Research Article

An Application of Classical Logic's Laws in Formulas of Fuzzy Implications

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The crucial role that fuzzy implications play in many applicable areas was our motivation to revisit the topic of them. In this paper, we apply classical logic's laws such as De Morgan's laws and the classical law of double negation in known formulas of fuzzy implications. These applications lead to new families of fuzzy implications. Although a duality in properties of the preliminary and induced families is expected, we will prove that this does not hold, in general. Moreover, we will prove that it is not ensured that these applications lead us to fuzzy implications, in general, without restrictions. We generate and study three induced families, the so-called D'-implications, QL'-implications, and R'-implications. Each family is the "closest" to its preliminary-"creator" family, and they both are simulating the same (or a similar) way of classical thinking.

1. Introduction

Although, in classical logic, the implication is uniquely determined, in fuzzy logic, there are not only several formulas but also families of fuzzy implications. Moreover, among fuzzy implications, there are several properties, which a fuzzy implication satisfies or violates. The necessity of this variety had been addressed by many authors [1–3], and it is the election of the proper fuzzy implication to any applied problem. More specifically, Mas et al. in [1] addressed the following:

Of course, all these expressions for implications are equivalent in any Boolean algebra and consequently in classical logic. However, in fuzzy logic, these four definitions yield to distinct classes of fuzzy implications. Thus, the following question naturally arises: why so many different models to perform this kind of operation? The main reason is because they are used to represent imprecise knowledge. Note that any "if then" rule in fuzzy systems is interpreted through one of these implication functions. So, depending on the context and on the proper rule and its behaviour, different implications can be adequate in any case.

Among the families or the construction methods of fuzzy implications, there are several types. There are fuzzy implications that are constructed by generalizations of classical tautologies, such as (S, N)-implications, QL-implications, D-implications [1, 3–6], and (T, N)-implications [2, 7, 8]. All these generalizations are not always an easy process. For example, in the case of (S, N)-implications [4, 6] and (T, N)-implications [2, 7, 8], the generalization is smooth and direct. On the contrary, in the case of QL- and D-operations [4, 6], the generalization holds under conditions, which need investigation and remain an open problem, when a QL- (respectively, D-) operation is a fuzzy implication [4]. There are generalizations from classical set theory, such as R-implications [4–6]. There are fuzzy implications that are constructed by generalizations of the aforementioned generalizations, such as (U, N)- and RU-implications [4]. There are fuzzy implications that are constructed by function generators with specific properties [4, 9–12]. Moreover, there are generation methods of fuzzy implications from known fuzzy implications [4, 13].

In this paper, we focus on these fuzzy implications, in which the formula contains at least a t-norm or a t-conorm.

Such families are (S, N)-implications, (T, N)-implications, QL-implications, D-implications, and R-implications.

We have to remind that, in classical logic, there are De Morgan's laws, which are expressed by tautologies $(p \wedge q)' \equiv p' \vee q'$ and $(p \vee q)' \equiv p' \wedge q'$ and the classical law of double negation, which is expressed by the tautology $(p')' \equiv p$.

So, the central idea is to apply De Morgan's laws and, if necessary, the classical law of double negation in known formulas of fuzzy implications and investigate the results. These are the generation of new families of fuzzy implications. As we will show in the following, each new induced family does not generally have the same properties as its preliminary-"creator" family. On the contrary, some duality in the properties is remarkable and expected, but this does not hold, in general. So, any case must be studied individually from the beginning.

2. Preliminaries

Definition 1 (see [4, 5, 14, 15]). A decreasing function $N\colon [0, 1] \longrightarrow [0, 1]$ is called fuzzy negation if $N(0) = 1$ and $N(1) = 0$. Moreover, a fuzzy negation N is called

(i) Strict if it is continuous and strictly decreasing.

(ii) Strong if it is an involution, i.e.,

$$N(N(x)) = x, \quad \text{for all } x \in [0, 1]. \tag{1}$$

(iii) Nonfilling if

$$N(x) = 1 \Longleftrightarrow x = 0. \tag{2}$$

Remark 1 (see [4])

(i) We call $N_C(x) = 1 - x$ the classical fuzzy negation, which is a strong negation.

(ii) Moreover, we will need the following fuzzy negations:

$$N_{D1}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0, \end{cases}$$
$$N_{D2}(x) = \begin{cases} 1, & \text{if } x < 1, \\ 0, & \text{if } x = 1. \end{cases} \tag{3}$$

Definition 2 (see [4]). A function $T\colon [0, 1]^2 \longrightarrow [0, 1]$ is called a triangular norm (shortly t-norm) if it satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

$$T(x, y) = T(y, x), \tag{4}$$

$$T(x, T(y, z)) = T(T(x, y), z), \tag{5}$$

if $y \le z$, then $T(x, y) \le T(x, z)$, i.e., $T(x, \cdot)$ is increasing, $\tag{6}$

$$T(x, 1) = x. \tag{7}$$

Dually, a function $S\colon [0, 1]^2 \longrightarrow [0, 1]$ is called a triangular conorm (shortly t-conorm) if it satisfies, for all $x, y, z \in [0, 1]$, conditions (4), (5), and (6), and additionally,

$$S(x, 0) = x. \tag{8}$$

Definition 3 (see [4]). A t-norm T (respectively, a t-conorm S) is called

(i) Continuous if it is continuous in both arguments

(ii) Idempotent if $T(x, x) = x$ (respectively, $S(x, x) = x$), for all $x \in [0, 1]$

(iii) Strict if it is continuous and strictly monotone, i.e., $T(x, y) < T(x, z)$ whenever $x > 0$ and $y < z$ (respectively, $S(x, y) < S(x, z)$ whenever $x < 1$ and $y < z$)

(iv) Positive if $T(x, y) = 0 \Longleftrightarrow x = 0$ or $y = 0$ (respectively, $S(x, y) = 1 \Longleftrightarrow x = 1$ or $y = 1$)

Tables 1 and 2 list a few of the common t-norms and t-conorms, respectively (see Tables 2.1 and 2.2 of [4]).

Definition 4 (see [4]). Let S be a t-conorm and N be a fuzzy negation. We say that the pair (S, N) satisfies the law of excluded middle if

$$S(N(x), x) = 1, \quad x \in [0, 1]. \tag{9}$$

Definition 5 (see [4]). Let T be a t-norm and N be a fuzzy negation. We say that the pair (T, N) satisfies the law of contradiction if

$$T(N(x), x) = 0, \quad x \in [0, 1]. \tag{10}$$

Definition 6 (see [4]). A triple (T, S, N), where N is a strong negation, is called a De Morgan triple if

$$T(x, y) = N(S(N(x), N(y))),$$
$$S(x, y) = N(T(N(x), N(y))), \quad x, y \in [0, 1]. \tag{11}$$

Moreover, in the first case, T is called N-dual of S, and in the second case, S is called N-dual of T.

Definition 7 (see [4, 16]). By Φ, we denote the family of all increasing bijections from $[0, 1]$ to $[0, 1]$. We say that functions $f, g\colon [0, 1]^n \longrightarrow [0, 1]$ are Φ-conjugate if there exists $\phi \in \Phi$ such that $g = f_\phi$, where

$$f_\phi(x_1, x_2, \ldots, x_n) = \phi^{-1}(f(\phi(x_1), \phi(x_2), \ldots, \phi(x_n))),$$
$$x_1, x_2, \ldots, x_n \in [0, 1]. \tag{12}$$

Remark 2 (see [4]). It is easy to prove that if $\phi \in \Phi$ and T is a t-norm, S is a t-conorm, and N is a fuzzy negation (respectively, strict and strong), then T_ϕ is a t-norm, S_ϕ is a t-conorm, and N_ϕ is a fuzzy negation (respectively, strict and strong).

TABLE 1: Basic t-norms.

Name	Formula
Minimum	$T_M(x, y) = \min\{x, y\}$
Algebraic product	$T_P(x, y) = x \cdot y$
Łukasiewicz	$T_L(x, y) = \max\{x + y - 1, 0\}$
Drastic product	$T_D(x, y) = \begin{cases} 0, & \text{if } x, y \in [0, 1), \\ \min\{x, y\}, & \text{otherwise} \end{cases}$
Nilpotent minimum	$T_{nM}(x, y) = \begin{cases} 0, & \text{if } x + y \leq 1, \\ \min\{x, y\}, & \text{otherwise} \end{cases}$

TABLE 2: Basic t-conorms.

Name	Formula
Maximum	$S_M(x, y) = \max\{x, y\}$
Probor	$S_P(x, y) = x + y - x \cdot y$
Łukasiewicz	$S_L(x, y) = \min\{x + y, 1\}$
Drastic sum	$S_D(x, y) = \begin{cases} 1, & \text{if } x, y \in (0, 1], \\ \max\{x, y\}, & \text{otherwise} \end{cases}$
Nilpotent maximum	$S_{nM}(x, y) = \begin{cases} 1, & \text{if } x + y \geq 1, \\ \max\{x, y\}, & \text{otherwise} \end{cases}$

Definition 8 (see [4, 5]). A function $I: [0, 1]^2 \longrightarrow [0, 1]$ is called a fuzzy implication if

$$I \text{ is decreasing with respect to the first variable,} \quad (13)$$

$$I \text{ is increasing with respect to the second variable,} \quad (14)$$

$$I(0, 0) = 1, \quad (15)$$

$$I(1, 1) = 1, \quad (16)$$

$$I(1, 0) = 0. \quad (17)$$

Remark 3 By axioms (14) and (15), we deduce the normality condition

$$I(0, 1) = 1. \quad (18)$$

Furthermore, by Definition 8, it is easy to prove the left and right boundary conditions [4]:

$$I(0, y) = 1, \quad y \in [0, 1], \quad (19)$$

$$I(x, 1) = 1, \quad x \in [0, 1]. \quad (20)$$

Table 3 lists a few of basic fuzzy implications (see Table 1.3 and Proposition 1.1.7 of [4]).

Definition 9 (see [4]). A fuzzy implication I is said to satisfy

(i) The left neutrality property if

$$I(1, y) = y, \quad y \in [0, 1]. \quad (21)$$

(ii) The exchange principle if

TABLE 3: Basic fuzzy implications.

Name	Formula
Łukasiewicz	$I_{LK}(x, y) = \min\{1, 1 - x + y\}.$
Reichenbach	$I_{RC}(x, y) = 1 - x + x \cdot y.$
Gödel	$I_{GD}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{if } x > y. \end{cases}$
Weber	$I_{WB}(x, y) = \begin{cases} 1, & \text{if } x < 1, \\ y, & \text{if } x = 1. \end{cases}$
Kleene–Dienes	$I_{KD}(x, y) = \max\{1 - x, y\}.$
Fodor	$I_{FD}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ \max\{1 - x, y\}, & \text{if } x > y. \end{cases}$
Yager	$I_{YG}(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ y^x, & \text{if } x > 0 \text{ or } y > 0. \end{cases}$
Dubois and Prade	$I_{DP}(x, y) = \begin{cases} 1 - x, & \text{if } y = 0, \\ y, & \text{if } x = 1, \\ 1, & \text{otherwise.} \end{cases}$
Goguen	$I_{GG}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ y/x, & \text{if } x > y. \end{cases}$
Rescher	$I_{RS}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x > y. \end{cases}$
The weakest	$I_0(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ 0, & \text{otherwise.} \end{cases}$
The strongest	$I_1(x, y) = \begin{cases} 0, & \text{if } x = 1 \text{ or } y = 0, \\ 1, & \text{otherwise.} \end{cases}$

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1]. \quad (22)$$

Remark 4 (see [4]). It is proved that if $\phi \in \Phi$ and $I: [0, 1]^2 \longrightarrow [0, 1]$ is a function, which satisfies axiom (13) (respectively, (14), (15), (16), and (17)), then $I_\phi: [0, 1]^2 \longrightarrow [0, 1]$ is a function, which also satisfies axiom (13) (respectively, (14), (15), (16), and (17)). So, if I is a fuzzy implication, then I_ϕ is also a fuzzy implication.

Proposition 1 (see [4]). *If a function $I: [0, 1]^2 \longrightarrow [0, 1]$ satisfies (13), (15), and (17), then the function $N_I: [0, 1] \longrightarrow [0, 1]$ is a fuzzy negation, where*

$$N_I(x) = I(x, 0), \quad x \in [0, 1]. \quad (23)$$

Definition 10 (see [4]). Let $I: [0, 1]^2 \longrightarrow [0, 1]$ be a fuzzy implication. The function N_I defined by Proposition 1 is called the natural negation of I.

Definition 11 (see [4]). Let N be a fuzzy negation and I be a fuzzy implication. A function $I_N: [0, 1]^2 \longrightarrow [0, 1]$ defined by

$$I_N(x, y) = I(N(y), N(x)), \quad x, y \in [0, 1], \quad (24)$$

is called the N-reciprocal of I. When N is the classical negation N_C, then I_N is called the reciprocal of I and is denoted by I'.

In Table 4, we present the known families of fuzzy implications that are generalizations from the classical logic (except R-implications that are generalizations from the classical set theory via the isomorphism that exists between classical two-valued logic and classical set theory and a set theoretic identity (see page 68 of [4])). We consider that the reader knows these families, as well as QL- and D-operations.

3. (T, N)-Implications

(T, N)-implications are mentioned as formulas by many authors [15, 17–19]. Baczyński and Jayaram in Corollary 2.5.31 of [4] related them with R-implications when the t-norm T is left continuous and N is a strong negation. Pradera et al. in Remark 30 of [21] mentioned the same formula, using aggregation functions, in general. Bedregal in Proposition 2.6 of [7] defined them for any t-norm and any fuzzy negation. They obtained their name in [2] and were studied by Pinheiro et al. in [2, 8] for any t-norm and any fuzzy negation.

(T, N)-implications are the "closest" to (S, N)-implications since for strong negations, they are the same family (see p. 234 of [15], [2] Theorem 3.1 of [8]). On the contrary, for nonstrong negations, they have no common implications since unlike (S, N)-implications (see Proposition 2.4.3 of [4]), (T, N)-implications violate (21) (see Proposition 3.3(i) of [2]). All these results lead us to Figure 1.

The aforementioned lead us to the claim that these two families simulate the same (or a similar) way of classical thinking. Firstly, we must explain the meaning of this simulation. The meaning is that there are tautologies in the classical logic, which can be proved without using truth tables or other rules, but only De Morgan's laws and the classical law of double negation.

So, firstly, we work on classical logic's tautologies, and secondly, we generalize them in fuzzy logic. Unfortunately, this generalization is not always an easy process, as we will show in the following.

Let us remark that (S, N)-implications are the generalization of the tautology

$$(p \Longrightarrow q) \equiv (p' \vee q), \tag{25}$$

and then by applying De Morgan's laws and the classical law of double negation, we get the following tautology:

$$(p \Longrightarrow q) \equiv (p' \vee q) \equiv \left[(p')' \wedge q' \right]' \equiv (p \wedge q')', \tag{26}$$

which leads us to (T, N)-implications.

Similarly, QL-operations (respectively, implications) are the generalization of the tautology

$$(p \Longrightarrow q) \equiv \left[p' \vee (p \wedge q) \right]. \tag{27}$$

At this point, let us make clear what is the meaning of "simulate the same (or a similar) way of classical thinking" via an example.

Example 1. Someone could claim that since

$$\left(p' \vee q \right) \equiv (p \Longrightarrow q) \equiv \left[p' \vee (p \wedge q) \right], \tag{28}$$

TABLE 4: Known families of fuzzy implications.

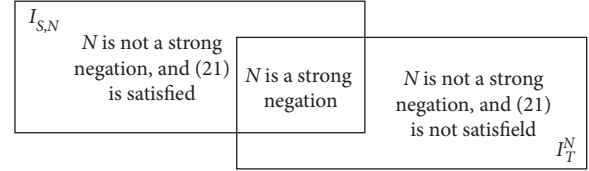| Name	Formula
(S, N) [1, 3–6]	$I_{S,N}(x, y) = S(N(x), y)$
(T, N) [2, 7, 8, 15, 17–19]	$I_T^N(x, y) = N(T(x, N(y)))$
QL [1, 3–6]	$I_{T,S,N}(x, y) = S(N(x), T(x, y))$, which satisfies (13)
D [3, 4, 6, 20]	$I^{T,S,N}(x, y) = S(T(N(x), N(y)), y)$, which satisfies (14)
R [3–6]	$I_T(x, y) = \sup\{t \in [0, 1] \mid T(x, t) \leq y\}$

FIGURE 1: Intersection between (S, N)- and (T, N)-implications.

(S, N)- and QL-implications simulate the same (or a similar) way of classical thinking. This is not acceptable since we cannot prove the tautology

$$(p' \vee q) \equiv \left[p' \vee (p \wedge q) \right], \tag{29}$$

using only De Morgan's laws and maybe the classical law of double negation.

4. D'-Implications

Similarly, D-operations (respectively, implications) are the generalization of the tautology

$$(p \Longrightarrow q) \equiv \left[(p' \wedge q') \vee q \right], \tag{30}$$

and after the application of De Morgan's laws and the classical law of double negation, we get the following tautology:

$$\begin{aligned} (p \Longrightarrow q) &\equiv \left[(p' \wedge q') \vee q \right] \equiv \left[(p \vee q)' \vee q \right] \\ &\equiv \left[((p \vee q)')' \wedge q' \right]' \equiv \left[(p \vee q) \wedge q' \right]', \end{aligned} \tag{31}$$

which leads us to a new family of fuzzy operations (respectively, implications) obtained by the following formula.

Definition 12. A function $I: [0, 1]^2 \longrightarrow [0, 1]$ is called a D'-operation if there exist a t-norm T, a t-conorm S, and a fuzzy negation N such that

$$I(x, y) = N(T(S(x, y), N(y))), x, y \in [0, 1]. \tag{32}$$

If I is a D'-operation generated by the triple (T, S, N), then we denote it by $I^{N,T,S}$.

Theorem 1. *Let $I^{N,T,S}$ be a D'-operation; then, $I^{N,T,S}$ satisfies (13)–(18) and (20), and if N is a strong negation, then $I^{N,T,S}$ satisfies (21). Furthermore, $N_{I^{N,T,S}} = N$, where $N_{I^{N,T,S}}(x) = I^{N,T,S}(x, 0), x \in [0, 1]$.*

Proof. Let $I^{N,T,S}$ be a D'-operation; then, for $x, y, z \in [0,1]$, if

$$x \leq y \implies S(x,z) \leq S(y,z)$$

$$\implies T(S(x,z), N(z)) \leq T(S(y,z), N(z))$$

$$\implies N(T(S(x,z), N(z))) \geq N(T(S(y,z), N(z)))$$

$$\implies I^{N,T,S}(x,z) \geq I^{N,T,S}(y,z), \tag{33}$$

which means that $I^{N,T,S}$ satisfies (13).

$I^{N,T,S}$ satisfies (15) since

$$I^{N,T,S}(0,0) = N(T(S(0,0), N(0))) = N(T(0,1)) \\ = N(0) = 1. \tag{34}$$

$I^{N,T,S}$ satisfies (16) since

$$I^{N,T,S}(1,1) = N(T(S(1,1), N(1))) = N(T(1,0)) \\ = N(0) = 1. \tag{35}$$

$I^{N,T,S}$ satisfies (17) since

$$I^{N,T,S}(1,0) = N(T(S(1,0), N(0))) = N(T(1,1)) \\ = N(1) = 0. \tag{36}$$

$I^{N,T,S}$ satisfies (18) since

$$I^{N,T,S}(0,1) = N(T(S(0,1), N(1))) = N(T(1,0)) \\ = N(0) = 1. \tag{37}$$

$I^{N,T,S}$ satisfies (20) since $\forall x \in [0,1]$, it is

$$I^{N,T,S}(x,1) = N(T(S(x,1), N(1))) = N(T(1,0)) \\ = N(0) = 1. \tag{38}$$

(21) is satisfied if N is a strong negation since $\forall x \in [0,1]$, it is

$$I^{N,T,S}(1,x) = N(T(S(1,x), N(x))) \\ = N(T(1, N(x))) \tag{39} \\ = N(N(x)) = x.$$

Lastly, we have

$$N_{I^{N,T,S}}(x) = I_{N,T,S}(x,0) \\ = N(T(S(x,0), N(0))) \\ = N(T(x,1)) \tag{40} \\ = N(x), \forall x \in [0,1].$$

By Theorem 1, it follows that a D'-operation is generated by a unique negation. The reason for which we use the name D'-operations instead of D'-implications is that they do not generally satisfy (14), so they are not always increasing with respect to the second variable, as it can be seen in the next example.

Example 2. Consider the triple (T_M, S_P, N_C). The obtained D'-operation is

$$I^{N_C,T_M,S_P}(x,y) = N_C(T_M(S_P(x,y), N_C(y))) \\ = 1 - \min\{x + y - x \cdot y, 1 - y\} \tag{41} \\ = \max\{1 - x - y + x \cdot y, y\},$$

which does not satisfy (14) since

$$0.1 \leq 0.2 \implies I^{N_C,T_M,S_P}(0.1, 0.1) = 0.81 > 0.72 \\ = I^{N_C,T_M,S_P}(0.1, 0.2). \tag{42}$$

Therefore, the first main problem is the characterization of those D'-operations, which satisfy (14). In this paper, we try to characterize these triples (T, S, N) that produce D'-operations, which satisfy (14). In the following, we will give only partial results as partial results are known in the literature for QL- and D-operations, too [4]. Following the terminology [8, 16], only if the D'-operation is a fuzzy implication, we use the term D'-implication. So, we have the following proposition, without proof.

Proposition 2. *A function $I: [0,1]^2 \longrightarrow [0,1]$ is called a D'-implication if it is a D'-operation and satisfies (14).*

All the above are useful, but they do not ensure that the set of D'-implications is nonempty. This is ensured by the next example.

Example 3. Consider the triple (T_L, S_M, N_C). The obtained D'-operation is

$$I^{N_C,T_L,S_M}(x,y) = N_C(T_L(S_M(x,y), N_C(y))) = \\ = 1 - \max\{\min\{x,y\} + 1 - y - 1, 0\} \tag{43} \\ = \min\{1 - x + y, 1\} = I_{LK}(x,y),$$

which is a D'-implication.

By Example 3, someone could automatically prove that the set of D'-implications is nonempty. Furthermore, it is very easy to observe and prove the following lemma. The proof is omitted due to its simplicity.

Lemma 1. *Let T_1, T_2, T be t-norms, S_1, S_2, S be t-conorms, and N be a fuzzy negation.*

(i) *If $T_1 \leq T_2$, then $I^{N,T_1,S}(x,y) \geq I^{N,T_2,S}(x,y)$*

(ii) *If $S_1 \leq S_2$, then $I^{N,T,S_1}(x,y) \geq I^{N,T,S_2}(x,y)$*

(iii) *If $T_1 \leq T_2$ and $S_1 \leq S_2$, then $I^{N,T_1,S_1}(x,y) \geq I^{N,T_2,S_2}(x,y)$*

Although Lemma 1 is simple, it is very useful for the following. It is $T_D \leq T$, for all t-norms T (see Remark 2.1.4.(ix), page 43, of [4]) and $S_M \leq S$, for all t-conorms S (see Remark 2.2.5.(viii), page 46, of [4]). Thus, it is easy to prove that the strongest D'-operation, which can be obtained by the fuzzy negation N_{D2}, is

$$I^{N_{D2}, T_D, S_M}(x, y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y < 1, \\ 1, & \text{otherwise,} \end{cases} \tag{44}$$

$$= I_4(x, y), \quad (\text{see } [4]),$$

which is a D'-implication, too. By this result, we can conclude that I_1 is not a D'-implication. That is, because its natural negation is N_{D2} and, furthermore, there are $x, y \in [0, 1]$, such that $I^{N_{D2}, T_D, S_M}(x, y) = I_4(x, y) < I_1(x, y)$. I_1 is stronger than the strongest D'-implication that has N_{D2} as its natural negation, meaning that is not a D'-implication.

The formula of D'-operations is complicated. Moreover, we have to check every time if axiom (14) is satisfied. If we use a nonfilling negation, the following theorem will be a useful tool to overcome these difficulties.

Theorem 2. *Let $I^{N,T,S}$ be a D'-implication and N be a nonfilling negation; then, it is $T(N(x), x) = 0, x \in [0, 1]$, that is, the pair (T, N) satisfies the law of contradiction (10).*

Proof. If $I^{N,T,S}$ is a D'-implication, then it satisfies (19), that is, for all $y \in [0, 1]$,

$$\begin{aligned} I^{N,T,S}(0, y) = 1 &\Longrightarrow N(T(S(0, y), N(y))) = 1 \\ &\Longrightarrow N(T(y, N(y))) = 1 \\ &\Longrightarrow T(N(y), y) = 0 \end{aligned} \tag{45}$$

since N is a nonfilling negation. So, $T(N(x), x) = 0, x \in [0, 1]$.

Remark 5. By Theorem 2, it is obvious that if N is a nonfilling negation and the pair (T, N) does not satisfy the law of contradiction (10), i.e., $T(N(x), x) \neq 0$, for some $x \in [0, 1]$, then the obtained $I^{N,T,S}$ D'-operation, for any t-conorm S, is not a fuzzy implication.

Theorem 3. *If $\phi \in \Phi$ and $I_{N,T,S}$ is a D'-operation (respectively, implication), then $(I^{N,T,S})_\phi$ is a D'-operation (respectively, implication), and moreover,*

$$\left(I^{N,T,S}\right)_\phi = I^{N_\phi, T_\phi, S_\phi}. \tag{46}$$

Proof. Let $I^{N,T,S}$ be a D'-operation (respectively, implication); then, $(I^{N,T,S})_\phi$ is a D'-operation (respectively, implication) according to Remark 4. Moreover, for all $x, y \in [0, 1]$, we deduce that

$$\begin{aligned} \left(I^{N,T,S}\right)_\phi(x, y) &= \phi^{-1}\left(I^{N,T,S}(\phi(x), \phi(y))\right) \\ &= \phi^{-1}\left(N(T(S(\phi(x), \phi(y)), N(\phi(y))))\right) \\ &= \phi^{-1}\left(N\left(\phi\left(\phi^{-1}\left(T(S(\phi(x), \phi(y)), N(\phi(y)))\right)\right)\right)\right) \\ &= N_\phi(\phi^{-1}(T(\phi(\phi^{-1}(S(\phi(x), \phi(y)))), \\ &\quad \phi(\phi^{-1}(N(\phi(y))))))) \\ &= N_\phi\left(T_\phi\left(S_\phi(x, y), N_\phi(y)\right)\right) \\ &= I^{N_\phi, T_\phi, S_\phi}(x, y). \end{aligned} \tag{47}$$

Since we have studied some general results for D'-operations (respectively, implications), we are going to study some more specific cases. This family has no interest if we use strong negations. This is explained by Theorem 4 after the following Proposition 3.

Proposition 3 (see Proposition 5 of [6]). *Let T be a t-norm, S be a t-conorm, and N be a strong negation. Then, the corresponding QL-operator, $I_{T,S,N}$, is a QL-implication if and only if the corresponding D-operator, $I^{T,S,N}$, is a D-implication.*

Theorem 4. *If N is a strong negation and $I^{N,T,S'}$ is a D'-operation, then*

(i) *$I^{N,T,S'} = I^{T',S,N} = (I_{T',S,N})_N$, where S, T' are N-dual of T, S', respectively, $I^{T',S,N}$ is a D-operation, and $(I_{T',S,N})_N$ is N-reciprocal of the QL-operation $I_{T',S,N}$*

(ii) *Moreover, if one of $I^{N,T,S'}$, $I^{T',S,N}$, and $I_{T',S,N}$ is a fuzzy implication, then the other two are fuzzy implications, too*

Proof

(i) Let $I^{N,T,S'}$ be a D'-operation. Then,

$$\begin{aligned} I^{N,T,S'}(x, y) &= N(T(S'(x, y), N(y))) \\ &= N(T(N(T'(N(x), N(y))), N(y))) \\ &= S(T'(N(x), N(y)), y) \\ &= I^{T',S,N}(x, y). \end{aligned} \tag{48}$$

On the contrary,

$$\begin{aligned} I^{N,T,S'}(x, y) &= N(T(S'(x, y), N(y))) \\ &= S(N(S'(x, y)), N(N(y))) \\ &= S(N(N(T'(N(x), N(y)))), y) \\ &= S(T'(N(x), N(y)), y) \\ &= S(y, T'(N(x), N(y))) \\ &= S(N(N(y)), T'(N(y), N(x))) \\ &= I_{T',S,N}(N(y), N(x)) \\ &= \left(I_{T',S,N}\right)_N(x, y). \end{aligned} \tag{49}$$

(ii) If one of $I_{T',S,N}$ and $I^{T',S,N}$ is a fuzzy implication, then the other one is a fuzzy implication too, according to Proposition 3. If $I^{N,T,S'}$ is a fuzzy implication, then because of (i), where $I^{N,T,S'} = I^{T',S,N}$, we conclude that $I^{T',S,N}$ is a fuzzy implication and vice versa.

Remark 6. By Theorem 4, it is obvious that if (T, S, N) is a De Morgan triple and $I^{N,T,S}$ is a D'-operation (respectively, implication), then $I^{N,T,S} = I^{T,S,N} = (I_{T,S,N})_N$, where $I^{T,S,N}$ is a D-operation (respectively, implication) and $(I_{T,S,N})_N$ is N-

reciprocal of the QL-operation (respectively, implication) $I_{T,S,N}$. The $(I_{T,S,N})_N$ N-reciprocal of the QL-operation $I_{T,S,N}$ is defined, respectively, to the N-reciprocal of the QL-implication $I_{T,S,N}$.

So, the first question arising is whether D'-operations (respectively,, implications) are or not a new family of operations (respectively, implications) or they are simply D-operations (respectively, implications). Although we have mentioned so many theorems and we have presented the whole rationale of our study, this is a question that seems not to have been answered yet.

It is known that if $I^{T,S,N}$ is a D-operation (respectively, implication), then the pair (S, N) must satisfy (9) (see Lemma 3 of [20]). Additionally, there does not exist any t-conorm S such that the pair (S, N_{D1}) satisfies (9) (see Remark 2.3.10(iii) of [4]). Thus, there is no D-implication generated by any triple (T, S, N_{D1}). Furthermore, it is easy to prove that if $I^{T,S,N}$ is a D-implication, then N is its natural negation. This proof is easy and similar to the proof of Theorem 1. To sum up, there is no D-implication which has N_{D1} as its natural negation. On the contrary, we have the following proposition.

Proposition 4. *By a triple of the form* (T, S, N_{D1}), *where* T *is any t-norm and* S *is any t-conorm, a* D'-*implication is obtained, which is* I_3 (*see* [22]).

Proof. By a triple of the form (T, S, N_{D1}), where T is any t-norm and S is any t-conorm, we obtain

$$
\begin{aligned}
I^{N_{D1},T,S}(x, y) &= N_{D1}\left(T\left(S(x, y), N_{D1}(y)\right)\right) \\
&= \begin{cases} N_{D1}\left(T\left(S(x, 0), N_{D1}(0)\right)\right), & \text{if } y = 0, \\ N_{D1}\left(T\left(S(x, y), N_{D1}(y)\right)\right), & \text{if } y > 0, \end{cases} \\
&= \begin{cases} N_{D1}\left(T(x, 1)\right), & \text{if } y = 0, \\ N_{D1}\left(T(S(x, y), 0)\right), & \text{if } y > 0, \end{cases} \\
&= \begin{cases} N_{D1}(x), & \text{if } y = 0, \\ N_{D1}(0), & \text{if } y > 0, \end{cases} \\
&= \begin{cases} 0, & \text{if } x > 0 \text{ and } y = 0, \\ 1, & \text{otherwise,} \end{cases} \\
&= I_3(x, y).
\end{aligned}
$$

(50)

Remark 7. The aforementioned Proposition 4 is very important, although it seems simple. Because of it, we conclude that the only D'-implication which has N_{D1} as its natural negation is I_3. So, automatically, we conclude that I_0, I_{GG}, I_{GD}, I_{RS}, and I_{YG} are not D'-implications since all of them have N_{D1} as their natural negation.

Proposition 4 is the answer of this first question. I_3 is a D'-implication, which is obviously not a D-implication since its natural negation is N_{D1}. So, it is proved that the family of D'-implications is not the same with D-implications' family.

Furthermore, due to tautology (5) and by our intuition, we expected duality properties of D'-operations and D-operations. So, the second question arising is whether or not this is true. For example, we mentioned that if $I^{T,S,N}$ is a D-operation (respectively, implication), then the pair (S, N) must satisfy (9). Does this imply that if $I^{N,T,S}$ is a D'-operation, then the pair (T, N) must satisfy (10)?

The answer seems to be negative, as we show in Theorem 2, since (10) is satisfied if we have a nonfilling negation N and not for sure, for any negation N. Finally, the answer is negative, according to the following proposition.

Proposition 5. *By a triple of the form* (T, S, N_{D2}), *where* T *is any t-norm and* S *is any idempotent, positive, or strict t-conorm, a* D'-*implication is obtained, which is* I_4 (*see* [22]).

Proof. It is known and easy to prove that the only idempotent t-conorm is S_M (see Remark 2.2.5(ii) of [4]), which is also positive (see Table 2.2, page 46, of [4]). Therefore, anything is proved for positive t-conorms is valid and for the idempotent too. Thus, by the triple (T, S, N_{D2}), where S is any positive or strict t-conorm, we obtain

$$
\begin{aligned}
I^{N_{D2},T,S}(x, y) &= N_{D2}\left(T\left(S(x, y), N_{D2}(y)\right)\right) \\
&= \begin{cases} N_{D2}\left(T\left(S(x, 1), N_{D2}(1)\right)\right), & \text{if } y = 1, \\ N_{D2}\left(T\left(S(x, y), N_{D2}(y)\right)\right), & \text{if } y < 1, \end{cases} \\
&= \begin{cases} N_{D2}\left(T(x, 0)\right), & \text{if } y = 1, \\ N_{D2}\left(T(S(x, y), 1)\right), & \text{if } y < 1, \end{cases} \\
&= \begin{cases} N_{D2}(0), & \text{if } y = 1, \\ N_{D2}(S(x, y)), & \text{if } y < 1, \end{cases} \\
&= \begin{cases} 1, & \text{if } y = 1, \\ 1, & \text{if } S(x, y) < 1 \text{ and } y < 1, \\ 0, & \text{if } S(x, y) = 1 \text{ and } y < 1, \end{cases} \\
&= \begin{cases} 0, & \text{if } x = 1 \text{ and } y < 1, \\ & \qquad = I_4(x, y). \\ 1, & \text{otherwise,} \end{cases}
\end{aligned}
$$

(51)

According to Proposition 5, I_4 is a D'-implication, which does not satisfy (10) since for any t-norm T, it is $T(N_{D2}(x), x) = x \neq 0$, for any $x \in (0, 1)$. This fact is exactly the reason why we have to study any new family generated by De Morgan's laws and, if necessary, the classical law of double negation from the beginning.

The third question arising is what is the relation between these families (D and D'). The answer is given partially in Theorem 1, where we proved that if N is a strong negation, then $I^{N,T,S}$ satisfies (21). Also, it is easy to prove the next proposition.

Proposition 6. *Let* $I^{N,T,S}$ *be a* D'-*operation, which satisfies* (21)*; then,* N *is a strong negation.*

Proof. Since the D'-operation $I^{N,T,S}$ satisfies (21), we have

$$I^{N,T,S}(1,x) = x \Longrightarrow N(T(S(1,x),N(x))) = x$$

$$\Longrightarrow N(T(1,N(x))) = x \qquad (52)$$

$$\Longrightarrow N(N(x)) = x, \quad \forall x \in [0,1].$$

Thus, N is a strong negation.

Clearly then, if N is a strong negation, then the corresponding D'-operation (respectively, implication) is a D-operation (respectively, implication) too, and vice versa. Therefore, in this case, every property of D-implications is valid to D'-implications. Since the properties are similar in both families, due to the duality of S, S', T, T', respectively, the existing theorems and propositions that have been studied so far (see [6]) are valid too, with some simple changes due to the mentioned duality.

For nonstrong negations, we have that D-operations (respectively, implications) are the operations (respectively, implications), which satisfy (21) (see Remark 1 of [6]), and on the contrary, D'-operations (respectively, implications) do not satisfy (21). All these results lead us to Figure 2.

In the following of this section, we present a useful theorem for the disqualification of some triples (T, S, N) that do not generate D'-implications.

Theorem 5. *By any triple (T, S, N), where S is any t-conorm, N is any continuous nonfilling fuzzy negation, and T is any idempotent, strict, or positive t-norm, any D'-implication cannot be obtained.*

Proof. Firstly, we have to mention that it is easy to prove that every continuous fuzzy negation N has a unique fixed point (see Theorem 1.4.7, page 15, of [4]). So, for any continuous fuzzy negation N, there exists exactly one $e \in (0,1)$ such that $N(e) = e$.

Moreover, the only idempotent t- norm is T_M (see Proposition 1.9 of [15]), and for any continuous nonfilling fuzzy negation N, we have $T_M(N(e),e) = T_M(e,e) = e \neq 0$. Therefore, according to Remark 5, any D'-implication cannot be obtained.

Even if T is a strict t-norm, then $T(N(e),e) = T(e,e) \neq 0$ because if $T(e,e) = 0 \Longleftrightarrow T(e,e) = T(e,0)$, a contradiction because $e > 0 \Longrightarrow T(e,e) > T(e,0)$. Therefore, according to Remark 5, any D'-implication cannot be obtained.

Finally, if T is any positive t-norm, then

$$T(x,y) = 0 \Longleftrightarrow x = 0 \text{ or } y = 0. \qquad (53)$$

So, $T(N(e),e) = T(e,e) \neq 0$, since $e > 0$, and again, according to Remark 5, any D'- implication cannot be obtained.

A very useful proposition is as follows.

Proposition 7. *By a triple of the form (T_D, S, N_C), where S is any idempotent or positive t-conorm, a D'-implication is obtained, which is I_{DP}.*
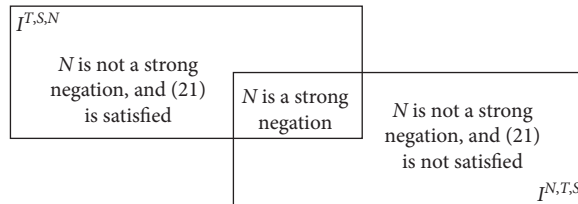

Figure 2: Intersection between families of D- and D'-operations (respectively, implications).

Proof. The only idempotent t-conorm is S_M (see Remark 2.2.5(ii) of [4]), which is also positive (see Table 2.2, page 46, of [4]). Therefore, by the triple (T_D, S, N_C), where S is any positive t-conorm, we obtain

$$I^{N_C, T_D, S}(x,y) = N_C(T_D(S(x,y), N_C(y)))$$

$$= 1 - T_D(S(x,y), 1-y)$$

$$= 1 - \begin{cases} 0, & \text{if } S(x,y) < 1 \text{ and } y > 0, \\ 1-y, & \text{if } x = 1 \text{ or } y = 1, \\ S(x,y), & \text{if } y = 0, \end{cases}$$

$$= 1 - \begin{cases} 1-y, & \text{if } x = 1, \\ S(x,0), & \text{if } y = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} y, & \text{if } x = 1, \\ 1-x, & \text{if } y = 0, \\ 1, & \text{otherwise,} \end{cases}$$

$$= I_{DP}(x,y).$$

$$(54)$$

Remark 8. It is easy to calculate D'-operations, which are obtained by their natural negation N_{D2}. They are always two-valued of the form

$$I_{N_{D2},T,S}(x,y) = \begin{cases} 0, & \text{if } S(x,y) = 1 \text{ and } y < 1, \\ 1, & \text{otherwise.} \end{cases} \qquad (55)$$

So, I_{WB}, which has N_{D2} as its natural negation, is not a D'-implication since it is not two-valued.

At the end of this section, we present Table 5, which contains basic D'-implications.

5. QL′-Implications

Similarly, QL-operations (respectively, implications) are the generalization of the tautology

$$(p \Longrightarrow q) \equiv [p' \vee (p \wedge q)], \qquad (56)$$

and after the application of De Morgan's laws and the classical law of double negation, we get the following tautology:

TABLE 5: Basic D'-implications.

T	S	N	Formula
T_L	S_M	N_C	I_{LK}
T_L	S_P	N_C	I_{RC}
T_L	S_L	N_C	I_{KD}
T_D	Any idempotent or positive S	N_C	I_{DP}
T_{nM}	S_M	N_C	I_{FD}
Any T	Any S	N_{D1}	I_3
Any T	Any idempotent, strict, or positive S	N_{D2}	I_4

$$(p \Longrightarrow q) \equiv [p' \vee (p \wedge q)] \equiv [(p')' \wedge (p' \vee q')]'$$
$$\equiv [p \wedge (p' \vee q')]', \tag{57}$$

which leads us to a new family of fuzzy operations (respectively, implications) obtained by the following formula.

Definition 13. A function $I: [0,1]^2 \longrightarrow [0,1]$ is called a QL'-operation if there exist a t-norm, a t-conorm S, and a fuzzy negation N such that

$$I(x, y) = N(T(x, S(N(x), N(y)))), \quad x, y \in [0,1]. \tag{58}$$

If I is a QL'-operation generated by the triple (T, S, N), then we denote it by $I_{N,T,S}$.

Theorem 6. Let $I_{N,T,S}$ be a QL'-operation; then, $I_{N,T,S}$ satisfies (14), (15), (16), (17), and (18), and if N is a strong negation, then $I_{N,T,S}$ satisfies (21). Furthermore, $N_{I_{N,T,S}} = N$, where $N_{I_{N,T,S}}(x) = I_{N,T,S}(x, 0), x \in [0,1]$.

Proof. Let $I_{N,T,S}$ be a QL'-operation; then, for $x, y, z \in [0,1]$, if

$$y \leq z \Longrightarrow N(y) \geq N(z)$$
$$\Longrightarrow S(N(x), N(y)) \geq S(N(x), N(z))$$
$$\Longrightarrow T(x, S(N(x), N(y))) \geq T(x, S(N(x), N(z)))$$
$$\Longrightarrow N(T(x, S(N(x), N(y))))$$
$$\leq N(T(x, S(N(x), N(z))))$$
$$\Longrightarrow I_{N,T,S}(x, y) \leq I_{N,T,S}(x, z), \tag{59}$$

which means that $I_{N,T,S}$ satisfies (14).
$I_{N,T,S}$ satisfies (15) since

$$I_{N,T,S}(0, 0) = N(T(0, S(N(0), N(0))))$$
$$= N(T(0, S(1, 1))) \tag{60}$$
$$= N(T(0, 1)) = N(0) = 1.$$

$I_{N,T,S}$ satisfies (16) since

$$I_{N,T,S}(1, 1) = N(T(1, S(N(1), N(1))))$$
$$= N(T(1, S(0, 0)))$$
$$= N(T(1, 0))$$
$$= N(0) = 1. \tag{61}$$

$I_{N,T,S}$ satisfies (17) since

$$I_{N,T,S}(1, 0) = N(T(1, S(N(1), N(0))))$$
$$= N(T(1, S(0, 1)))$$
$$= N(T(1, 1))$$
$$= N(1) = 0. \tag{62}$$

$I_{N,T,S}$ satisfies (18) since

$$I_{N,T,S}(0, 1) = N(T(0, S(N(0), N(1))))$$
$$= N(T(0, S(1, 0)))$$
$$= N(T(0, 1))$$
$$= N(0) = 1. \tag{63}$$

(21) is satisfied if N is a strong negation since

$$I_{N,T,S}(1, x) = N(T(1, S(N(1), N(x))))$$
$$= N(T(1, S(0, N(x))))$$
$$= N(T(1, N(x)))$$
$$= N(N(x)) = x, \quad \forall x \in [0,1]. \tag{64}$$

Lastly, we have

$$N_{I_{N,T,S}}(x) = I_{N,T,S}(x, 0)$$
$$= N(T(x, S(N(x), N(0))))$$
$$= N(T(x, S(N(x), 1)))$$
$$= N(T(x, 1)) \tag{65}$$
$$= N(x), \quad \forall x \in [0,1].$$

By Theorem 6, it follows that a QL'-operation is generated by a unique negation. Although it is not proved that $I_{N,T,S}$ satisfies (13), $N_{I_{N,T,S}}$ is obviously a fuzzy negation since $N_{I_{N,T,S}} = N$. Moreover, we use the name QL'-operations instead of QL'-implications because they do not generally satisfy (13), so they are not always decreasing with respect to the first variable, as it can be seen in the next example.

Example 4. Consider the triple (T_M, S_P, N_C). The obtained QL'-operation is

$$I_{N_C,T_M,S_P}(x, y) = N_C(T_M(x, S_P(N_C(x), N_C(y))))$$
$$= 1 - \min\{x, N_C(x) + N_C(y) - N_C(x) \cdot N_C(y)\}$$
$$= 1 - \min\{x, 1 - x + 1 - y - (1 - x) \cdot (1 - y)\}$$
$$= 1 - \min\{x, 1 - x \cdot y\}$$
$$= \max\{1 - x, x \cdot y\}, \tag{66}$$

which does not satisfy (13) since

$$0.2 \le 0.9 \Longrightarrow I_{N_C,T_M,S_P}(0.2, 1) = 0.8 < 0.9 = I_{N_C,T_M,S_P}(0.9, 1).$$

$$(67)$$

Therefore, the first main problem is the characterization of those QL'-operations, which satisfy (13). Similar to the previous section, only partial results will be proved.

Proposition 8. *A function $I: [0,1]^2 \longrightarrow [0,1]$ is called a QL'-implication if it is a QL'-operation and satisfies (13).*

Proposition 9. *Let $I_{N,T,S}$ be a QL'-operation, which satisfies (21); then, N is a strong negation.*

Proof. The proof is similar to the proof of Proposition 6.

Proposition 10. *Let $I_{N,T,S}$ be a QL'-operation. $I_{N,T,S}$ satisfies (20) if the pair (T, N) satisfies the law of contradiction (10).*

Proof. We assume that the pair (T, N) satisfies the law of contradiction (10). Hence, $I_{N,T,S}$ satisfies (20) since

$$\begin{aligned} I_{N,T,S}(x, 1) &= N(T(x, S(N(x), N(1)))) \\ &= N(T(x, S(N(x), 0))) \\ &= N(T(x, N(x))) \\ &= N(0) = 1, \quad \forall x \in [0, 1]. \end{aligned}$$

$$(68)$$

Proposition 11. *Let $I_{N,T,S}$ be a QL'-operation generated by a nonfilling negation N. Moreover, if $I_{N,T,S}$ satisfies (20), then the pair (T, N) satisfies the law of contradiction (10).*

Proof. Let $I_{N,T,S}$ be a QL'-operation generated by a nonfilling negation N and satisfy (20). Therefore, it is

$$\begin{aligned} I_{N,T,S}(x, 1) = 1 &\Longrightarrow N(T(x, S(N(x), N(1)))) = 1 \\ &\Longrightarrow N(T(x, S(N(x), 0))) = 1 \\ &\Longrightarrow N(T(x, N(x))) = 1 \\ &\Longrightarrow T(x, N(x)) = 0, \forall x \in [0, 1] \end{aligned}$$

$$(69)$$

since N is a nonfilling negation. Thus, the pair (T, N) satisfies (10).

The set of QL'-implications is nonempty. This is ensured with the next theorem and some remarks.

Theorem 7. *If N is a strong negation and $I_{N,T,S'}$ is a QL'-operation, then*

   (i) *$I_{N,T,S'} = I_{T',S,N} = (I^{T',S,N})_N = (I^{N,T,S'})_N$, where S, T are N-dual of T, S', respectively, $I^{T',S,N}$ is a QL-operation, and $(I_{T',S,N})_N$ is the N-reciprocal of the QL-operation $I_{T',S,N}$.*

   (ii) *Moreover, if one of $I_{N,T,S'}$, $I^{N,T,S'}$, $I^{T',S,N}$, and $I_{T',S,N}$ is a fuzzy implication, then the other three are fuzzy implications, too.*

Proof

   (i) Let $I_{N,T,S'}$ be a QL'-operation. Then,

$$\begin{aligned} I_{N,T,S'}(x, y) &= N(T(x, S'(N(x), N(y)))) \\ &= N(T(N(N(x)), N(T'(x, y)))) \\ &= S(N(x), T'(x, y)) \\ &= S(T'(x, y), N(x)) \\ &= S(T'(N(N(x)), N(N(y))), N(x)) \\ &= S(T'(N(N(y)), N(N(x))), N(x)) \\ &= I^{T',S,N}(N(y), N(x)) \\ &= \left(I^{T',S,N}\right)_N(x, y). \end{aligned}$$

$$(70)$$

By Theorem 4, we deduce that

$$I_{N,T,S'} = \left(I^{T',S,N}\right)_N = \left(I^{N,T,S'}\right)_N = \left(\left(I_{T',S,N}\right)_N\right)_N = I_{T',S,N},$$

(see [1] (1.18)).

$$(71)$$

   (ii) If one of $I^{N,T,S'}$, $I^{T',S,N}$, and $I_{T',S,N}$ is a fuzzy implication, then the other two are also fuzzy implications, according to Theorem 4 (ii). Moreover, since $I_{T',S,N} = I_{N,T,S'}$, $I_{N,T,S'}$ is a fuzzy implication. Vice versa, if $I_{N,T,S'}$ is a fuzzy implication, then $I_{T',S,N}$ is also a fuzzy implication, so $I^{T',S,N}$ and $I^{N,T,S'}$ are fuzzy implications.

Remark 9

   (i) By Theorem 7, it is obvious that if (T, S, N) is a De Morgan triple and $I_{N,T,S}$ is the corresponding QL'-operation (respectively, implication), then

$$I_{N,T,S} = I_{T,S,N} = \left(I^{T,S,N}\right)_N = \left(I^{N,T,S}\right)_N.$$

$$(72)$$

   (ii) By (i) and Table 5, we deduce that $I_{N_C,T_L,S_L} = (I^{N_C,T_L,S_L})_{N_C} = (I_{KD}')' = I_{KD}$. Thus, the set of QL'-implications is nonempty.

Theorem 8. *Let $I_{N,T,S}$ be a QL'-implication and N be a nonfilling negation; then, it is $T(N(x), x) = 0, x \in [0, 1]$, that is, the pair (T, N) satisfies the law of contradiction (10).*

Proof. If $I_{N,T,S}$ is a QL'-implication, then it satisfies (20), and if N is a nonfilling negation, then the proof is given by Proposition 10.

Remark 10. By Theorem 8, it is obvious that if N is a nonfilling negation and the pair (T, N) does not satisfy the law of contradiction (10), i.e., $T(N(x), x) \ne 0$, for some $x \in [0, 1]$, then the corresponding QL'-operation, for any t-conorm S, is not a fuzzy implication.

Theorem 9. *If $\phi \in \Phi$ and $I_{N,T,S}$ is a QL'-operation (respectively, implication), then $(I_{N,T,S})_\phi$ is a QL'-operation (respectively, implication), and moreover, $(I_{N,T,S})_\phi = I_{N_\phi,T_\phi,S_\phi}$.*

Proof. Let $I_{N,T,S}$ be a QL'-operation (respectively, implication); then, $(I_{N,T,S})_\phi$ is a QL'-operation (respectively, implication), according to Remark 4. Moreover, for all $x, y \in [0,1]$, we deduce that

$$
\begin{aligned}
\left(I_{N,T,S}\right)_\phi(x,y) &= \phi^{-1}\left(I_{N,T,S}(\phi(x),\phi(y))\right) \\
&= \phi^{-1}\left(N\left(T\left(\phi(x), S\left(N\left(\phi(x)\right), N\left(\phi(y)\right)\right)\right)\right)\right) \\
&= \phi^{-1}\left(N\left(\phi\left(\phi^{-1}\left(T\left(\phi(x), S\left(\phi\left(\phi^{-1}\left(N\left(\phi(x)\right)\right)\right), \phi\left(\phi^{-1}\left(N\left(\phi(y)\right)\right)\right)\right)\right)\right)\right)\right)\right) \\
&= N_\phi\left(\phi^{-1}\left(T\left(\phi(x), \phi\left(\phi^{-1}\left(S\left(\phi\left(N_\phi(x)\right), \phi\left(N_\phi(y)\right)\right)\right)\right)\right)\right)\right) \\
&= N_\phi\left(T_\phi\left(x, S_\phi\left(N_\phi(x), N_\phi(y)\right)\right)\right) \\
&= I_{N_\phi,T_\phi,S_\phi}(x,y).
\end{aligned}
\tag{73}
$$

Since we have studied some general results for QL'-operations (respectively, implications), we are going to study some more specific cases. Several questions arising are similar to those we answer for D'-operations (respectively, implications). Answers to many of them are given above, and they are similar to previous answers we gave for D'-operations (respectively, implications). Finally, we have to answer the next question. Are QL'-operations (respectively, implications) a new family of operations (respectively, implications)? It is known that there is no QL-implication generated by N_{D1}. Moreover, there is no QL-implication which has N_{D1} as its natural negation (see Remark 2.6.6(i) of [4]). On the contrary, we have the next proposition.

Proposition 12. *By a triple of the form (T, S, N_{D1}), where T is any t-norm and S is any t-conorm, a QL'-implication is obtained, which is I_3 (see [22]).*

Proof. By a triple of the form (T, S, N_{D1}), where T is any t-norm and S is any t-conorm, we obtain

$$
\begin{aligned}
I_{N_{D1},T,S}(x,y) &= N_{D1}\left(T\left(x, S\left(N_{D1}(x), N_{D1}(y)\right)\right)\right) \\
&= \begin{cases} N_{D1}\left(T\left(x, S\left(N_{D1}(x), 1\right)\right)\right), & \text{if } y = 0, \\ N_{D1}\left(T\left(x, S\left(N_{D1}(x), 0\right)\right)\right), & \text{if } y > 0, \end{cases} \\
&= \begin{cases} N_{D1}\left(T(x, 1)\right), & \text{if } y = 0, \\ N_{D1}\left(T\left(x, N_{D1}(x)\right)\right), & \text{if } y > 0, \end{cases} \\
&= \begin{cases} N_{D1}(x), & \text{if } y = 0, \\ N_{D1}(0), & \text{if } y > 0, \end{cases} \\
&= \begin{cases} N_{D1}(x), & \text{if } y = 0, \\ 1, & \text{if } y > 0, \end{cases} \\
&= \begin{cases} 0, & \text{if } x > 0 \text{ and } y = 0, \\ 1, & \text{otherwise}, \end{cases} \\
&= I_3(x,y).
\end{aligned}
\tag{74}
$$

Remark 11. Because of Proposition 12, we conclude that the only QL'-implication, which has N_{D1} as its natural negation, is I_3. So, automatically, we conclude that I_0, I_{GG}, I_{GD}, I_{RS},

and I_{YG} are not QL'-implications since all of them have N_{D1} as their natural negation.

Although it seems simple, Proposition 12 is very important. I_3 is a QL'-implication, which is obviously not a QL-implication since its natural negation is N_{D1}. So, it is proved that the family of QL'-implications is not the same with the QL-implications' family.

Proposition 13. *By a triple of the form (T, S, N_{D2}), where T is any t-norm and S is any t-conorm, a QL'-implication is obtained, which is I_4 (see [22]).*

Proof. By a triple of the form (T, S, N_{D2}), where T is any t-norm and S is any t-conorm, we obtain

$$
\begin{aligned}
I_{N_{D2},T,S}(x,y) &= N_{D2}\left(T\left(x, S\left(N_{D2}(x), N_{D2}(y)\right)\right)\right) \\
&= \begin{cases} N_{D2}\left(T\left(x, S\left(N_{D2}(x), 1\right)\right)\right), & \text{if } y < 1, \\ N_{D2}\left(T\left(x, S\left(N_{D2}(x), 0\right)\right)\right), & \text{if } y = 1, \end{cases} \\
&= \begin{cases} N_{D2}\left(T(x, 1)\right), & \text{if } y < 1, \\ N_{D2}\left(T\left(x, N_{D2}(x)\right)\right), & \text{if } y = 1, \end{cases} \\
&= \begin{cases} N_{D2}(0), & \text{if } x = 1 \text{ and } y = 1, \\ N_{D2}(x), & \text{otherwise}, \end{cases} \\
&= \begin{cases} 1, & \text{if } x = 1 \text{ and } y = 1, \\ 1, & \text{if } x < 1, \\ 0, & \text{if } x = 1 \text{ and } y < 1, \end{cases} \\
&= \begin{cases} 0, & \text{if } x = 1 \text{ and } y < 1, \\ 1, & \text{otherwise}. \end{cases} \\
&= I_4(x,y).
\end{aligned}
\tag{75}
$$

Remark 12. Because of Proposition 13, we conclude that the only QL'-implication, which has N_{D2} as its natural negation, is I_4. So, automatically, we conclude that I_1 and I_{WB} are not QL'-implications since all of them have N_{D2} as their natural negation.

The question we need to answer is the relation between these families (QL and QL'). An answer is given partially in

Proposition 9 where it is proved that if $I^{N,T,S}$ satisfies (21), then N is a strong negation. Clearly by Theorem 7, if N is a strong negation, then the corresponding QL'-operation (respectively, implication) is also a QL-operation (respectively, implication) and vice versa. Moreover, for nonstrong negations, the corresponding QL-operations (respectively, implications) satisfy (21) (see Proposition 2.6.2 of [4]), and the corresponding QL'-operations (respectively, implications) do not satisfy (21). All these results lead to Figure 3.

In the following of this section, we present a useful theorem for the disqualification of some triples (T, S, N) that do not generate QL'-implications.

Theorem 10. *By any triple (T, S, N), where S is any t-conorm, N is any continuous nonfilling fuzzy negation, and T is any idempotent, strict, or positive t-norm, any QL'-implication cannot be obtained.*

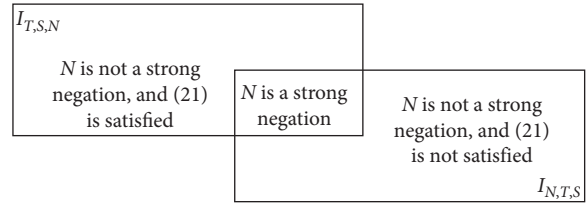*Proof.* The proof is similar to the proof of Theorem 5. It results by Remark 10.

FIGURE 3: Intersection between families of QL- and QL'-operations (respectively, implications).

A very useful proposition is as follows.

Proposition 14. *By a triple of the form (T_D, S, N_C), where S is any idempotent or positive t-conorm, a QL'-implication is obtained, which is I_{DP}.*

Proof. As we mentioned before, the only idempotent t-conorm is S_M, which is also positive. By the triple (T_D, S, N_C), where S is any positive t-conorm, we obtain

$$
\begin{aligned}
I_{N_C, T_D, S}(x, y) &= N_C(T_D(x, S(N_C(x), N_C(y)))) \\
&= 1 - T_D(x, S(1 - x, 1 - y)) \\
&= 1 - \begin{cases} 0, & \text{if } x < 1 \text{ and } S(1-x, 1-y) < 1, \\ x, & \text{if } S(1-x, 1-y) = 1, \\ S(1-x, 1-y), & \text{if } x = 1, \end{cases} \\
&= 1 - \begin{cases} 0, & \text{if } x < 1 \text{ and } S(1-x, 1-y) < 1, \\ x, & \text{if } 1-x = 1 \text{ or } 1-y = 1, \\ S(0, 1-y), & \text{if } x = 1, \end{cases} \\
&= 1 - \begin{cases} 0, & \text{if } x < 1 \text{ and } S(1-x, 1-y) < 1, \\ x, & \text{if } x = 0 \text{ or } y = 0, \\ 1-y, & \text{if } x = 1, \end{cases} \\
&= 1 - \begin{cases} 1-y, & \text{if } x = 1, \\ x, & \text{if } y = 0, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} y, & \text{if } x = 1, \\ 1-x, & \text{if } y = 0, \\ 1, & \text{otherwise}, \end{cases} \quad = I_{DP}(x, y).
\end{aligned}
\tag{76}
$$

Lastly, in this section, we present Table 6, which contains basic QL'-implications.

6. R'-Implications

The obtained generalization via De Morgan's laws of R-implications leads to the next definition, which refers to the family of R'-operations. This generalization is a counterexample for the fact that axioms (13)–(17) are not invariant via an application of De Morgan's laws. To be more precise, axiom (17) is not invariant since R'-operations violate it, as we will show in the following. This is the reason we use the term operations, rather than implications.

Definition 14. A function $I: [0, 1]^2 \longrightarrow [0, 1]$ is called an R'-operation if there exist a t-conorm S and a fuzzy negation N such that

$$
\begin{aligned}
I(x, y) = \sup\{t \in [0, 1] | N(S(N(x), N(t))) \le y\}, \\
x, y \in [0, 1].
\end{aligned}
\tag{77}
$$

If I is an R'-operation generated by a t-conorm S and a fuzzy negation N, then we denote it by I_S^N.

The induced family of R'-operations is a special case of residual implicator of the conjunctor $C(x, y) = N(S(N(x), N(y)))$, where N is a fuzzy negation and S is a t-conorm (see [23]). Also, this formula is a special case of

TABLE 6: Basic QL'-implications.

T	S	N	Formula
T_L	S_M	N_C	I_{LK}
T_L	S_P	N_C	I_{RC}
T_L	S_L	N_C	I_{KD}
T_D	Any idempotent or positive S	N_C	I_{DP}
T_{nM}	S_M	N_C	I_{FD}
Any T	Any S	N_{D1}	I_3
Any T	Any S	N_{D2}	I_4

formula (2) in Lemma 1 of [24]. However, we will study this special family because of the crucial results we will obtain.

Theorem 11. *Let I_S^N be an R'-operation; then, I_S^N satisfies (13), (14), (15), and (16).*

Proof. Let I_S^N be an R'-implication; then, for $x, y, z \in [0,1]$, we have to show that if $x \leq y$, then $I_S^N(x,z) \geq I_S^N(y,z)$, which is equivalent with the inequality

$$\sup\{t \in [0,1] | N(S(N(x), N(t))) \leq z\}$$
$$\geq \sup\{t \in [0,1] | N(S(N(y), N(t))) \leq z\}, \tag{78}$$

which means that we have to show the inclusion

$$\{t \in [0,1] | N(S(N(x), N(t))) \leq z\} \supset \{t \in [0,1] | N(S(N(y), N(t))) \leq z\}. \tag{79}$$

So, for any $t \in [0,1]$ such that $N(S(N(y), N(t)) \leq z$, it is obvious that

$$x \leq y \Longrightarrow N(x) \geq N(y)$$
$$\Longrightarrow S(N(x), N(t)) \geq S(N(y), N(t)) \tag{80}$$
$$\Longrightarrow N(S(N(x), N(t))) \leq N(S(N(y), N(t))).$$

Therefore, I_S^N satisfies (13).

Secondly, we have to prove (14); hence, for $x, y, z \in [0,1]$, we have to show that if $y \leq z$, then $I_S^N(x,y) \leq I_S^N(x,z)$, which is equivalent with the inequality

$$\sup\{t \in [0,1] | N(S(N(x), N(t))) \leq y\}$$
$$\leq \sup\{t \in [0,1] | N(S(N(x), N(t))) \leq z\}, \tag{81}$$

which is obvious that it holds. Therefore, I_S^N satisfies (14).

I_S^N satisfies (15) since

$$I_S^N(0,0) = \sup\{t \in [0,1] | N(S(N(0), N(t))) \leq 0\}$$
$$= \sup\{t \in [0,1] | N(S(1, N(t))) \leq 0\}$$
$$= \sup\{t \in [0,1] | N(1) \leq 0\} \tag{82}$$
$$= \sup\{t \in [0,1] | 0 \leq 0\} = 1.$$

I_S^N satisfies (16) since

$$I_S^N(1,1) = \sup\{t \in [0,1] | N(S(N(1), N(t))) \leq 1\}$$
$$= \sup\{t \in [0,1] | N(S(0, N(t))) \leq 1\} \tag{83}$$
$$= \sup\{t \in [0,1] | N(N(t)) \leq 1\} = 1,$$

since $N(N(1)) = N(0) = 1$.

The formula of R'-operations is very complicated and makes their study really difficult. Furthermore, R'-operations do not always satisfy (17) since

$$I_S^{N_{D2}}(1,0) = \sup\{t \in [0,1] | N_{D2}(S(N_{D2}(1), N_{D2}(t))) \leq 0\}$$
$$= \sup\{t \in [0,1] | N_{D2}(S(0, N_{D2}(t))) \leq 0\}$$
$$= \sup\{t \in [0,1] | N_{D2}(N_{D2}(t)) \leq 0\}$$
$$= \sup\{t \in [0,1] | N_{D2}(N_{D2}(t)) = 0\}$$
$$= \sup\{t \in [0,1] | N_{D2}(t) = 1\}$$
$$= \sup\{t \in [0,1] | t < 1\} = 1. \tag{84}$$

So, it is obvious that there does not exist any R'-implication generated by N_{D2}. All these results above lead to the next proposition without the proof because it is obvious.

Proposition 15. *A function $I: [0,1]^2 \longrightarrow [0,1]$ is called an R'-implication if it is an R'-operation and satisfies (17).*

The question arising is when an R'-operation is a fuzzy implication. The next theorem is the answer.

Theorem 12. *Let I_S^N be an R'-operation. I_S^N is an R'-implication if and only if the negation N satisfies the equivalence $N(N(x)) = 0 \Longleftrightarrow x = 0$.*

Proof. If the negation N satisfies the equivalence $N(N(x)) = 0 \Longleftrightarrow x = 0$, then I_S^N satisfies (17) since

$$I_S^N(1,0) = \sup\{t \in [0,1] | N(S(N(1), N(t))) \leq 0\}$$
$$= \sup\{t \in [0,1] | N(S(0, N(t))) \leq 0\}$$
$$= \sup\{t \in [0,1] | N(N(t)) \leq 0\} \tag{85}$$
$$= \sup\{t \in [0,1] | N(N(t)) = 0\}$$
$$= \sup\{t \in [0,1] | t = 0\} = 0.$$

Hence, by Proposition 15, we deduce that I_S^N is a fuzzy implication. On the contrary, if I_S^N is an R'-implication, then (17) is satisfied. Hence,

$$I_S^N(1,0) = 0 \Longleftrightarrow \sup\{t \in [0,1] | N(S(N(1), N(t))) \leq 0\} = 0$$
$$\Longleftrightarrow \sup\{t \in [0,1] | N(S(0, N(t))) \leq 0\} = 0$$
$$\Longleftrightarrow \sup\{t \in [0,1] | N(N(t)) \leq 0\} = 0$$
$$\Longleftrightarrow \sup\{t \in [0,1] | N(N(t)) = 0\} = 0, \tag{86}$$

which means $N(N(x)) = 0 \Longleftrightarrow x = 0$.

Remark 13. By Theorem 12, we deduce one more time that there does not exist any R'-implication generated by N_{D2} since $N_{D2}(N_{D2}(x)) = 0 \Longleftrightarrow x < 1$.

Proposition 16. *The R'- implication genrated from N_{D1} and any t- conorm S is I_0.*

Proof. The R'- implication genrated from N_{D1} and any t-conorm S is

$$
\begin{aligned}
I_S^{N_{D1}}(x, y) &= \sup\{t \in [0,1] | N_{D1}(S(N_{D1}(x), N_{D1}(t))) \le y\} \\
&= \begin{cases} \sup\{t \in [0,1] | N_{D1}(S(1, N_{D1}(t))) \le y\}, & \text{if } x = 0, \\ \sup\{t \in [0,1] | N_{D1}(S(0, N_{D1}(t))) \le y\}, & \text{if } x > 0, \end{cases} \\
&= \begin{cases} \sup\{t \in [0,1] | N_{D1}(1) \le y\}, & \text{if } x = 0, \\ \sup\{t \in [0,1] | N_{D1}(N_{D1}(t)) \le y\}, & \text{if } x > 0, \end{cases} \\
&= \begin{cases} \sup\{t \in [0,1] | 0 \le y\}, & \text{if } x = 0, \\ \sup\{t \in [0,1] | N_{D1}(N_{D1}(t)) < 1\}, & \text{if } x > 0 \text{ and } y < 1, \\ \sup\{t \in [0,1] | N_{D1}(N_{D1}(t)) \le 1\}, & \text{if } x > 0 \text{ and } y = 1, \end{cases} \\
&= \begin{cases} 1, & \text{if } x = 0, \\ \sup\{t \in [0,1] | N_{D1}(t) > 0\}, & \text{if } x > 0 \text{ and } y < 1, \\ \sup\{t \in [0,1] | t \ge 0\}, & \text{if } x > 0 \text{ and } y = 1, \end{cases} \\
&= \begin{cases} 1, & \text{if } x = 0, \\ \sup\{t \in [0,1] | N_{D1}(t) = 1\}, & \text{if } x > 0 \text{ and } y < 1, \\ 1, & \text{if } x > 0 \text{ and } y = 1, \end{cases} \\
&= \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0 \text{ and } y < 1, \\ 1, & \text{if } x > 0 \text{ and } y = 1, \end{cases} \\
&= \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 1, \\ 0, & \text{otherwise}, \end{cases} \\
&= I_0(x, y).
\end{aligned}
\tag{87}
$$

Remark 14. By Proposition 16, we conclude that the set of R'-implications is nonempty. Moreover, R'-implications are a different family than the family of R-implications. This happens because I_0 is an R'-implication. On the contrary, it is not an R-implication since it does not satisfy (21) (see Theorem 2.5.4 of [4]).

Theorem 13. *Let I_S^N be an R'-operation. If I_S^N satisfies (21), then $N(N(x)) \le x$, for any $x \in [0, 1]$.*

Proof. Let I_S^N be an R'-operation, which satisfies (21); then, we have

$$
\begin{aligned}
I_S^N(1, x) = x &\implies \sup\{t \in [0,1] | N(S(N(1), N(t))) \le x\} = x \\
&\implies N(S(N(1), N(x))) \le x \\
&\implies N(S(0, N(x))) \le x \\
&\implies N(N(x)) \le x,
\end{aligned}
\tag{88}
$$

for any $x \in [0, 1]$.

Theorem 14. *If N is a strictly decreasing nonstrong negation, then the obtained I_S^N is an R'-implication, which does not satisfy (21).*

Proof. Let I_S^N be an R'-operation. Since N is a strictly decreasing nonstrong negation, we have that

$$
\begin{aligned}
N(N(x)) = 0 &\iff N(N(x)) = N(1) \\
&\iff N(x) = 1 \\
&\iff N(x) = N(0) \\
&\iff x = 0.
\end{aligned}
\tag{89}
$$

So, by Theorem 12, we deduce that I_S^N is an R'-implication. If we assume that I_S^N satisfies (21), then by Theorem 13, we have $N(N(x)) \le x$, for any $x \in [0, 1]$. Since N is not a strong

negation, there is at least one $x_0 \in (0,1)$ such that $N(N(x_0)) \neq x_0$, so $N(N(x_0)) < x_0$. N is a strictly decreasing function, so $N(N(N(x_0))) > N(x_0)$, which means there is $y_0 = N(x_0) \in (0,1)$ such that $N(N(y_0)) > y_0$, which is a contradiction. Thus, I_S^N does not satisfy (21).

Theorem 15. *If N is a strong negation and I_S^N is an R'-implication, then $I_S^N = I_T$, where the t-norm T is the N-dual of t-conorm S.*

Proof. The proof is obvious.

Theorem 15 leads us to have no interest about R'-implications generated by strong negations. Moreover, we have to mention that R-implications always satisfy (21) (see Theorem 2.5.4 of [4]). So, some results are visualized in Figure 4.

Theorem 16. *If $\phi \in \Phi$ and I_S^N is an R'-implication, then $(I_S^N)_\phi$ is an R'-implication. Moreover,*

$$\left(I_S^N\right)_\phi = I_{S_\phi}^{N_\phi}. \tag{90}$$

Proof. Let I_S^N be an R'-implication; then, $(I_S^N)_\phi$ is an R'-implication according to Remark 4. Now, from the continuity of the bijection ϕ, we have

$$
\begin{aligned}
\left(I_S^N\right)_\phi(x,y) &= \phi^{-1}\left(I_S^N(\phi(x),\phi(y))\right) \\
&= \phi^{-1}\left(\sup\{t \in [0,1] | N(S(N(\phi(x)),N(t))) \leq \phi(y)\}\right) \\
&= \sup\left\{\phi^{-1}(t) \in [0,1] | \phi^{-1}(N(S(N(\phi(x)),N(t)))) \leq y\right\} \\
&= \sup\left\{t \in [0,1] | \phi^{-1}(N(S(N(\phi(x)),N(\phi(t))))) \leq y\right\} \\
&= \sup\left\{t \in [0,1] | \phi^{-1}\left(N\left(S\left(\phi\left(\phi^{-1}(N(\phi(x)))\right),\phi\left(\phi^{-1}(N(\phi(t)))\right)\right)\right)\right) \leq y\right\} \\
&= \sup\left\{t \in [0,1] | \phi^{-1}\left(N\left(\phi\left(\phi^{-1}\left(S(\phi(N_\phi(x)),\phi(N_\phi(t)))\right)\right)\right)\right) \leq y\right\} \\
&= \sup\left\{t \in [0,1] | N_\phi\left(S_\phi\left(N_\phi(x),N_\phi(t)\right)\right) \leq y\right\} \\
&= I_{S_\phi}^{N_\phi}(x,y),
\end{aligned}
\tag{91}
$$

for any $x,y \in [0,1]$.

Lastly, in this section, we present Table 7, which contains basic R'-implications. The formulas of R'-implications which are generated from N_C are calculated by Theorem 15 and Table 2.6 of [4].

7. Some Results before Conclusions

According to Figures 1–3, we know that, for nonstrong negations, (T,N)-, D'-, and QL'-implications do not satisfy (21). The same result holds for R'-implications when we use a strictly decreasing negation N according to Theorem 14. On the contrary, (S,N)-, D-, QL-, and R-implications always satisfy (21).

Furthermore, there are other known families of fuzzy implications, such as

(i) Yager's f-generated and g-generated implications as they are defined by Yager [12] (see also [4, 9]),

(ii) h-implications as they are defined by Jayaram [9, 25, 26],

(iii) h-implications as they are defined by Massanet and Torrens [10], and

(iv) Fuzzy implications through fuzzy negations as they are defined by Souliotis and Papadopoulos [11].

All these aforementioned families satisfy (21) (see Theorems 3.1.7 and 3.2.8 of [4], Theorem 9(i) of [9],

Theorem 5 of [10], and Proposition 6 of [11]), so it is obvious that D'-, QL'- and R'-implications are different families from the aforementioned.

Also, there are (U,N)- and RU-implications (see [4]) and (h,e)-implications as they are defined by Massanet and Torrens [10]. These families do not satisfy (21) (see Remarks 5.3.8 and 5.4.6 of [4] and Theorem 13 of [10]), unlike D'-, QL'-, and R'-implications, which satisfy (21) when N is a strong negation, according to Theorems 1, 6, and 15 (in combination with Theorem 2.5.4 of [4]). Thus, D'-, QL'-, and R'-implications are different families from the aforementioned.

Lastly, we investigate the relation between the families of (T,N)-, D'-, QL'-, and R'-implications. As sets of operations, they are different since (T,N)-implications satisfy (13)–(17), D'-operations do not always satisfy (14), QL'-operations do not always satisfy (13), and R'-operations do not always satisfy (17).

Now, let S_{SS}^λ be the Schweizer–Sklar t-conorm; then, for $\lambda = 2$, it is

$$S_{SS}^2 = 1 - \sqrt{\max\{(1-x)^2 + (1-y)^2 - 1, 0\}}. \tag{92}$$

The obtained QL-implication by the triple (T_P, S_{SS}^2, N_C) is

$$
\begin{aligned}
I_{T_P, S_{SS}^2, N_C}(x,y) &= I_{PC}(x,y) \\
&= 1 - \sqrt{\max\{x \cdot (x + x \cdot y^2 - 2y), 0\}},
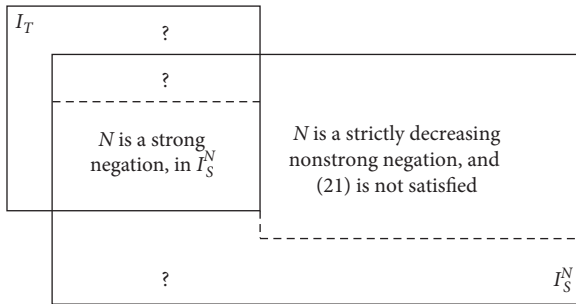\end{aligned}
\tag{93}
$$

Figure 4: Intersection between families of R- and R'-operations (respectively, implications).

Table 7: Basic R'-implications.

S	N	Formula
S_M	N_C	I_{GD}
S_P	N_C	I_{GG}
S_L	N_C	I_{LK}
S_D	N_C	I_{WB}
S_{nM}	N_C	I_{FD}
Any S	N_{D1}	I_0

which is not an (S, N)-implication since I_{PC} does not satisfy (22) (see Example 2.6.15, page 135, and Figure 4.2 of [4]).
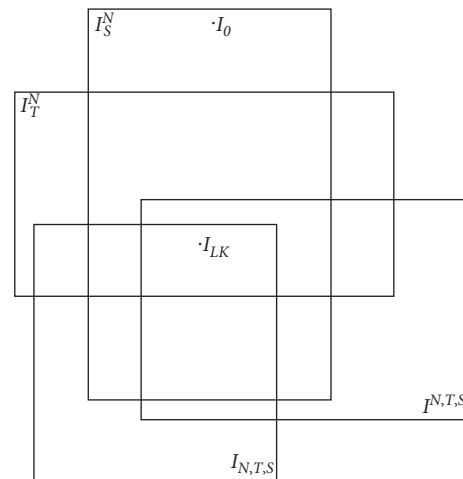
$$I^{N_C, T_{SS}^2, S_P}(x, y) = \left(I_{T_P, S_{SS}^2, N_C}\right)_{N_C}(x, y) = I_{PC}'(x, y)$$
$$= 1 - \sqrt{\max\left\{(1 - y) \cdot \left[(1 - y) + (1 - y) \cdot (1 - x)^2 - 2 \cdot (1 - x)\right], 0\right\}}, \tag{94}$$

which does not satisfy (22) since

$$I_{PC}'(0.7, I_{PC}'(0.8, 0.2)) = 0.84515 \neq 0.80071$$
$$= I_{PC}'(0.8, I_{PC}'(0.7, 0.2)), \tag{95}$$

so the D'-implication $I^{N_C, T_{SS}^2, S_P} = I_{PC}'$ is not an (S, N)-implication; hence, it is not a (T, N)-implication.

On the contrary, I_{LK} belongs to all families that are generated and defined in this paper and those that are mentioned in Table 4 since

$$I_{S_L, N_C} = I_{T_L}^{N_C} = I_{T_M, S_L, N_C} = I^{T_M, S_L, N_C} = I_{T_L}$$
$$= I^{N_C, T_L, S_M} = I_{N_C, T_L, S_M} = I_{S_L}^{N_C} = I_{LK}. \tag{96}$$

Moreover, it is obvious that D'-implications are the reciprocals of QL'-implications for strong negations since they are the same sets with their preliminary families, respectively. Surprisingly, sometimes, this property holds for nonstrong negations (for example, I_3 and I_4 in Tables 6 and 7). The exact relation of these two families needs more investigation. The same happens for the relation between R- and R'-implications.

Finally, some of these results are presented in Figures 5–7, where we can see that the intersection between (T, N)-, D'-, QL'-, and R'-implications is nonempty. Moreover, D'-, QL'-,

Figure 5: Sets of (T, N)-, D'-, QL'-, and R'-implications.

Since the natural negation of I_{PC} is the strong negation N_C, obviously, I_{PC} is not a (T, N)-implication, and it is a QL'-implication. More specifically, $I_{N_C, T_{SS}^2, S_P} = I_{PC}$, where the t-norm T_{SS}^2 is the N_C-dual of t-conorm S_{SS}^2.

Moreover, it is

and R'-implications contain at least one fuzzy implication that (T, N)-implications do not contain.

8. Conclusions

Many families of fuzzy implications can be produced by well-known generalizations of the notion of implication from classical to fuzzy logic. Moreover, there are fuzzy implications, which, in their formula, contain at least a t-norm (such as R-implications) or a t-conorm (such as (S, N)-implications). De Morgan's laws and, if necessary, the classical law of double negation are useful tools to transform these already known families of fuzzy implications. As a result, new families of fuzzy implications are arising. These new families are the "closest" of their preliminary-"creator" families. At this point, we should remark that the preliminary and the induced family of fuzzy implications are different sets, at least in any family we mention in this paper. The induced family is an expansion of its preliminary family, in a field where all these fuzzy implications simulate the same (or a similar) way of classical thinking.

In this paper, we specifically mentioned a known family, the so-called (T, N)-implications, and we have studied three new families, the so-called D'-, QL'-, and R'-implications.
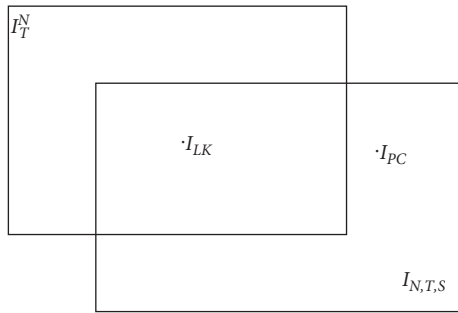
FIGURE 6: Sets of (T, N)- and QL'-implications.
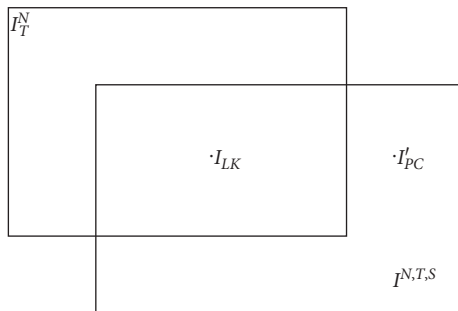
FIGURE 7: Sets of (T, N)- and D'-implications.

The first three families give us the following results. For strong negations, the sets of (S, N)- and (T, N)-implications are the same set. The same result holds for the sets of QL- and QL'-implications and for D- and D'-implications, respectively. However, if the negation we use is not strong, then the preliminary families (S, N)-, QL-, and D-implications satisfy (21), and the corresponding induced families (T, N)-, QL'-, and D'-implications do not satisfy (21). Although these new families are common with their preliminary families when N is strong, they have no common implications when N is not strong. These families are very important since they are the "first" generalized generators from classical to fuzzy logic, where for nonstrong negations, (21) is not satisfied.

Furthermore, the last family, the so-called R'-implications, gives the following results. If we use a strong negation N in the formula of R'-implications, then R- and R'-implications are the same set. Moreover, for strictly decreasing and nonstrong negations N, the obtained R'-operations are fuzzy implications that do not satisfy (21), a property that R-implications always satisfy. This family is a counterexample that axioms (13)–(17) are not invariant, or, more specifically, axiom (17) is surely not invariant, via the application of De Morgan's laws.

However, the characterization of triples (T, S, N), such that a D'-operation or a QL'-operation is a fuzzy implication, is still unsolved. On the contrary, the condition under which an R'-operation is a fuzzy implication has been proved.

Another result is that the expected duality of the properties does not hold, in general, via this application of classical logic's laws, but under some conditions.

Unfortunately, the induced families must be studied individually every time since there is no general theory that seems to hold for every induced family.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

D. S. G. contributed to writing the original draft. B. K. P. supervised the study and contributed to writing, reviewing, and editing. Both authors read and agreed to the published version of the manuscript.

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