, m) \right) \frac{t^n}{n!}, \end{aligned} \quad (43)$$

and thus, by (3), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x_1) d\mu_0(x) - \int_{\mathbb{Z}_p} f(x) d\mu_0(x) &= \frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{1/\lambda} - 1} (1 + \lambda \log(1 + t))^{1/\lambda} \\ &\quad - \frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{1/\lambda} - 1} \\ &= \sum_{n=0}^{\infty} (D_n(1|\lambda) - D_n(\lambda)) \frac{t^n}{n!}. \end{aligned} \quad (44)$$

Moreover,

$$\begin{aligned} ((1/\lambda) \log(1 + \lambda \log(1 + t))) \sum_{l=0}^{n-1} (1 + \lambda \log(1 + t))^{l/\lambda} &= \frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{1/\lambda}} (1 + \lambda \log(1 + t))^n \\ &\quad - \frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{1/\lambda}} \\ &= \sum_{r=0}^{\infty} \{D_r(n|\lambda) - D_r(\lambda)\} \frac{t^r}{r!}, \end{aligned} \quad (45)$$

and, by (43), we obtain

$$\begin{aligned}
& (1/\lambda)\log(1+\lambda\log(1+t)) \sum_{l=0}^{n-1} (1+\lambda\log(1+t))^{l/\lambda} \\
&= \left(\sum_{p=0}^{\infty} \sum_{m=1}^p (-\lambda)^{m-1} (m-1)! S_1(p, m) \frac{t^p}{p!} \right) \\
&\cdot \left(\sum_{l=0}^{n-1} \sum_{s=0}^{\infty} \sum_{q=0}^s (l)_{q,\lambda} S_1(s, q) \frac{t^s}{s!} \right) \quad (46) \\
&= \sum_{r=0}^{\infty} \left(\sum_{l=0}^{n-1} \sum_{p=0}^r \sum_{m=1}^p \sum_{q=0}^{r-p} \binom{r}{p} (l)_{q,\lambda} (-\lambda)^{m-1} \right. \\
&\quad \cdot (m-1)! S_1(p, m) S_1(r-p, q) \frac{t^r}{r!} \cdot
\end{aligned}$$

From (43), (44), and (46), we obtain the following theorem which represents a recurrence relations between degenerate Daehee polynomials of the second kind and degenerate Daehee numbers of the second kind.

Theorem 4. For each nonnegative integer r , we have

$$D_r(1|\lambda) - D_r(\lambda) = \sum_{m=1}^r (-\lambda)^{m-1} (m-1)! S_1(r, m). \quad (47)$$

Moreover, for each positive integer $n \geq 2$,

$$\begin{aligned}
D_r(n|\lambda) - D_r(\lambda) &= \sum_{l=0}^{n-1} \sum_{p=0}^r \sum_{m=1}^p \sum_{q=0}^{r-p} \binom{r}{p} (l)_{q,\lambda} (-\lambda)^{m-1} \\
&\cdot (m-1)! S_1(p, m) S_1(r-p, q). \quad (48)
\end{aligned}$$

3. Higher-Order Degenerate Daehee Polynomials of the Second Kind

In this section, we consider the *higher-order degenerate Daehee polynomials of the second kind* given by the generating function as follows: for the given positive real number r ,

$$\begin{aligned}
& \left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda} - 1} \right)^r (1+\lambda\log(1+t))^{x/\lambda} \\
&= \sum_{n=0}^{\infty} D_n^{(r)}(x|\lambda) \frac{t^n}{n!}. \quad (49)
\end{aligned}$$

In particular, if $x = 0$, $D_n^{(r)}(0|\lambda) = D_n^{(r)}(\lambda)$ are called the *higher-order degenerate Daehee numbers of the second kind*. Note that

$$\begin{aligned}
\lim_{\lambda \rightarrow 0} \sum_{n=0}^{\infty} D_n^{(r)}(x|\lambda) \frac{t^n}{n!} &= \lim_{\lambda \rightarrow 0} \left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda} - 1} \right)^r \\
&\cdot (1+\lambda\log(1+t))^{x/\lambda} \\
&= \left(\frac{\log(1+t)}{t} \right)^r (1+t)^x \\
&= \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}. \quad (50)
\end{aligned}$$

From (35), we note that

$$\begin{aligned}
& \left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda} - 1} \right)^r (1+\lambda\log(1+t))^{x/\lambda} \\
&= \sum_{n=0}^{\infty} B_n^{(r)}(x|\lambda) \frac{(\log(1+t))^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{B_n^{(r)}(x|\lambda)}{n!} n! \sum_{l=n}^{\infty} S_1(l, n) \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n B_m^{(r)}(x|\lambda) S_1(n, m) \right) \frac{t^n}{n!}. \quad (51)
\end{aligned}$$

In addition, by replacing t by $e^t - 1$ in (49), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} D_n^{(r)}(x|\lambda) \frac{(e^t - 1)^n}{n!} &= \sum_{n=0}^{\infty} \frac{D_n^{(r)}(x|\lambda)}{n!} n! \sum_{m=n}^{\infty} S_2(n, m) \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n D_m^{(r)}(x|\lambda) S_2(n, m) \right) \frac{t^n}{n!}, \quad (52)
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} D_n^{(r)}(x|\lambda) \frac{(e^t - 1)^n}{n!} &= \left(\frac{(1/\lambda)\log(1+\lambda t)}{(1+\lambda t)^{1/\lambda} - 1} \right)^r (1+\lambda t)^{x/\lambda} \\
&= \sum_{n=0}^{\infty} B_n^{(r)}(x|\lambda) \frac{t^n}{n!}. \quad (53)
\end{aligned}$$

Hence, by (51)–(53), we obtain the following theorem.

Theorem 5. For $n \geq 0$, we have

$$D_n^{(r)}(x|\lambda) = \sum_{m=0}^n B_m^{(r)}(x|\lambda) S_1(n, m). \quad (54)$$

Moreover,

$$B_n^{(r)}(x|\lambda) = \sum_{m=0}^n D_m^{(r)}(x|\lambda) S_2(n, m). \quad (55)$$

In particular, if $r = 1$, then we know that Theorem 5 is a generalization of Theorem 1 and Theorem 2.

Note that for each $k, r \in \mathbb{N}$,

$$\begin{aligned} & \left(\frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{1/\lambda} - 1} \right)^r (1 + \lambda \log(1 + t))^{x/\lambda} \\ &= \left(\frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{1/\lambda} - 1} \right)^k \\ & \quad \cdot \left(\frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{1/\lambda} - 1} \right)^{r-k} (1 + \lambda \log(1 + t))^{x/\lambda} \\ &= \left(\sum_{n=0}^{\infty} D_n^{(k)}(\lambda) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} D_n^{(r-k)}(x|\lambda) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} D_m^{(k)}(\lambda) D_{n-m}^{(r-k)}(x|\lambda) \right) \frac{t^n}{n!}. \end{aligned} \quad (56)$$

It is well known that for each $k \in \mathbb{Z}$,

$$\left(\frac{t}{\log(1 + t)} \right)^k (1 + t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-k+1)}(x) \frac{t^n}{n!}. \quad (57)$$

By (23), (7), and (57), we have

$$\begin{aligned} & \left(\frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{1/\lambda} - 1} \right)^r (1 + \lambda \log(1 + t))^{x/\lambda} \\ &= \left(\frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{1/\lambda} - 1} \right)^{-r} (1 + \lambda \log(1 + t))^{x/\lambda} \\ &= \sum_{n=0}^{\infty} B_n^{(n+r+1)}(x+1) \frac{1}{n!} ((1 + \lambda \log(1 + t))^{1/\lambda} - 1)^n \\ &= \left(\sum_{n=0}^{\infty} B_n^{(n+r+1)}(x+1) \right) \left(\sum_{m=n}^{\infty} S_{2,\lambda}(m, n) \frac{(\log(1 + t))^m}{m!} \right) \\ &= \sum_{p=0}^{\infty} \sum_{n=0}^p B_n^{(n+r+1)}(x+1) S_{2,\lambda}(p, n) \frac{1}{p!} (\log(1 + t))^p \\ &= \sum_{q=0}^{\infty} \left(\sum_{p=0}^q \sum_{n=0}^p B_n^{(n+r+1)}(x+1) S_{2,\lambda}(q-l, n) S_1(q, p) \right) \frac{t^q}{q!}. \end{aligned} \quad (58)$$

By (56) and (58), we obtain the following theorem.

Theorem 6. For each $n, q \geq 0$, we have

$$\begin{aligned} D_n^{(r)}(x|\lambda) &= \sum_{m=0}^n \binom{n}{m} D_m^{(k)}(\lambda) D_{n-m}^{(r-k)}(x|\lambda), \\ D_q^{(r)}(x|\lambda) &= \sum_{p=0}^q \sum_{n=0}^p B_n^{(n+r+1)}(x+1) S_{2,\lambda}(q-l, n) S_1(q, p). \end{aligned} \quad (59)$$

Note that by (3),

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \lambda \log(1 + t))^{x+x_1+\cdots+x_r/\lambda} d\mu_0(x_1) \cdots d\mu_0(x_r) \\ &= \left(\frac{(1/\lambda) \log(1 + \lambda \log(1 + t))}{(1 + \lambda \log(1 + t))^{1/\lambda} - 1} \right)^r (1 + \lambda \log(1 + t))^{x/\lambda} \\ &= \sum_{n=0}^{\infty} D_n^{(r)}(x|\lambda) \frac{t^n}{n!}. \end{aligned} \quad (60)$$

By (17) and (60), we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \lambda \log(1 + t))^{x+x_1+\cdots+x_r/\lambda} d\mu_0(x_1) \cdots d\mu_0(x_r) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x+x_1+\cdots+x_r}{n} \lambda^n \\ & \quad \cdot (\log(1 + t))^n d\mu(x_1) \cdots d\mu(x_r) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x+x_1+\cdots+x_r)_{n,\lambda} \frac{1}{n!} n! \sum_{l=n}^{\infty} S_1(l, n) \frac{x^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x+x_1+\cdots+x_r)_{m,\lambda} S_1(n, m) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n S_1(n, m) \beta_m^{(r)}(x) \right) \frac{t^n}{n!}. \end{aligned} \quad (61)$$

Thus, by (60) and (61), we obtain the following theorem which shows that higher-order degenerate Daehee polynomials of the second kind are represented by linear combination of the higher-order Carlitz's type degenerate Bernoulli polynomials.

Theorem 7. For each nonnegative integer n and each integer r ,

$$D_n^{(r)}(x|\lambda) = \sum_{m=0}^n S_1(n, m) \beta_m^{(r)}(x). \quad (62)$$

By (7) and (23), we have

$$\begin{aligned}
& (1 + \lambda \log(1+t))^{x+x_1+\dots+x_r/\lambda} \\
&= \left((1 + \lambda \log(1+t))^{1/\lambda} - 1 + 1 \right)^{x+x_1+\dots+x_r} \\
&= \sum_{m=0}^{\infty} \binom{x+x_1+\dots+x_r}{m} \left((1 + \lambda \log(1+t))^{1/\lambda} - 1 \right)^m \\
&= \sum_{m=0}^{\infty} (x+x_1+\dots+x_r)_m \sum_{n=m}^{\infty} S_{2,\lambda}(n, m) \frac{(\log(1+t))^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^n (x+x_1+\dots+x_r)_m S_{2,\lambda}(n, m) \frac{(\log(1+t))^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^n (x+x_1+\dots+x_r)_m S_{2,\lambda}(n, m) \frac{1}{n!} \sum_{l=n}^{\infty} S_1(l, n) \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{m=0}^k (x+x_1+\dots+x_r)_m S_{2,\lambda}(k, m) S_1(n, k) \right) \frac{t^n}{n!}, \quad (63)
\end{aligned}$$

and so by (22), (60), and (63), we obtain

$$\begin{aligned}
& \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \lambda \log(1+t))^{x+x_1+\dots+x_r/\lambda} d\mu_0(x_1) \cdots d\mu_0(x_r) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{m=0}^k S_{2,\lambda}(k, m) S_1(n, k) \int_{\mathbb{Z}_p} \right. \\
& \quad \left. \cdots \int_{\mathbb{Z}_p} (x+x_1+\dots+x_r)_m d\mu_0(x_1) \cdots d\mu_0(x_r) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{m=0}^k S_{2,\lambda}(k, m) S_1(n, k) D_m^{(r)}(x) \right) \frac{t^n}{n!}. \quad (64)
\end{aligned}$$

By (60) and (64), we obtain the following theorem.

Theorem 8. For each nonnegative integer m ,

$$D_n^{(r)}(x|\lambda) = \sum_{k=0}^n \sum_{m=0}^k S_{2,\lambda}(k, m) S_1(n, k) D_m^{(r)}(x). \quad (65)$$

Theorem 8 shows that higher-order degenerate Daehee polynomials are related closely to Daehee polynomials of order r .

4. Conclusion

In the past two decades, the degenerations of special functions and their applications have been studied as a new area of mathematics. In this paper, we considered the degenerate Daehee numbers and polynomials by using p -adic invariant integral on $\mathbb{Z}_p$ which are different from Kim's

degenerate Daehee polynomials. We derive some new and interesting properties of those polynomials.

Next, from the definition of the higher-order degenerate Daehee numbers and Daehee polynomials of the second kind, we found the relationship between the degenerate Bernoulli polynomials, the first and second Stirling numbers, the Bernoulli polynomials, degenerate Stirling numbers of the second kind, and those numbers and polynomials.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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