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Research Article

Reduction to a Canonical Form of a Third-Order Polynomial Matrix with One Characteristic Root by means of Semiscalarly Equivalent Transformations

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For the selected class of polynomial matrices of order three with one characteristic root with respect to the transformation of semiscalar equivalence, special triangular forms are established. The theorems of their uniqueness are proved. This gives reason to consider such canonical forms.

1. Introduction

In [1], it is proved that the matrix $F(x) \in M(n, C[x])$ of full rank by means of transformation:

$$F(x) \longrightarrow PF(x)Q(x) = G(x), \qquad (1)$$

where $P \in GL(n, C)$ and $Q(x) \in GL(n, C[x])$ can be reduced to the lower triangular form with invariant factors on the principal diagonal. Subdiagonal elements in a matrix of this form are ambiguously defined. The matrices $F(x), G(x)$ which are related by the transformation (1) are called semiscalarly equivalent [1]. In [2], the specified triangular form for polynomial 3×3 matrices with one characteristic root is a little simplified. The resulting matrix of a simplified triangular form is called a *reduced*. In [2], the invariants of the reduced matrix are established. In particular, the invariance of the location of zero subdiagonal elements is proved. In [3], the reduced matrix, if there are some zero elements under its principal diagonal, by means of transformations of the form (1) (i.e., by means of *semiscalarly equivalent transformations*) is reduced to such matrices, which are uniquely defined. This gives grounds to consider the obtained matrices canonical for the selected class of matrices. This article introduces canonical forms for reduced matrices with all nonzero subdiagonal elements.

2. Previous Information

Here are some definitions and notations that will be used in this article, which are known from [2, 3]. This applies to the definitions of the *younger degree*, *of the younger term, of the younger coefficient*, *q-monomial, and q-coefficient of the* polynomial and others. For example, monomial $4x^2$ and its degree 2 are, respectively, a younger term and younger degree of polynomial $f(x) = -3x^7 + 6x^5 - x^4 + 4x^2$, and 4 is the younger coefficient of this polynomial. Monomial $6x^5$and its coefficient 6 are, respectively, a 5-monomial and 5-coefficient of polynomial $f(x)$.

Let all the roots of the characteristic polynomial $\det F(x)$ (= characteristic roots) of the matrix $F(x)$ be equal to each other; that is, the matrix $F(x)$ has only one (without taking into account the multiplicity) characteristic root. Without loss of generality, we assume that the only characteristic root is zero and the first invariant factor of matrix $F(x)$ is equal to one. With such assumptions, it is proved in [2] that, by means of semiscalarly equivalent transformations, the matrix $F(x)$ is reduced to the matrix of the form

$$A(x)=\begin{Vmatrix}1 & 0 & 0\\ a_1(x) & x^{k_1} & 0\\ a_3(x) & a_2(x) & x^{k_2}\end{Vmatrix}, \tag{2}$$

which satisfies the following conditions:

(1) $\deg a_1 < k_1$, $\deg a_2, \deg a_3 < k_2$, $a_2(x) = x^{k_1} a_2'(x) = 0$, $a_1(0) = a_3(0) = a_2'(0) = 0$.

(2) $co\deg a_3 \neq co\deg a_1$, $co\deg a_3 \neq co\deg a_2'$, if $co\deg a_3 < co\deg a_2$.

(3) $co\deg a_3 \neq 2co\deg a_1 + co\deg a_2'$ and $(2co\deg a_1)$-monomial in $a_1(x)$ is absent if $co\deg a_3 \geq co\deg a_2$.

(4) Younger coefficients in $a_1(x)$ and $a_2(x)$ are units.

The matrix $A(x)$ of the form (2) with conditions (1)–(4) in [2] is called the reduced matrix. Next, we consider the situation where the last two invariant multipliers of the matrix $A(x)$ do not coincide, that is, $k_1 < k_2$. The case $k_1 = k_2$ was considered in [4]. The notation $A(x) \approx B(x)$ means that the matrices $A(x)$ and $B(x)$ are semiscalarly equivalent. It should be noted that the problem of classification with respect to semiscalar equivalence of matrices of the second order is solved in the article [5]. Thus, this article discusses other situations that differ from [4, 5]. In [2], it is proved that, in case $A(x) \approx B(x)$, we can choose the left transformation matrix in the transition from $A(x)$ to the reduced matrix

$$B(x)=\begin{Vmatrix}1 & 0 & 0\\ b_1(x) & x^{k_1} & 0\\ b_3(x) & b_2(x) & x^{k_2}\end{Vmatrix}, \tag{3}$$

of the lower triangular form. We will then apply semiscalarly equivalent transforms $A(x) \longrightarrow SA(x)R(x) = B(x)$ to the matrix $A(x)$ to obtain a reduced matrix $B(x)$ of the form (3) with predefined properties. Let us show that, by a given reduced matrix $A(x)$ of form (2) and a matrix

$$S=\begin{Vmatrix}1 & s_{12} & s_{13}\\ 0 & 1 & s_{23}\\ 0 & 0 & 1\end{Vmatrix}, \tag{4}$$

we can find the matrix $B(x)$ and the right transformation matrix $R(x)$ so that $A(x) \approx B(x) = SA(x)R(x)$. Using the method of uncertain coefficients for given elements $a_1(x), a_2(x), a_3(x)$ and s_{12}, s_{13}, s_{23} of matrices $A(x)$ and S, respectively, with congruence

$$\begin{aligned}&a_1(x) + s_{23}a_3(x) - b_1(x)\left(1 + s_{12}a_1(x) + s_{13}a_3(x)\right)\\ &\quad \equiv 0\left(\bmod x^{k_1}\right),\end{aligned} \tag{5}$$

we find $b_1(x) \in C[x]$, $\deg b_1 < k_1$. We denote such elements by $r(x)_{uv}, u, v = 1, 2$:

$$\begin{aligned}
r_{11}(x) &= 1 + s_{12}a_1(x) + s_{13}a_3(x),\\
r_{12}(x) &= s_{12}x^{k_1} + s_{13}a_2(x),\\
r_{21}(x) &= \frac{a_1(x) + s_{23}a_3(x) - b_1(x)r_{11}(x)}{x^{k_1}} \in C[x],\\
r_{22}(x) &= 1 + s_{23}a_2'(x) - s_{12}b_1(x) - s_{13}b_1(x)a_2'(x).
\end{aligned} \tag{6}$$

Here $a_2'(x) = a_2(x)/x^{k_1} \in C[x]$. We form the matrix $\|r(x)_{uv}\|_1^2$ and consider the congruence

$$\left\| b_3(x) \;\; b_2(x) \right\| \left\| r_{uv}(x) \right\|_1^2 \equiv \left\| a_3(x) \;\; a_2(x) \right\| \left(\bmod x^{k_2}\right), \tag{7}$$

with the unknown $b_2(x), b_3(x)$. Since the free member of a matrix polynomial $\|r(x)_{uv}\|_1^2$ is a unit matrix, we can use the method of uncertain coefficients to solve this congruence and find $b_2(x), b_3(x) \in C[x]$, $\deg b_2, \deg b_3 < k_2$. We can check that $b_2'(x) = b_2(x)/x^{k_1} \in C[x]$. In addition to the above, we also denote

$$\begin{aligned}
r_{13}(x) &= s_{13}x^{k_2},\\
r_{23}(x) &= s_{23}x^{k_2-k_1} - b_1(x)r_{13}(x),\\
r_{31}(x) &= \frac{a_3(x) - b_3(x)r_{11}(x) - b_2(x)r_{21}(x)}{x^{k_2}} \in C[x],\\
r_{32}(x) &= \frac{a_2(x) - b_3(x)r_{12}(x) - b_2(x)r_{22}(x)}{x^{k_2}} \in C[x],\\
r_{33}(x) &= \frac{1 - b_3(x)r_{13}(x) - b_2(x)r_{23}(x)}{x^{k_2}} \in C[x].
\end{aligned} \tag{8}$$

By the above $r_{ij}(x)\, i, j = 1, 2, 3$, and from the congruences (5) and (7) $b_i(x)$, we construct $\|r_{ij}(x)\|_1^3$ and a matrix $B(x)$ of the form (3), respectively. We can be convinced of equality $SA(x) = B(x)\|r_{ij}(x)\|_1^3$. This means that $\|r_{ij}(x)\|_1^3$ is reversible and its inverted matrix together with the matrix S reduces $A(x)$ to $B(x)$. If the matrix S (4) in the transition from $A(x)$ to $B(x)$ has one of the following views:

$$\begin{Vmatrix}1 & 0 & 0\\ 0 & 1 & s_{23}\\ 0 & 0 & 1\end{Vmatrix},\quad \begin{Vmatrix}1 & s_{12} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{Vmatrix},\quad \text{or} \begin{Vmatrix}1 & 0 & s_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{Vmatrix}, \tag{9}$$

then we will say that transformations of type I, transformations of type II, or transformations of type III, respectively, are applied to the matrix $A(x)$. We shall use the following notation for matrices $A(x)$ of form (2) and $B(x)$ of form (3):

$$\begin{aligned}\delta_A(x):&=\det\begin{Vmatrix} a_1(x) & 1 \\ a_3(x) & a_2'(x)\end{Vmatrix},\\ \delta_B(x):&=\det\begin{Vmatrix} b_1(x) & 1 \\ b_3(x) & b_2'(x)\end{Vmatrix}.\end{aligned} \tag{10}$$

3. The Main Results

Theorem 1. *Suppose that in the reduced matrix $A(x)$ of the form (2), we have $a_1(x), a_2(x), a_3(x)\neq 0$, $a_2(x)=x^{k_1}a_2'(x)$, $q_1:=codega_1$, $q_2:=codega_2'$, $q_3:=codega_3$, $n_j:=\left\{\begin{matrix} q_3, j=1 \\ q_2+q_3, j=3\end{matrix}\right\}$, $m_j:=q_j+q_3$, $j=1,3$. Then, the matrix $A(x)$ is semiscalarly equivalent to the reduced matrix $B(x)$ of the form (3), where elements $b_1(x), b_2(x), b_3(x)\neq 0$ satisfy one of the following conditions:*

(1) $(2q_3)$-monomial is absent in $b_3(x)$, if $q_3<q_1$ and $q_3<q_2$.

(2) $(2q_3)$- and (q_1+q_3)-monomials are absent in $b_3(x)$ if $q_3<q_1$ and $q_3>q_2$.

(3) If $q_3>q_1$ and $q_3<q_2$, then in the first of polynomials $b_j(x)$, $j=1,3$, which satisfies condition $n_j<k_j$, n_j-monomial is absent, and in the first of these polynomials, which satisfies condition $m_j<k_j$, m_j-monomial is absent.

The matrix $B(x)$ is uniquely defined.

Proof. Existence.

(1) If $2q_3\geq k_2$, then $A(x)$ is the desired matrix. Otherwise, we denote by d_0 and d_1, respectively, the lower coefficient and the $(2q_3)$-coefficient of the polynomial $a_3(x)$ and apply to $A(x)$ transformations of the type III. In the left transformation matrix (see (9)), we put $s_{13}=d_1/d_0^2$. The elements $b_i(x)$, $i=1,2,3$, of the obtained in this way matrix $B(x)$ satisfy the congruence:

$$a_1(x)-b_1(x)-s_{13}a_3(x)b_1(x)\equiv 0\left(\operatorname{mod}x^{k_1}\right), \tag{11}$$

$$a_2(x)-b_2(x)-s_{13}a_2(x)\delta_B(x)\equiv 0\left(\operatorname{mod}x^{k_2}\right), \tag{12}$$

$$\begin{aligned}a_3(x)+\delta_B(x)+s_{13}a_3(x)\delta_B(x)\\ -a_1(x)b_2'(x)\equiv 0\left(\operatorname{mod}x^{k_2}\right).\end{aligned} \tag{13}$$

First, we obtain from (11) and (12) that the lower terms in $b_1(x), b_2(x)$ are identical with the lower terms in $b_1(x), b_2(x)$, respectively. Further note that the younger terms in $\delta_B(x)$ and $b_3(x)$ coincide, the lower degrees of the last two additions in the left-hand side (13) exceed $codegb_3=q_3$, and inequality $codeg(a_3(x)\delta_B(x))=2q_3<codeg(a_1(x)b_2'(x))$ holds. Therefore, by comparing the $(2q_3)$-coefficients in both parts of (13), we obtain zero for such $(2q_3)$-coefficient in $b_3(x)$. So, there $B(x)$ is a desired matrix.

(2) If in matrix $A(x)$ has $2q_3\geq k_2$, then everything is proven—this matrix is the desired one. Otherwise, we will apply to it the transformation mentioned in Section 1. To show the absence of the $(2q_3)$-monomial in $b_3(x)$, one must take into account that $codeg(a_1(x)-b_1(x))\geq q_1+q_3$ (see (11)).Therefore, in (13), we have $codeg(b_2'(x)(a_1(x)b_1(x))>2q_3)$. The remaining considerations are the same as in paragraph 1. In order to not introduce new notations, we further assume that there is no $(2q_3)$-monomial in the element $a_3(x)$ of original matrix $A(x)$. If $q_1+q_3\geq k_2$, then everything is proven—the matrix $A(x)$ is the desired one. Otherwise to $A(x)$, we apply transformation of the type II. At the same time, in the left transformation matrix (see (9)), we put $s_{12}=d_2/d_0$, where d_0 and d_2 are, respectively, the lower coefficient and (q_1+q_3)-coefficient of the polynomial $a_3(x)$. The elements $b_i(x)$, $i=1,2,3$, of the reduced matrix $B(x)$ obtained in this way satisfy the congruence:

$$a_1(x)-b_1(x)-s_{12}a_1(x)b_1(x)\equiv 0\left(\operatorname{mod}x^{k_1}\right), \tag{14}$$

$$a_2(x)-b_2(x)+s_{12}\Delta_B(x)\equiv 0\left(\operatorname{mod}x^{k_2}\right), \tag{15}$$

$$\begin{aligned}a_3(x)+\delta_B(x)+s_{12}a_1(x)\delta_B(x)\\ -a_1(x)b_2'(x)\equiv 0\left(\operatorname{mod}x^{k_2}\right).\end{aligned} \tag{16}$$

It can be seen from (14) and (15) that the younger terms in $b_1(x), b_2(x)$ are the same as the lower terms in $a_1(x), a_2(x)$, respectively (their coefficients are equal to one).

Let us write (16) as follows:

$$\begin{aligned}a_3(x)-b_3(x)+s_{12}a_1(x)\left(b_1(x)b_2'(x)-b_3(x)\right)-\\ -\left(a_1(x)-b_1(x)\right)b_2'(x)\equiv 0\left(\operatorname{mod}x^{k_2}\right).\end{aligned} \tag{17}$$

From (14), we have $codeg(a_1(x)-b_1(x))\geq 2q_1$. Because

$$\begin{aligned}codeg\left(b_2'(x)a_1(x)b_1(x)\right)&=2q_1+q_2,\\ codeg\left(b_2'(x)\left(a_1(x)-b_1(x)\right)\right)&\geq 2q_1+q_2,\end{aligned} \tag{18}$$

and $codeg(a_1(x)b_3(x))=q_1+q_3<2q_1+q_2$, then by comparing the (q_1+q_3)-coefficients in both parts of (17), we find that $b_3(x)$ contains no (q_1+q_3)-monomial. And because $2q_3<q_1+q_3$, then in $b_3(x)$, as in $a_3(x)$, there is no $(2q_3)$-monomial.

(3) Suppose that conditions $q_3 > q_1$ and $q_3 < q_2$ are satisfied in matrix $A(x)$.

(1) If $q_3 \geq k_1$ and $2q_3 \geq k_2$, then all is proved—matrix $A(x)$ is the desired one.

(2) Let $q_3 \geq k_1$ and $2q_3 < k_2$. Since $q_3 < co\deg a_2$, then q_3 (as well as q_1) is invariant (see Proposition 6 [2]). We apply to $A(x)$ the transformation specified in Section 1. As a result, we obtain the matrix $B(x)$ in the form (3). Its elements satisfy the congruences (11)–(13). Since $q_1 + q_3 > k_1$, then from (11), we have $a_1(x) = b_1(x)$. It can be seen from (12) that $a_2(x) = b_2(x)$. Now we can represent (13) as

$$a_3(x) - b_3(x) - s_{13}a_3(x)\left(b_1(x)b_2'(x) - b_3(x)\right) \equiv 0\left(\bmod x^{k_2}\right). \tag{19}$$

From the last congruence, we have $co\deg b_3 = q_3$. Therefore, $B(x)$ is a reduced matrix. If we take into account $q_1 + q_2 > q_3$, then by comparing the $(2q_3)$-coefficients at both times (19), we will conclude that, in $b_3(x)$, there is no $(2q_3)$-monomial. If $q_2 + q_3 \geq k_2$, then everything is proved-matrix $B(x)$ is the desired one. Otherwise, we take another step. In order not to introduce new notations, we will assume that element $a_3(x)$ of matrix $A(x)$ does not contain $(2q_3)$-monomial. We apply to $A(x)$ transformations of the type I. In the left transformation matrix (see (9)), we put $s_{23} = d_2/d_0$, where d_0 and d_2 are, respectively, the lower coefficient and the $(q_1 + q_3)$ coefficient of polynomial $a_3(x)$. The elements $b_i(x)$, $i = 1, 2, 3$ of the resulting matrix $B(x)$ satisfy the congruence:

$$a_1(x) - b_1(x) + s_{23}a_3(x) \equiv 0\left(\bmod x^{k_1}\right), \tag{20}$$

$$a_2(x) - b_2(x) - s_{23}a_2'(x)b_2(x) \equiv 0\left(\bmod x^{k_2}\right), \tag{21}$$

$$a_3(x) - b_3(x) + (b_1(x) - a_1(x))b_2'(x) - s_{23}b_2'(x)a_3(x) \equiv 0\left(\bmod x^{k_2}\right). \tag{22}$$

From (20), we have that $a_1(x) = b_1(x)$, and from (21), it follows $co\deg a_2 = co\deg b_2$. Then, from (22), we get

$$a_3(x) - b_3(x) - s_{23}b_2'(x)a_3(x) \equiv 0\left(\bmod x^{k_2}\right), \tag{23}$$

where we can get $co\deg b_3 = q_3$. Comparing the coefficients in both parts of (23), we conclude that there is no the monomial of degree $q_1 + q_3$ in polynomial $b_3(x)$. At the same time, in $b_3(x)$, as in $a_3(x)$, there is no $(2q_3)$-monomial.

(3) Now suppose that, in matrix $A(x)$, we have $q_3 < k_1$ and $2q_3 \geq k_2$. Apply to $A(x)$ transformations of the type I. In the left transformation matrix (see (9)), we put $s_{23} = -d_3/d_0$, where d_0 and d_3 are, respectively, the lower coefficient of the polynomial $a_3(x)$ and the q_3-coefficient of the polynomial $a_1(x)$. As a result, we obtain a matrix $B(x)$ of the form (3) whose elements $b_i(x)$, $i = 1, 2, 3$ satisfy the congruences (20)–(22). From (20), we have that $co\deg b_1 = q_1$ and q_3-monomial in $b_1(x)$ is absent. From (21), $co\deg b_2 = co\deg a_2$, and from (22), we have $co\deg b_3 = q_3$. It also follows from (20)–(22) that the lower coefficients in $a_i(x)$ and $b_i(x)$, $i = 1, 2, 3$ coincide. That is, $B(x)$ is a reduced matrix. If $q_1 + q_3 \geq k_1$, then everything is already proven. Then, $B(x)$ is the desired matrix. Otherwise, in order to not introduce new notations, we consider the q_3-coefficient in the element $a_1(x)$ of the matrix $A(x)$ null. Denote by d_0 and d_4, respectively, the lower polynomial coefficient of the polynomial $a_3(x)$and $(q_1 + q_3)$-coefficient of the polynomial $a_1(x)$. We perform over the matrix $A(x)$ transformation of the type III. For this, we put $s_{13} = d_4/d_0$ in the left transformation matrix (see (9)). The elements of the resulting matrix $B(x)$ satisfy the congruences (11)–(13). From (21), we obtain that $co\deg b_1 = q_1$, the lower coefficient of the polynomial $b_1(x)$ is 1, and its q_3- and $(q_1 + q_3)$-coefficients are zero. From (12), it is seen that the lower coefficient in $b_2(x)$, as in $a_2(x)$, is equal to 1. Therefore, the matrix $B(x)$ has the necessary properties.

(4) Let $q_3 < k_1$ and $2q_3 < k_2$. We can assume that the q_3-coefficient in $a_1(x)$ of matrix $A(x)$ is zero. If this is not the case, then to $A(x)$, we will apply transformation of the type I described in Section 3. If $q_1 + q_3 < k_1$, then to $A(x)$, we apply transformation of the type III described in Section 3. Then, the resulting matrix will be zero $(q_1 + q_3)$-coefficient and will remain zero q_3-coefficient of the polynomial in position $(2, 1)$. If $q_1 + q_3 \geq k_1$, then from the matrix $A(x)$ by means of transformations of the type III referred to in item 1, we go to the redundant matrix $B(x)$, in which $2q_3$-monomial of polynomial $b_3(x)$ is absent. Then, q_3-factor in $b_1(x)$ will also remain zero. This proves the first part of the theorem (existence). □

3.1. Uniqueness of the Matrix in Theorem 1

(1) Suppose that, for the reducible matrices $A(x)$, $B(x)$of forms (2) and (3), condition 1 of theorem holds, and, in addition, we have $A(x) \approx B(x)$. Then, the left transformative matrix S in the equality $SA(x)R(x) = B(x)$ can be chosen in the form (9) (see Corollary 1 and Remark 1 [2]) and elements $a_i(x)$ and $b_i(x)$, $i = 1, 2, 3$, of these matrices satisfy the congruence

$$a_3(x) + \delta_B(x) - s_{13}a_3(x)\delta_B(x) - a_1(x)b_2'(x) \equiv 0\left(\bmod x^{k_2}\right). \tag{24}$$

We have $co\deg(\delta_B(x)a_3(x)) = 2q_3 < q_1 + q_2$. If $2q_3 \geq k_2$, then from (24), $a_3(x) - b_3(x) \equiv 0\left(\bmod x^{k_2}\right)$ follows. Otherwise, in (24), we have $s_{13} = 0$ since

$(2q_3)$-monomials in $a_3(x)$ and $b_3(x)$ are absent. In any case, $A(x) = B(x)$.

(2) Suppose that the reduced matrices $A(x)$, $B(x)$ of the forms (2) and (3) satisfy condition 2 of theorems and $A(x) \approx B(x)$. Then, in the left transformative matrix $S(4)$, in the transition from $A(x)$ to $B(x)$, we have $s_{23} = 0$ (see Corollary 1 and Remark 1 [2]) and the elements $a_i(x)$ and $b_i(x)$ of the matrices $A(x)$ and $B(x)$ satisfy the congruences:

$$\begin{aligned} a_1(x) - b_1(x)\left(1 + s_{12}a_1(x) + s_{13}a_3(x)\right) &\equiv 0\left(\operatorname{mod} x^{k_1}\right), \\ a_2(x) - b_2(x) + s_{12}\Delta_B(x) + s_{13}a_2'(x)\Delta_B(x) &\equiv 0\left(\operatorname{mod} x^{k_2}\right), \\ a_3(x) + \delta_B(x) + \delta_B(x)\left(s_{12}a_1(x) + s_{13}a_3(x)\right) - a_1(x)b_2'(x) &\equiv 0\left(\operatorname{mod} x^{k_2}\right). \end{aligned} \tag{25}$$

From (25), we can write

$$\begin{aligned} &a_3(x) - b_3(x) + \delta_B(x)\left(s_{12}a_1(x) + s_{13}a_3(x)\right) \\ &\quad + b_2'(x)\left(b_1(x) - a_1(x)\right) \equiv 0\left(\operatorname{mod} x^{k_2}\right). \end{aligned} \tag{26}$$

From (25), we have $\operatorname{codeg}(a_1(x) - b_1(x)) \geq q_1 + q_3$. It is easy to see that

$$\begin{aligned} \operatorname{codeg}\left(\delta_B(x)a_3(x)\right) &= 2q_3 < q_1 + q_3 = \\ &= \operatorname{codeg}\left(\delta_B(x)a_1(x)\right) \\ &< \operatorname{codeg}\left(b_2'(x)\left(b_1(x) - a_1(x)\right)\right). \end{aligned} \tag{27}$$

If $2q_3 \geq k_2$, then from (26), we have $a_3(x) - b_3(x) \equiv 0(\operatorname{mod} x^{k_2})$; hence, it follows $a_3(x) = b_3(x)$.

Since $2q_3 < \operatorname{codeg}\Delta_A < \operatorname{codeg}(a_2'(x)\Delta_A(x))$, then from (26), $a_2(x) - b_2(x) \equiv 0(\operatorname{mod} x^{k_2})$ follows, whence $a_2(x) = b_2(x)$. From (25), taking into account $2q_3 < q_3 + q_1 < \operatorname{codeg}(a_1(x))^2$, we get $a_1(x) - b_1(x) \equiv 0(\operatorname{mod} x^{k_1})$ from where $a_1(x) = b_1(x)$. So, we have $A(x) = B(x)$.

If $2q_3 < k_2$, then from (25), we get $s_{13} = 0$. If $q_1 + q_3 \geq k_2$, then taking into account $q_1 + q_3 < \operatorname{codeg}\delta_A + k_1$ and $q_1 + q_3 < 2q_1$ from (25) and (26), we have $a_j(x) - b_j(x) \equiv 0(\operatorname{mod} x^{k_j})$, $j = 1, 2$, and $a_3(x) - b_3(x) \equiv 0(\operatorname{mod} x^{k_2})$. Therefore, $A(x), B(x)$ coincide. If $q_1 + q_3 < k_2$, then from (26), we obtain $s_{12} = 0$. Hence, in this case, the matrices $A(x), B(x)$ also coincide.

(3) Suppose that the reduced matrices $A(x)$, $B(x)$ of the forms (2) and (3) satisfy condition 3 of theorems and $A(x) \approx B(x)$. Then, for the elements of these matrices, we can write the congruences:

$$\begin{aligned} a_1(x) - b_1(x) + s_{23}a_3(x) - s_{13}a_3(x)b_1(x) &\equiv 0\left(\operatorname{mod} x^{k_1}\right), \\ a_2(x) - b_2(x) - s_{23}a_2'(x)b_2(x) + s_{13}a_2(x)\delta_B(x) &\equiv 0\left(\operatorname{mod} x^{k_2}\right), \\ a_3(x) + \delta_B(x) - a_3(x)\left(s_{23}b_2'(x) + s_{13}\delta_B(x)\right) - a_1(x)b_2'(x) &\equiv 0\left(\operatorname{mod} x^{k_2}\right). \end{aligned} \tag{28}$$

If $q_3 \geq k_1$, then $q_1 + q_3 > k_1$, and from (28), we get $a_1(x) = b_1(x)$. Then, (28) will take the form

$$a_3(x) - b_3(x) - a_3(x)\left(s_{23}b_2'(x) + s_{13}\delta_B(x)\right) \equiv 0\left(\operatorname{mod} x^{k_2}\right). \tag{29}$$

Obviously, $\operatorname{codeg}(a_3(x)\delta_B(x)) = 2q_3$. If $2q_3 \geq k_2$, then (29) implies $a_3(x) = b_3(x)$ since $\operatorname{codeg}(a_3(x)b_2'(x)) > \operatorname{codeg}(a_3(x)\delta_B(x))$.

Then, from (28), we get $a_2(x) = b_2(x)$ since

$$\begin{aligned} \operatorname{codeg}\left(b_2(x)a_2'(x)\right) &> \operatorname{codeg}\left(b_2'(x)a_3(x)\right), \\ \operatorname{codeg}\left(\Delta_B(x)a_2'(x)\right) &> \operatorname{codeg}\left(\delta_B(x)a_3(x)\right). \end{aligned} \tag{30}$$

If $2q_3 < k_2$, then (29) implies $s_{13} = 0$. If, moreover, $\operatorname{codeg}(a_3(x)b_2'(x)) < k_2$, then from (29), it yields $s_{23} = 0$ and all is proved. If $\operatorname{codeg}(a_3(x)b_2'(x)) \geq k_2$, then all the same from (28) and (29), we have $a_3(x) = b_3(x)$ and $a_2(x) = b_2(x)$, respectively.

If $q_3 < k_1$, then from (28), we get $s_{23} = 0$. If in addition $q_1 + q_3 < k_1$, then from (28), it follows also $s_{13} = 0$ and all is proved. If $q_1 + q_3 \geq k_1$, then $a_1(x) = b_1(x)$, and again from (28), we go to (29). It follows from this that $s_{13} = 0$, if $\operatorname{codeg}(a_3(x)\delta_B(x)) < k_2$. And if $\operatorname{codeg}(a_3(x)\delta_B(x)) \geq k_2$, then immediately from (28) and (29), we have $a_3(x) = b_3(x)$ and $a_2(x) = b_2(x)$, respectively. Theorem is proved.

Suppose that, in the reduced matrices $A(x), B(x)$ of the forms (2) and (3), we have $a_1(x), a_2(x), a_3(x), b_1(x), b_2(x), b_3(x) \neq 0$. Let us keep the notation given in theorem:

$$\begin{aligned} q_1 &:= \operatorname{codeg} a_1, \\ q_2 &:= \operatorname{codeg} a_2', \\ q_3 &:= \operatorname{codeg} a_3, \\ a_2'(x) &= \frac{a_2(x)}{x^{k_1}} \in C[x], \\ b_2'(x) &= \frac{b_2(x)}{x^{k_1}} \in C[x]. \end{aligned} \tag{31}$$

We define polynomials:

$$\begin{aligned} a_{11}(x) &:\equiv (a_1(x))^2 \left(\operatorname{mod} x^{k_1}\right), \\ a_{22}(x) &:\equiv (a_2'(x))^2 \left(\operatorname{mod} x^{k_2-k_1}\right), \\ a_{32}(x) &:\equiv a_3(x) a_2'(x) \left(\operatorname{mod} x^{k_2}\right), \\ a_{04}(x) &:\equiv \delta_A(x) \left(\operatorname{mod} x^{k_2-k_1}\right). \\ a_{14}(x) &:\equiv a_1(x)\delta_A(x) \left(\operatorname{mod} x^{k_2}\right), \\ a_{34}(x) &:\equiv a_3(x)\delta_A(x) \left(\operatorname{mod} x^{k_2}\right). \end{aligned} \tag{32}$$

From the coefficients of each of the polynomials $a_1(x)$, $a_3(x)$, and $a_{11}(x)$, we form, respectively, columns $\overline{a}_1$, $\overline{a}_{03}$, and $\overline{a}_{11}$ of height $k_1 - q_1$. In the first place, in these columns, we put q_1-coefficients, and below in order of increasing degrees, we place the rest of their coefficients, up to degree $k_1 - 1$ inclusive. We denote by $\overline{a}_2$, $\overline{a}_{22}$, and $\overline{a}_{04}$, the columns of height $k_2 - k_1 - q_2$, constructed from the coefficients of polynomials $a_2(x)$, $a_{22}(x)$, and $a_{04}(x)$, respectively. In the first place in each of these columns, we put q_2-coefficients. Below we place the rest of their coefficients (including zero) up to the degree $k_2 - k_1 - 1$. Similarly, from the coefficients of polynomials $a_3(x)$, $a_{32}(x)$, $a_{34}(x)$, and $a_{14}(x)$, we form columns $\overline{a}_3$, $\overline{a}_{32}$, $\overline{a}_{34}$, and $\overline{a}_{14}$ and height $k_2 - q_3$. Here, we also put in the first place q_3-coefficients, and then, in the order of increasing degrees, we place all other coefficients. In the last places, there will be $(k_2 - 1)$-coefficients. For $A(x)$, by the columns formed, we construct the matrices of the following form:

$$\begin{aligned} K_A &= \left\| \begin{matrix} \overline{a}_1 & \\ \overline{a}_2 & \\ \overline{a}_3 & K_{0A} \end{matrix} \right\|, \\ K_{0A} &= \left\| \begin{matrix} K_{1A} \\ K_{2A} \\ K_{3A} \end{matrix} \right\|, \end{aligned} \tag{33}$$

$$\begin{aligned} K_{1A} &= \left\| -\overline{a}_{03} \;\; \overline{0} \;\; \overline{a}_{11} \right\|, \\ K_{2A} &= \left\| \overline{a}_{22} \;\; \overline{0} \;\; -\overline{a}_{04} \right\|, \\ K_{3A} &= \left\| \overline{a}_{32} \;\; -\overline{a}_{34} \;\; -\overline{a}_{14} \right\|. \end{aligned} \tag{34}$$

In complete analogy for $B(x)$, we construct matrices of the following form:

$$\begin{aligned} K_B &= \left\| \begin{matrix} \overline{b}_1 & \\ \overline{b}_2 & \\ \overline{b}_3 & K_{0B} \end{matrix} \right\|, \\ K_{0B} &= \left\| \begin{matrix} K_{1B} \\ K_{2B} \\ K_{3B} \end{matrix} \right\|, \end{aligned} \tag{35}$$

$$\begin{aligned} K_{1B} &= \left\| -\overline{b}_{03} \;\; \overline{0} \;\; \overline{b}_{11} \right\|, \\ K_{2B} &= \left\| \overline{b}_{22} \;\; \overline{0} \;\; -\overline{b}_{04} \right\|, \\ K_{3B} &= \left\| \overline{b}_{32} \;\; -\overline{b}_{34} \;\; -\overline{b}_{14} \right\|. \end{aligned} \tag{36}$$

Obviously, in these matrices, each row consists of monomial coefficients of the same degrees.

Theorem 2. *Let in the reduced matrix $A(x)$ of the form (2), we have $a_1(x), a_2(x), a_3(x) \neq 0$, $q_3 > q_1$, $q_3 > q_2$ and $n_1 := q_1 + q_3$, $n_2 := q_2 + \operatorname{codeg}\delta_A + k_1$. Then, $A(x) \approx B(x)$, where in the reduced matrix $B(x)$ of the form (3), all elements $b_1(x), b_2(x), b_3(x)$ are nonzero, polynomial $b_3(x)$ does not contain n_1-monomial if $n_1 < k_1$, and polynomial $\delta_B(x)$ does not contain $(n_2 - k_1)$-monomial if $n_2 < k_2$.*

In addition, one of the following conditions is true:

(1) In $b_1(x)$, n_1-monomial is absent, if $n_1 < k_1$ and $n_2 < k_2$.
(2) In $b_2(x)$, $(\operatorname{codeg}\delta_A + k_1)$- and n_2-monomials are absent, if $n_1 \geq k_1$ and $n_2 < k_2$.
(3) In $b_1(x)$, q_3- and n_1-monomials are absent, if $n_1 < k_1$ and $n_2 \geq k_2$.
(4) In the first column of the matrix K_B (35), the coefficients of the polynomials $b_1(x), b_2(x), b_3(x)$ are zero elements that correspond to the maximum system of the first linearly independent rows of the submatrix K_{0B}, if $n_1 \geq k_1$ and $n_2 \geq k_2$.

The matrix $B(x)$ is uniquely defined.

Proof. Existence. Let $n_1 < k_1$.

We apply to $A(x)$ transformation of the type II with the left transformation matrix of the form (9). At the same time, we put $s_{12} = d_1/d_0$, where d_0 is the younger coefficient and d_1 is the n_1-coefficient in $a_3(x)$. The elements $b_i(x)$, $i = 1, 2, 3$, of the thus obtained reduced matrix $B(x)$ satisfy the congruences (14)–(16). We write (16) in the form

$$a_3(x) - b_3(x) - s_{12}a_1(x)b_3(x) - b_2(x)r_{21}(x) \equiv 0\left(\operatorname{mod} x^{k_2}\right), \tag{37}$$

where $r_{21}(x) = a_1(x) - b_1(x) - s_{12}a_1(x)b_1(x)/x^{k_1} \in C[x]$. Comparing the n_1-coefficients in both parts of the last congruence, we have that $b_3(x)$ does not contain n_1-monomial. We further assume that element $a_3(x)$ of the matrix $A(x)$ does not contain n_1-monomial (if $n_1 < k_1$).

Let $n_2 < k_2$. Denote by c_0 and c_1, respectively, the junior and $(n_2 - k_1)$-coefficients of the polynomial $\delta_A(x)$. Apply to $A(x)$ transformations of the type I with the left transformation matrix of the form (9), while putting $s_{23} = c_1/c_0$.

The elements $b_i(x)$, $i = 1, 2, 3$, of the thus obtained reduced matrix $B(x)$ satisfy the congruences (20)–(22) (with the one listed here s_{23}). From (21) and (22), we obtain

$$\delta_A(x) - \delta_B(x) - s_{23}b_2'(x)\delta_A(x) \equiv 0\left(\mathrm{mod}\, x^{k_2 - k_1}\right). \tag{38}$$

If we compare the $(n_2 - k_1)$-coefficients in both parts of the last congruence, we will conclude that $\delta_B(x)$ does not contain n_2-monomial:

(1) Suppose that, in element $a_3(x)$ of matrix $A(x)$, there is no monomial of degree $n_1 < k_1$, and in polynomial $\delta_A(x)$, there is no monomial of degree $n_2 < k_2$. Denote by d_0 and c_1, respectively, the lower coefficient in $a_3(x)$ and n_1-coefficient in $a_1(x)$. With the help transformation of the type III, we pass from $A(x)$ to the reduced matrix $B(x)$. In the left transformation matrix (see (9)), we put $s_{13} = c_1/d_0$. The elements of the resulting matrix $B(x)$ satisfy the congruences (11)–(13) (with the one specified here s_{13}). From (11), we get that, in element $b_1(x)$, n_1-monomial is missing.

We write (13) in the form

$$\begin{aligned} a_3(x) - b_3(x) - s_{13}a_3(x)b_3(x) \\ - b_2(x)r_{21}(x) \equiv 0\left(\mathrm{mod}\, x^{k_2}\right), \end{aligned} \tag{39}$$

where $r_{21}(x) = a_1(x) - b_1(x) - s_{13}a_3(x)b_1(x)/x^{k_1} \in C[x]$. Since $2q_3 > n_1$, then, as seen from the last congruence, in $b_3(x)$, as in $a_3(x)$, there is no n_1-monomial. Also in $\delta_B(x)$, as in $\delta_A(x)$, there is no $(n_2 - k_1)$-monomial. This is evident from the congruence

$$\delta_A(x) - \delta_B(x) + s_{13}\delta_A(x)\delta_B(x) \equiv 0\left(\mathrm{mod}\, x^{k_2 - k_1}\right), \tag{40}$$

which is recorded on the basis of (12) and (13) since $co\deg(\delta_A(x)\delta_B(x)) > n_2 - k_1$. This proves the existence of matrix $B(x)$ with condition (1) specified in theorem.

(2) Suppose that conditions $n_1 \ge k_1$, $n_2 < k_2$, are satisfied in matrix $A(x)$, and $(n_2 - k_1)$-monomial is absent in polynomial $\delta_A(x)$. We denote by c_0 and d_2, respectively, the lower coefficient in $\delta_A(x)$ and the $(co\deg\delta_A + k_1)$-coefficient in $a_2(x)$. Let us do over matrix $A(x)$ transformation of the type II. To do this we put $s_{12} = -d_2/c_0$ in the left transformation matrix (see (9)). We obtain a reduced matrix $B(x)$ whose elements satisfy the congruences of the form (14)–(16) (with s_{12} indicated here). Taking into account that the lower coefficients in $\delta_A(x)$ and $\delta_B(x)$ coincide, then from (15) we find that $(co\deg\delta_A + k_1)$-monomial is absent in $b_2(x)$. From (15) and (16), we have $\delta_A(x) - \delta_B(x) \equiv 0\,(\mathrm{mod}\, x^{k_2 - k_1})$. It follows that, in $\delta_B(x)$, as in $\delta_A(x)$, there are no monomials of degree $(n_2 - k_1)$.

Next, we consider the absence of $(co\deg\delta_A + k_1)$-monomial in element $a_2(x)$ of the matrix $A(x)$. Denote by c_0 and d_3, respectively, the lower coefficient in $\delta_A(x)$ and n_2-coefficient in $a_2(x)$. Above the matrix $A(x)$, we carry out the transformation of the type III. Here, we put $s_{13} = d_3/c_0$ in the left transformation matrix (see (9)). The elements of the obtained reduced matrix $B(x)$ satisfy the congruences of the form (11)–(13) (with s_{13} indicated here). It can be seen from (12) that n_2-monomial is absent in $b_2(x)$. Also $(co\deg\delta_A + k_1)$-coefficient in $b_2(x)$ will remain zero since $co\deg\delta_A < n_2 - k_1$. As can be seen from (40), in $\delta_B(x)$, as in $\delta_A(x)$, $(n_2 - k_1)$-monomial is absent since $co\deg(\delta_A(x)\delta_B(x)) > n_2 - k_1$.

The existence of the required matrix $B(x)$ with condition (2) is proved.

(3) Let $n_2 \ge k_2$ for $A(x)$ and in $a_3(x)$ be absent monomial of degree $n_1 < k_1$. In the first step, we apply to the matrix $A(x)$ transformation of the type I with the left transformative matrix (see (9)), in which $s_{23} = -c_2/d_0$, where d_0 and c_2 are, respectively, the lower coefficient in the $a_3(x)$ and the q_3-coefficient in $a_1(x)$. As a result, we obtain a reduced matrix $B(x)$of the form (3) whose elements satisfy the conditions of the form (20)–(22) (with s_{23} selected here). From (20), it is seen that, in $b_1(x)$, the q_3-monomial is absent. From (20) and (22), it can be written as

$$a_3(x) - b_3(x) - b_2(x)r_{21}(x) \equiv 0\left(\mathrm{mod}\, x^{k_2}\right), \tag{41}$$

where $r_{21}(x) = a_1(x) - b_1(x) + s_{23}a_3(x)/x^{k_1} \in C[x]$. From the last congruence, it can be seen that n_1-monomial is absent in $b_3(x)$ as in $a_3(x)$.

Let already $a_1(x)$ in $A(x)$ not contain q_3-monomial. Denote by d_0 and c_3, respectively, the lower coefficient in $a_3(x)$ and the n_1-coefficient in $a_1(x)$ and let $s_{13} = c_3/d_0$. In the second step, with the help of the transformation of the type III with the specified s_{13} in the left transformative matrix (see (9)), we pass from $A(x)$ to some reduced matrix $B(x)$ of the form (3). For elements of the matrix $B(x)$, conditions (11)–(13) (with the specified here s_{13}) are satisfied. From (11), it follows that, in $b_1(x)$, there is no n_1-monomial. In addition, $b_1(x)$ does not contain q_3-monomial. On the basis of (11) and (13), we can write the congruence of the form (41) in which $r_{21}(x) = a_1(x) - b_1(x) - s_{13}a_3(x)b_1(x)/x^{k_1} \in C[x]$. It shows that, in $b_3(x)$, in comparison with $a_3(x)$, the zero coefficient of n_1-monomial is preserved. This proves the existence for the matrix $A(x)$ a semiscalarly equivalent reduced matrix $B(x)$ with condition 3.

(4) Suppose that conditions $n_1 \ge k_1$, $n_2 \ge k_2$, are satisfied in the reduced matrix $A(x)$. If $K_{0A} = 0$ in K_A (33),

then the desired matrix is $A(x)$ and everything is already proven. Otherwise, in the first step, we fix in the matrix K_{0A} the first nonzero row $\overline{u}_1 = \| d_{11}\ \ d_{12}\ \ d_{13} \|$ and the corresponding row $\| d_1\ \ d_{11}\ \ d_{12}\ \ d_{13} \|$ in K_A. Let $\overline{u}_1$ consist of h_1-coefficients and be the l_1-rd row in K_{0A}. We find an arbitrary solution $\| x_{10}\ \ x_{20}\ \ x_{30} \|^t$ of the equation

$$\| d_{11}\ \ d_{12}\ \ d_{13} \| \| x_1\ \ x_2\ \ x_3 \|^t = d_1. \tag{42}$$

We apply to $A(x)$ a semiscalarly equivalent transformation with the left transformative matrix S of the form (4). At the same time, in S, we put $s_{23} = x_{10}$, $s_{13} = x_{20}$, and $s_{12} = x_{30}$. The elements $b_i(x)$, $i = 1, 2, 3$, of the obtained reduced matrix $B(x)$ of the form (3) satisfy the congruence:

$$\begin{aligned}
a_1(x) - b_1(x) + s_{23}a_3(x) - s_{12}a_1(x)b_1(x) &\equiv 0\left(\operatorname{mod} x^{k_1}\right),\\
a_2'(x) - b_2'(x) - s_{23}a_2'(x)b_2'(x) + s_{12}\delta_B(x) &\equiv 0\left(\operatorname{mod} x^{k_2-k_1}\right),\\
a_3(x) - b_3(x) - s_{23}a_3(x)b_2'(x) + \left(s_{12}a_1(x) + s_{13}a_3(x)\right)\delta_B(x) &\equiv 0\left(\operatorname{mod} x^{k_2}\right).
\end{aligned} \tag{43}$$

Depending on which of the matrices K_{1A}, K_{2A}, or K_{3A} (see (34)) row $\overline{u}_1$ belongs, let us consider the congruence (43), respectively. By comparing the h_1-coefficients in both parts of that congruence, we conclude that the l_1-th element of the first column of matrix K_B (35) is zero. In addition, all rows in K_B, which precede the l_1-th, coincide with the corresponding rows of the matrix K_A.

If $rank K_{0A} = 1$, then everything is already proven. Matrix $B(x)$ is the desired one. Otherwise, we assume that the l_1-th element of the first column of the matrix K_A is zero. In the second step, we fix in K_{0A} the first linearly independent of $\overline{u}_1$ row $\overline{u}_2 = \| d_{21}\ \ d_{22}\ \ d_{23} \|$, as well as the corresponding to it row $\| d_2\ \ d_{21}\ \ d_{22}\ \ d_{23} \|$ in K_A and the degree h_2 of monomials, the coefficients of which form these rows. Also let $\overline{u}_2$ be the l_2-th row in K_{0A} $l_2 > l_1$.

We find some solution $\| y_{10}\ \ y_{20}\ \ y_{30} \|^t$ of the equation

$$\left\| \begin{matrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{matrix} \right\| \| y_1\ \ y_2\ \ y_3 \|^t = \| 0\ \ d_2 \|^t. \tag{44}$$

We apply to $A(x)$ a semiscalarly equivalent transformation with the left transformative matrix S of the form (4), putting $s_{23} = y_{10}$, $s_{13} = y_{20}$, and $s_{12} = y_{30}$. We obtain a reduced matrix $B(x)$ of the form (3).

Again, as in the previous step, we consider one of the congruences (43) depending on which of the matrices K_{1A}, K_{2A}, or K_{3A} (see (34)) contains row $\overline{u}_2$. In both parts of this congruence, we compare the coefficients of the h_2-monomials and conclude that the l_2-th element of the first column of the matrix K_B (34) is equal to zero. Also from this and the previous congruences, we get that every row preceding the l_2-th in K_A coincides with the corresponding row in K_B. If $rank K_{0A} = 2$, then everything is already proven. Then, matrix $B(x)$ is the desired one. Otherwise, in order to not introduce new designations, we assume that the first column of matrix K_A has zero l_1-th and l_2-th elements. In matrix K_{0A}, we fix the l_3-th row, which is the first linearly independent of $\overline{u}_1$, $\overline{u}_2$ $(l_3 > l_2 > l_1)$. Let this be line $\overline{u}_3 = \| d_{31}\ \ d_{32}\ \ d_{33} \|$. To him, K_A corresponds to $\| d_3\ \ d_{31}\ \ d_{32}\ \ d_{33} \|$. Also let h_3 be the exponent that corresponds to these rows. We find the (unique) solution $\| z_{10}\ \ z_{20}\ \ z_{30} \|^t$ of the equation

$$\left\| \begin{matrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{matrix} \right\| \| z_1\ \ z_2\ \ z_3 \|^t = \| 0\ \ 0\ \ d_3 \|^t. \tag{45}$$

We apply to $A(x)$ a semiscalarly equivalent transformation with the left transformation matrix S of the form (4) putting $s_{23} = z_{10}$, $s_{13} = z_{20}$, and $s_{12} = z_{30}$. We obtain the matrix $B(x)$. The above considerations show that $B(x)$ is the desired matrix. □

3.2. Uniqueness of the Matrix in Theorem 2. Suppose that, for the reduced matrices $A(x)$, $B(x)$ of forms (2) and (3), we have $A(x) \approx B(x)$. Suppose also that elements $a_3(x)$, $b_3(x)$ of these matrices do not contain n_1-monomials if $n_1 < k_1$, and in polynomials $\delta_A(x)$, $\delta_B(x)$, there are no $(n_2 - k_1)$-monomials if $n_2 < k_2$. Let us first show that the matrix S in the transition from $A(x)$ to $B(x)$ can be selected in the form

$$S = \left\| \begin{matrix} 1 & 0 & s_{13} \\ 0 & 1 & s_{23} \\ 0 & 0 & 1 \end{matrix} \right\|, \tag{46}$$

if $n_1 < k_1$, or in the form

$$S = \left\| \begin{matrix} 1 & s_{12} & s_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\|, \tag{47}$$

if $n_2 < k_2$.

Indeed, the elements of the matrices $A(x)$, $B(x)$ satisfy the congruence

$$\begin{aligned} &a_3(x) - b_3(x) - \left(s_{12}a_1(x) + s_{13}a_3(x)\right)b_3(x) - b_2(x)r_{21}(x) \\ &\quad \equiv 0\left(\operatorname{mod} x^{k_2}\right). \end{aligned} \tag{48}$$

If we compare the coefficients of the monomers of degree $n_1 < k_1$ in both parts of this congruence, we get $s_{12} = 0$. Also, from equivalence $A(x) \approx B(x)$, it is easy to get congruence

$$\begin{aligned} &\delta_A(x) - \delta_B(x) + s_{13}\delta_A(x)\delta_B(x) - s_{23}\delta_A(x)b_2'(x) \\ &\quad \equiv 0\left(\operatorname{mod} x^{k_2-k_1}\right). \end{aligned} \tag{49}$$

If we compare the coefficients of the monomers of degree $n_2 < k_2$ in both parts of the last congruence, then we come to $s_{23} = 0$:

(1) In case $n_1 < k_1$, $n_2 < k_2$, the transition matrix S from $A(x)$ to $B(x)$ has the form (46) and (47) simultaneously. Therefore, we have $s_{12} = s_{23} = 0$. Elements $a_1(x)$, $b_1(x)$ in $A(x)$, $B(x)$ satisfy (11). From here, we get $s_{13} = 0$. For this reason, matrices 1 and 2 coincide.

(2) Since $n_2 < k_2$, then matrix S of the transition from $A(x)$ to $B(x)$ has the form (47), and the elements $a_2(x)$, $b_2(x)$ in $A(x)$, $B(x)$ satisfy (25). In $a_2(x)$, $b_2(x)$, there are no $(co\deg\delta_A + k_1)$- and n_2-monomials, so from (25), we get $s_{12} = s_{13} = 0$. So, $A(x) = B(x)$.

(3) If $n_1 < k_1$, then the matrix S of the transition from $A(x)$ to $B(x)$ has the form (46). Elements $a_1(x)$, $b_1(x)$ in $A(x)$, $B(x)$ satisfy the congruence (28). From it, we have $s_{23} = s_{13} = 0$, since in $a_1(x)$, as in $b_1(x)$, there are no q_3-and n_1-monomials. Therefore, in this case, $A(x)$, $B(x)$ coincide.

(4) Suppose that matrix $B(x)$ satisfies condition 4, that is, in K_B, the elements of the first column corresponding to the maximum system of the first linearly independent rows of the submatrix K_{0B} are zero. Suppose that matrix $A(x)$ also has the same property, and in addition, condition $A(x) \approx B(x)$ holds. Then, the elements $a_i(x)$, $b_i(x)$, $i = 1, 2, 3$, of these matrices satisfy the congruences (43). If in K_A, we have $K_{0A} = 0$, then

$$\begin{aligned} \min\left(q_3, q_1^2\right) &\geq k_1, \\ \min\left(q_1^2, co\deg\delta_A\right) &\geq k_2 - k_1, \\ \min\left(q_2 + q_3, co\deg\left(a_1(x)\delta_A(x)\right)\right) &\geq k_2. \end{aligned} \tag{50}$$

Therefore, as can be seen from (43), $a_i(x) = b_i(x)$, $i = 1, 2, 3$.

If in K_A, we have $K_{0A} \neq 0$, and l_1 is the number of the first nonzero row $\overline{u}_1$ in K_{0A}, then the first l_1 elements in the first column of the matrix K_A coincide with the corresponding elements in the matrix K_B; moreover, l_1-th elements are zero. Therefore, in K_{0A}, the first $l_1 + 1$ rows coincide with the corresponding rows of the matrix K_{0B}. In addition, from congruences (43), we have $\overline{u}_1 \left\| s_{23} \;\; s_{13} \;\; s_{12} \right\|^t = 0$. If the next after $\overline{u}_1$ row $\overline{v}$ in K_{0A} (or in K_{0B}) is linearly dependent on $\overline{u}_1$, then

$$\overline{v} \left\| s_{23} \;\; s_{13} \;\; s_{12} \right\|^t = 0. \tag{51}$$

Then, from (43), we obtain that the first $l_1 + 1$ elements in the first column of the matrix K_A coincide with the corresponding elements in K_B. If $\overline{u}_1$ and $\overline{v}$ are linearly independent, then (51) is still satisfied since in this case, the $(l_1 + 1)$-th elements in the first columns of matrices K_A and K_B are zero. Then, the $l_1 + 2$ th row in K_{0A} coincides with the corresponding row of the matrix K_{0B}. We think of this row in the same way as it was done above with row $\overline{v}$. Let $\overline{u}_2$ be the first linearly independent of row $\overline{u}_1$ and l_2 be its number in K_{0A}. Then, this row coincides with the l_2-th row in K_{0B}, and the first l_2 elements of the first column in K_A coincide with the corresponding elements in K_B, with l_2th elements being zero. Then from (43), we have $\overline{u}_2 \left\| s_{23} \;\; s_{13} \;\; s_{12} \right\|^t = 0$. If $\overline{w}$ is the $(l_2 + 1)$-th row in K_{0A}, then the corresponding $(l_2 + 1)$-th row in K_{0B} is also $\overline{w}$. If $\overline{w}$ is linearly dependent on the system $\overline{u}_1$, $\overline{u}_2$, then

$$\overline{w} \left\| s_{23} \;\; s_{13} \;\; s_{12} \right\|^t = 0, \tag{52}$$

and the $(l_2 + 1)$-th elements in the first columns of matrices K_A,K_B coincide. Otherwise, these elements also coincide because they are null. Continuing our considerations, we show that, in K_A, K_B, the first columns coincide, or at some steps, we will get $s_{12} = s_{13} = s_{23} = 0$. In each case, $A(x) = B(x)$. Theorem is proved.

Example 1. Matrices $A(x) = \begin{Vmatrix} 1 & 0 & 0 \\ x^3 & x^4 & 0 \\ x^6 + x^4 + x^2 & x^7 & x^8 \end{Vmatrix}$, $B(x) = \begin{Vmatrix} 1 & 0 & 0 \\ x^3 & x^4 & 0 \\ x^2 & x^7 & x^8 \end{Vmatrix}$, and $C(x) = \begin{Vmatrix} x^4 - x^2 & x^2 - 1 & -x \\ 0 & x^3 & 0 \\ x^4 & x^2 & x^5 + x^2 \end{Vmatrix}$ are semiscalarly equivalent. In this case, $A(x)$ is a reduced, and $B(x)$ is a canonical matrix for $C(x)$.

4. Conclusion

The matrices $B(x)$, whose existence is established in Theorems 1 and 2, can be considered canonical in the class of semiscalarly equivalent matrices. The method of their construction follows from the proof of the first parts of these theorems. This completes the study of semiscalar equivalence of third-order polynomial matrices with one characteristic root, started in the previous works of the author.

The results obtained in this article, as well as the results of the works cited here, are applicable to the study of the simultaneous similarity of sets of numerical matrices. In this context, the works of [6–9] should be noted. These results also have utility in solving Sylvester-type matrix equations over polynomial rings. Such equations often arise in applied problems.

Data Availability

Data from previous studies were used to support this study. They are cited at relevant places within the text as references.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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