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Research Article

A_α-Spectral Characterizations of Some Joins

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Let G be a graph with n vertices. For every real $\alpha \in [0,1]$, write $A_\alpha(G)$ for the matrix $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $A(G)$ and $D(G)$ denote the adjacency matrix and the degree matrix of G, respectively. The collection of eigenvalues of $A_\alpha(G)$ together with multiplicities are called the A_α-spectrum of G. A graph G is said to be determined by its A_α-spectrum if all graphs having the same A_α-spectrum as G are isomorphic to G. In this paper, we show that some joins are determined by their A_α-spectra for $\alpha \in (0,1/2)$ or $(1/2, t1)$.

1. Introduction

We use G to denote a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The degree of a vertex $v \in V(G)$ is denoted by $\mathrm{d}(v)$. For a subgraph H of G, let $G - E(H)$ denote the subgraph obtained from G by deleting the edges of H. Let $c_i(G)$ and $p_i(G)$ denote, respectively, the numbers of i-cycles and i-vertex paths in G. Let $c_3(G_v)$ denote the number of triangles containing the vertex v of G. Let $G \cup H$ be the union of two graphs G and H which have no common vertices. For any positive integer l, let lG denote be the union of l disjoint copies of graph G. The join of two disjoint graphs G and H, denoted by $G \vee H$, is the graph obtained by joining each vertex of G to each vertex of H. For convenience, the complete graph, path, cycle and star on n vertices are denoted by K_n, P_n, C_n, and $K_{1,n-1}$, respectively.

Let $A(G)$ and $D(G)$ denote, respectively, the adjacency matrix and degree matrix of G. For every real $\alpha \in [0,1]$, write $A_\alpha(G)$ for the matrix $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$. Note that $A_0(G) = A(G)$ and $2A_{1/2}(G) = Q(G)$, where $Q(G)$ is the signless Laplacian matrix of G. The polynomial

$$\phi(G) = \phi(G,x) = \det\left(xI - A_\alpha(G)\right) = \sum_{i=1}^{n} c_{\alpha i}(G) x^{n-i}, \quad (1)$$

is called A_α-characteristic polynomial, where I is the identity matrix of order n. The theory of A_α-characteristic polynomial of a graph is well elaborated [1–8].

The A_α-spectrum of G is a collection of roots of $\phi(G)$ together with multiplicities. Two graphs are said to be A_α-cospectral if they have the same A_α-spectrum. A graph is called an A_α-DS graph if it is determined by its A_α-spectrum, meaning that there exists no other graph that is non-isomorphic to it but A_α-cospectral with it.

It is interesting to characterize which graph is determined by some graph spectrum [9–11]. The problem was raised by Günthard and Primas [12] in 1956 with motivations from chemistry. In recent years, although many graphs have been proved to be DS graphs, the problem of determining DS graphs is still far from being completely solved [13, 14]. Recently, Lin et al. [15] considered the problem which graph is determined by its A_α-spectrum? And they gave some characterizing properties of A_α-spectrum and proposed the following problem.

Problem 1. Characterizing graphs G determined by their A_α-spectra such that $G \vee K_m\,(m \geq 1)$ is also determined by their A_α-spectra for $\alpha \in (0,1/2)$ or $(1/2, t1)$.

Liu and Lu [16] discussed the problem which join graph is determined by its Q-spectrum? And they pointed out the following problem.

Problem 2. Prove or disprove that $\overline{K_n} \vee K_m$ is determined by its Q-spectrum for $m \geq 3$.

In this paper, we focus on Problem 1 above, and we prove that some join graphs are A_α-DS graphs. Furthermore, we also give a special solution for Problem 2. The rest of this paper is organized as follows. In Section 2, we present some characterizing properties of the A_α-spectrum of graphs and give the formula to compute $c_3(G_v)$ in $K_n - E(H)$, where H is a subgraph of K_n with l edges. In Section 3, we give a solution for Problem 1.

2. Preliminaries

Let G_n denote the set of graphs each of which is obtained from K_n by removing five or fewer edges. For $n \geq 10$, there exist exactly 45 nonisomorphic graphs each of which is obtained from K_n by removing five or fewer edges [17]. These graphs are labeled by G_{ij}, $1 \leq i \leq 5$ and $0 \leq j \leq 25$ and illustrated in Figure 1. Checking the structure of G_{ij}, we know that $G_{ij} = H \vee K_m$, where H is a graph obtained form K_t deleting some edges, and $t + m = n$, e.g., $G_{44} = (K_4 - E(C_4)) \vee K_{n-4}$.

Cámara and Haemers [18] discussed the problem which $G_{ij} \in \mathrm{G}_n$ is determined by its A_0-spectrum. And they gave the following result.

Theorem 1 (see [18]). *Let $G \in \mathrm{G}_n$ be a graph with $n \neq 7$ vertices. Then, G is A_0-DS graph.*

Lemma 1 (see [19]). *Let $H \subseteq K_n$ be a graph with l edges and let $G = K_n - E(H)$. Then,*

$$c_3(G) = \binom{n}{3} - l(n-2) + \sum_{v \in V(H)} \binom{\mathrm{d}(v)}{2} - c_3(H). \quad (2)$$

By Lemma 1, the number of triangles of some $G \in \mathrm{G}_n$ is calculated [17], see Table 1.

Lemma 2 (see [17]). *Let $H \subseteq K_n$ be a graph with l edges and let $G = K_n - E(H)$. Then,*

$$c_4(G) = 3\binom{n}{4} - 2l\binom{n-2}{2} + \left[2\binom{l}{2} + (n-5)\sum_{v \in V(H)} \binom{\mathrm{d}(v)}{2}\right] - p_4(H) + c_4(H). \quad (3)$$

By Lemma 2, the number of quadrangles of some $G \in \mathrm{G}_n$ is calculated [17], see Table 2.

Using the Principle of Inclusion-Exclusion, we can obtain the following result.

Lemma 3. *Let $H \subseteq K_n$ be a graph with k edges and let $G = K_n - E(H)$. Let $v \in V(G)$, and let v be an endpoint of $l(\leq k)$ edges in $E(H)$. Then,*

$$c_3(G_v) = \binom{n-1}{2} - (k-l) - l(n-1-l) + c_3(\overline{G}_v) + |P_3| - \binom{l}{2}. \quad (4)$$

Proof. Let $E(H) = \{e_1, e_2, \ldots, e_k\}$. Let S_i denote the set of triangles of K_n containing $e_i\,(i = 1, 2, \ldots, k)$ and v. Thus, there exists exactly $\binom{n-1}{2}$ triangles containing v in K_n. By the Inclusion-Exclusion Principle, we have

$$c_3(G_v) = \binom{n-1}{2} - \sum_{i=1}^{l} |S_i| + \sum_{i<j} |S_i \cap S_j| - \sum_{i<j<k} |S_i \cap S_j \cap S_k|. \quad (5)$$

For any edge e_i, if v is an endpoint of e_i, then there exists $n-1-l$ triangles containing e_i. Otherwise, there exists $k-l$ triangles containing e_i. So, $\sum_{i=1}^{l} |S_i| = l(n-1-l) + k - l$. For any given e_i and e_j, if v is a common endpoint of e_i and e_j, then there exists $c_3(\overline{G}_v)$ triangles containing e_i and e_j. Otherwise, there exists $|P_3|$ triangles containing e_i and e_j in $\overline{G}$, where P_3 is a path which v is origin endpoint and $|P_3|$ is the number of vertices with length 2 to v. Thus, $\sum_{i<j} |S_i \cap S_j| = c_3(\overline{G}_v) + |P_3|$. Since any two edges in l edges induce a triangle, $\sum_{i<j<k} |S_i \cap S_j \cap S_k| = \binom{l}{2}$. By the above arguments, we arrive in equation (4). □

Lemma 4 (see [20] and [5]). *Let G be a graph with n vertices and m edges, and let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of G. Suppose that $\phi(A_\alpha(G), x) = \sum_j c_{\alpha j} x^{n-j}$. Then,*

(i) $c_{\alpha 0} = 1$

(ii) $c_{\alpha 1} = -2\alpha m$

(iii) $c_{\alpha 2} = 2\alpha^2 m^2 - (1-\alpha)^2 m - 1/2\alpha^2 \sum_i d_i^2$

(iv) $c_{\alpha 3} = -2(1-\alpha)^3 c_3(G) + 2\alpha(1-\alpha)^2 m^2 - \alpha(1-\alpha)^2 \sum_i d_i^2 - 1/3\alpha^3 (4m^3 - 3m\sum_i d_i^2 + \sum_i d_i^3)$

(v) $c_{\alpha 4} = -1/4\alpha^4 \sum_i d_i^4 - \alpha^2(1-\alpha)^2 \sum_i d_i^3 + 2/3\alpha^4 m \sum_i d_i^3 + 5/2\alpha^2(1-\alpha)^2 m \sum_i d_i^2 - \alpha^4 m^2 \sum_i d_i^2 - 1/2(1-\alpha)^4 \sum_i d_i^2 + 1/8\alpha^4 (\sum_i d_i^2)^2 - \alpha^2 (1-\alpha)^2 \sum_{(v_i v_j) \in E(G)} d_i d_j - 2\alpha(1-\alpha)^3 \sum_i d_i c_3(G_{v_i}) + 4\alpha(1-\alpha)^3 m c_3(G) - 2(1-\alpha)^4 c_4(G) + 2/3\alpha^4 m^4 - 2\alpha^2(1-\alpha)^2 m^3 + 1/2(1-\alpha)^4 m^2 + 1/2(1-\alpha)^4 m$

For convenience, by Lemmas 3 and 4, we calculate the value $\sum_i d_i c_3(G_{v_i})$ of some graphs in G_n, see Table 3.

Lemma 5 (see [21]). *Let G and H be two graphs with n vertices. For $\alpha \in [0, 1]$, if G and H are A_α-cospectral, then the following statements hold:*

(i) $|V(G)| = |V(H)|$.

(ii) $|E(G)| = |E(H)|$.

(iii) If G is r-regular, then H is r-regular.

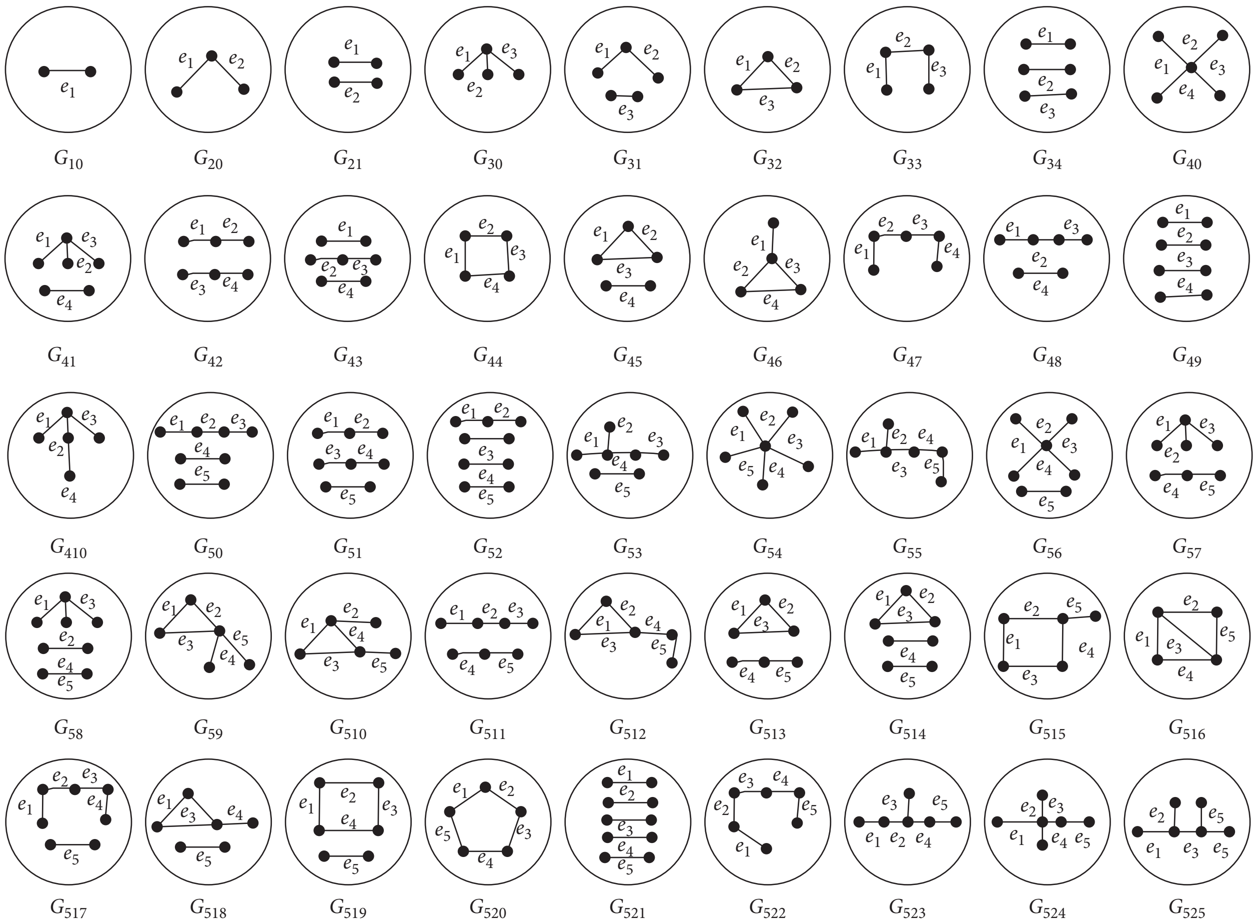

FIGURE 1: The graphs obtained from K_n by deleting five or fewer edges drawn as lines in a disk.

TABLE 1: The numbers of triangles of some graphs in G_n.

Graph	$c_3(G)$	Graph	$c_3(G)$
G_{30}	$\binom{n}{3} - 3n + 9$	G_{59}, G_{524}	$\binom{n}{3} - 5n + 17$
G_{31}	$\binom{n}{3} - 3n + 7$	G_{42}, G_{45}, G_{48}	$\binom{n}{3} - 4n + 10$
G_{40}	$\binom{n}{3} - 4n + 14$	G_{44}, G_{46}, G_{410}	$\binom{n}{3} - 4n + 12$
G_{43}	$\binom{n}{3} - 4n + 9$	G_{50}, G_{51}, G_{514}	$\binom{n}{3} - 5n + 12$
G_{52}	$\binom{n}{3} - 5n + 11$	$G_{55}, G_{512}, G_{520}, G_{523}$	$\binom{n}{3} - 5n + 15$
G_{54}	$\binom{n}{3} - 5n + 20$	$G_{58}, G_{511}, G_{513}, G_{517}$	$\binom{n}{3} - 5n + 13$
G_{32}, G_{33}	$\binom{n}{3} - 3n + 8$	$G_{53}, G_{57}, G_{518}, G_{519}, G_{522}$	$\binom{n}{3} - 5n + 14$
G_{41}, G_{47}	$\binom{n}{3} - 4n + 11$	$G_{56}, G_{510}, G_{515}, G_{516}, G_{525}$	$\binom{n}{3} - 5n + 16$

Table 2: The numbers of quadrangles of some graphs in G_n.

Graph	$c_4(G)$	Graph	$c_4(G)$
G_{42}	$3\binom{n}{4} - 4n^2 + 22n - 22$	G_{48}	$3\binom{n}{4} - 4n^2 + 22n - 23$
G_{50}	$3\binom{n}{4} - 5n^2 + 27n - 21$	G_{51}	$3\binom{n}{4} - 5n^2 + 27n - 20$
G_{53}	$3\binom{n}{4} - 5n^2 + 29n - 32$	G_{57}	$3\binom{n}{4} - 5n^2 + 29n - 30$
G_{55}	$3\binom{n}{4} - 5n^2 + 30n - 38$	G_{523}	$3\binom{n}{4} - 5n^2 + 30n - 39$
G_{511}	$3\binom{n}{4} - 5n^2 + 28n - 26$	G_{517}	$3\binom{n}{4} - 5n^2 + 28n - 27$
G_{519}	$3\binom{n}{4} - 5n^2 + 29n - 33$	G_{522}	$3\binom{n}{4} - 5n^2 + 29n - 33$

Suppose that $d_1 \geq d_2 \geq \cdots \geq d_n$ and $d_1' \geq d_2' \geq \cdots \geq d_n'$ are the degree sequences of G and H, respectively. If G and H are A_α-cospectral with $\alpha \in (0,1]$, then

(iv) $\sum_{1\leq i<j\leq n} d_i d_j = \sum_{1\leq i<j\leq n} d_i' d_j'$.

(v) $\sum_{1\leq i\leq n} d_i^2 = \sum_{1\leq i\leq n} d_i'^2$.

Lemma 6 (see [21]). *The complete graph K_n is determined by its A_α-spectrum.*

Lemma 7 (see [21]). *The graph $\overline{kK_2 \cup (n-2k)K_1}$ is determined by its A_α-spectrum, where $1 \leq k \leq \lfloor n/2 \rfloor$ and $0 \leq \alpha \leq 1$.*

By Lemma 7, we can obtain a corollary as follows.

Corollary 1. *Graphs G_{10}, G_{21}, G_{34}, G_{49}, and G_{521} are determined by their A_α-spectra, where $0 \leq \alpha \leq 1$.*

The M-coronal of an $n \times n$ square matrix M, denoted by $\Gamma_M(x)$, is defined to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is,

$$\Gamma_M(x) = 1_n^T (xI_n - M)^{-1} 1_n, \quad (6)$$

where 1_n denotes the column vector of size n with all the entries equal to one and 1_n^T means the transpose of 1_n ([22, 23]).

Lemma 8 (see [16]). *If G is an arbitrary graph and H_1 and H_2 are Q-cospectral graphs with $\Gamma_{Q(H_1)}(x) = \Gamma_{Q(H_2)}(x)$, then $G \vee H_1$ and $G \vee H_2$ are Q-cospectral.*

By Lemma 8, we obtain directly the following corollary.

Corollary 2. *If G is an arbitrary graph and H_1 and H_2 are Q-cospectral graphs with $\Gamma_{Q(H_1)}(x) = \Gamma_{Q(H_2)}(x)$, then $G \vee H_1$ and $G \vee H_2$ are $A_{1/2}$-cospectral.*

Lemma 9. *The each of following holds:*

(i) $(K_m - E(lK_2)) \vee (K_4 - E(K_{1,3}))$ *and* $(K_m - E(lK_2)) \vee (K_4 - E(K_3))$ *are $A_{1/2}$-cospectral, where $0 \leq l \leq \lfloor m/2 \rfloor$*

(ii) $(K_m - E(P_l)) \vee (K_4 - E(K_{1,3}))$ *and* $(K_m - E(P_l)) \vee (K_4 - E(K_3))$ *are $A_{1/2}$-cospectral, where $2 \leq l \leq m$*

Proof. Directly calculating the signless Laplacian polynomials of $K_4 - E(K_{1,3})$ and $K_4 - E(K_3)$ yield $\det|xI - Q(K_4 - E(K_{1,3})| = \det|xI - Q(K_4 - E(K_3)| = x^4 - 6x^3 + 9x^2 + 4x))$. Furthermore, by simple computations, we have $\Gamma_Q(K_4 - E(K_{1,3}))(x) = \Gamma_Q(K_4 - E(K_3))(x) = 4\ (x-1)/x(x-4)$. By Corollary 2, it is easy to see that the results in Lemma 9 hold.

By Lemma 9, we obtain some $A_{1/2}$-cospectral mates in G_n. □

Corollary 3. *The following results hold:*

(i) *Graphs G_{30} and G_{32} are $A_{1/2}$-cospectral*

(ii) *Graphs G_{41} and G_{45} are $A_{1/2}$-cospectral*

(iii) *Graphs G_{58} and G_{514} are $A_{1/2}$-cospectral*

(iv) *Graphs G_{57} and G_{513} are $A_{1/2}$-cospectral*

Remark 1. By Corollaries 1 and 3, we know that $\overline{K_2} \vee K_m = G_{10}$ is a Q-DS graph, and $\overline{K_3} \vee K_m = G_{30}$ and G_{32} are Q-cospectral. These results answer the special case of Problem 2.

3. Main Results

In this section, we show that all graphs in G_n are determined by their A_α-spectra.

Theorem 2. *Graphs G_{20} and G_{21} are A_α-DS graphs, where $0 < \alpha \leq 1$.*

Proof. The result follows from Lemma 5 and Corollary 1. □

Theorem 3. *Let G be a graph obtained from K_n by deleting three edges, and then G is determined by the A_α-spectra when $\alpha \in (0, 1/2) \cup (1/2, 1]$.*

Table 3: The value $\sum_i d_i c_3(G_{v_i})$ of some graphs in $\mathbb{G}_n$.

Graph	$\sum_i d_i c_3(G_{v_i})$	Graph	$\sum_i d_i c_3(G_{v_i})$
G_{42}	$1/2n^4 - 2n^3 - 27/2n^2 + 65n - 50$	G_{48}	$1/2n^4 - 2n^3 - 27/2n^2 + 65n - 52$
G_{50}	$1/2n^4 - 2n^3 - 35/2n^2 + 79n - 48$	G_{51}	$1/2n^4 - 2n^3 - 35/2n^2 + 79n - 46$
G_{53}	$1/2n^4 - 2n^3 - 35/2n^2 + 89n - 79$	G_{57}	$1/2n^4 - 2n^3 - 35/2n^2 + 89n - 75$
G_{519}	$1/2n^4 - 2n^3 - 35/2n^2 + 89n - 80$	G_{522}	$1/2n^4 - 2n^3 - 35/2n^2 + 89n - 78$
G_{511}	$1/2n^4 - 2n^3 - 35/2n^2 + 84n - 61$	G_{517}	$1/2n^4 - 2n^3 - 35/2n^2 + 84n - 63$
G_{55}	$1/2n^4 - 2n^3 - 35/2n^2 + 94n - 94$	G_{523}	$1/2n^4 - 2n^3 - 35/2n^2 + 94n - 96$
G_{56}	$1/2n^4 - 2n^3 - 35/2n^2 + 99n - 110$	G_{525}	$1/2n^4 - 2n^3 - 35/2n^2 + 99n - 112$

Proof. Checking Figure 1, we know that G is isomorphic to one of $\{G_{30}, G_{31}, G_{32}, G_{33}, G_{34}\}$. Directly computing yields $\sum_{i=1}^n d_i^2(G_{31}) = n^3 - 2n^2 - 11n + 20$, $\sum_{i=1}^n d_i^2(G_{33}) = n^3 - 2n^2 - 11n + 22$, and $\sum_{i=1}^n d_i^2(G_{30}) = \sum_{i=1}^n d_i^2(G_{32}) = n^3 - 2n^2 - 11n + 24$. By Lemma 4 (iv) and Table 1, we have

$$\begin{aligned} c_{\alpha 3}(G_{32}) - c_{\alpha 3}(G_{30}) &= 2(1-\alpha)^3 (c_3(G_{30}) - c_3(G_{32})) \\ &\quad + \frac{1}{3}\alpha^3 \left(\sum_i d_i^3(G_{30}) - \sum_i d_i^3(G_{32}) \right) \\ &= 2(1-\alpha)^3 - 2\alpha^3 = 2 - 6\alpha + 6\alpha^2 - 4\alpha^3. \end{aligned} \quad (7)$$

Solving equation

$$4\alpha^3 - 6\alpha^2 + 6\alpha - 2 = 0, \quad (8)$$

we have $\alpha = 1/2$, $1/2 + \sqrt{3}i/2$, or $1/2 - \sqrt{3}i/2$. This implies that $c_{\alpha 3}(G_{32}) \neq c_{\alpha 3}(G_{30})$ for $\alpha \in (0, 1/2) \cup (1/2, 1]$.

By Corollaries 1 and 3 (i) and Lemma 7 (i), (ii), and (v), the result in Theorem 3 holds. □

Remark 2. By the proof of Theorem 3, it can be known that G_{31}, G_{33}, and G_{34} are determined by their Q-spectra.

Lemma 10. *Each of the following holds:*

(i) Graphs G_{44} and G_{410} are not A_α-cospectral, where $\alpha \in (0, 1/2) \cup (1/2, t1)$

(ii) Graphs G_{42} and G_{48} are not A_α-cospectral, where $\alpha \in (0, 1/2) \cup (1/2, t1)$

(iii) Graphs G_{41}, G_{45}, and G_{47} are not pairwise A_α-cospectral, where $\alpha \in (0, 1/2) \cup (1/2, t1)$

Proof

(i) By Lemma 4 (iv) and Table 1, we have $c_{\alpha 3}(G_{44}) - c_{\alpha 3}(G_{410}) = 2\alpha^3$. Solving equation

$$2\alpha^3 = 0, \quad (9)$$

we obtain $\alpha = 0$, 0 or 0. It implies that G_{42} and G_{48} are not A_α-cospectral, when $\alpha \in (0, 1/2) \cup (1/2, t1)$.

(ii) By Lemma 4 (v) and Tables 1–3, we obtain that

$$c_{\alpha 4}(G_{48}) - c_{\alpha 4}(G_{42}) = -2(\alpha - 1)^4 - \alpha^2(\alpha - 1)^2 + 4\alpha(\alpha - 1)^3. \quad (10)$$

Solving equation

$$-2(\alpha - 1)^4 - \alpha^2(\alpha - 1)^2 + 4\alpha(\alpha - 1)^3 = 0, \quad (11)$$

we have $\alpha = 1, 1, \sqrt{2}$, or $-\sqrt{2}$. This indicates that G_{42} and G_{48} are not A_α-cospectral when $\alpha \in (0, 1/2) \cup (1/2, t1)$.

(iii) Similarly, by Lemma 4 (iv) and Table 1, we obtain that

$$\begin{aligned} c_{\alpha 3}(G_{41}) - c_{\alpha 3}(G_{45}) &= 4\alpha^3 - 6\alpha^2 + 6\alpha - 2, \\ c_{\alpha 3}(G_{41}) - c_{\alpha 3}(G_{47}) &= 2\alpha^3, \\ c_{\alpha 3}(G_{45}) - c_{\alpha 3}(G_{47}) &= -2\alpha^3 + 6\alpha^2 - 6\alpha + 2. \end{aligned} \quad (12)$$

Solving equation

$$-2\alpha^3 + 6\alpha^2 - 6\alpha + 2 = 0, \quad (13)$$

we obtain $\alpha = 1$, 1 or 1. By the roots of equations (8), (9), and (13), we know that G_{41}, G_{45}, and G_{47} are not pairwise A_α-cospectral when $\alpha \in (0, 1/2) \cup (1/2, t1)$. □

Theorem 4. *Graphs G_{40}, G_{41}, G_{42}, G_{43}, G_{44}, G_{45}, G_{46}, G_{47}, G_{48}, G_{49}, and G_{410} are determined by their A_α-spectra, respectively, where $\alpha \in (0, 1/2) \cup (1/2, t1)$.*

Proof. By simple computations, we obtain that $\sum_{i=1}^n d_i^2(G_{40}) = n^3 - 2n^2 - 15n + 36$, $\sum_{i=1}^n d_i^2(G_{43}) = n^3 - 2n^2 - 15n + 26$, $\sum_{i=1}^n d_i^2(G_{46}) = n^3 - 2n^2 - 15n + 34$, $\sum_{i=1}^n d_i^2(G_{42}) = \sum_{i=1}^n d_i^2(G_{48}) = n^3 - 2n^2 - 15n + 28$, $\sum_{i=1}^n d_i^2(G_{41}) = \sum_{i=1}^n d_i^2(G_{45}) = \sum_{i=1}^n d_i^2(G_{47}) = n^3 - 2n^2 - 15n + 30$, and $\sum_{i=1}^n d_i^2(G_{44}) = \sum_{i=1}^n d_i^2(G_{410}) = n^3 - 2n^2 - 15n + 32$.

By Corollaries 1 and 3 (ii) and Lemmas 5 and 10, graphs G_{40}, G_{41}, G_{42}, G_{43}, G_{44}, G_{45}, G_{46}, G_{47}, G_{48}, G_{49}, and G_{410} are A_α-DS graphs, where $\alpha \in (0, 1/2) \cup (1/2, t1)$. □

Lemma 11. *Each of the following holds:*

(i) Graphs G_{50} and G_{51} are not A_α-cospectral, where $\alpha \in (0, 1/2) \cup (1/2, t1)$

(ii) Graphs G_{59} and G_{516} are not A_α-cospectral, where $\alpha \in (1/2, t1)$

(iii) Graphs G_{510} and G_{524} are not A_α-cospectral, where $\alpha \in (1/2, t1)$

(iv) Graphs G_{53}, G_{57}, G_{513}, G_{519}, and G_{522} are not pairwise A_α-cospectral, where $\alpha \in (0,1/2) \cup (1/2, t1)$.

(v) Graphs G_{55}, G_{518}, G_{520}, and G_{523} are not pairwise A_α-cospectral, where $\alpha \in (0,1/2) \cup (1/2, t1)$

(vi) Graphs G_{56}, G_{512}, G_{515}, and G_{525} are not pairwise A_α-cospectral, where $\alpha \in (1/2, t1)$

(vii) Graphs G_{58}, G_{511}, G_{514}, and G_{517} are not pairwise A_α-cospectral, where $\alpha \in (0,1/2) \cup (1/2, t1)$

Proof

(i) By Lemma 4 (v) and Tables 1–3, we have

$$c_{\alpha 4}(G_{50}) - c_{\alpha 4}(G_{51}) = 2(\alpha-1)^4 + \alpha^2(\alpha-1)^2 - 4\alpha(\alpha-1)^3. \tag{14}$$

By the roots of equations (11), we know that G_{50} and G_{51} not A_α-cospectral when $\alpha \in (0,1/2) \cup (1/2, t1)$.

(ii) By Lemma 4 (iv) and Table 1, we have $c_{\alpha 3}(G_{59}) - c_{\alpha 3}(G_{516}) = 6\alpha^3 - 6\alpha^2 + 6\alpha - 2$. Solving equation

$$6\alpha^3 - 6\alpha^2 + 6\alpha - 2 = 0, \tag{15}$$

we have $\alpha = \sqrt[3]{4} + \sqrt[3]{-2} + 1/3 < 1/2$, $2\sqrt[3]{4} + \sqrt[3]{-2}(-\sqrt{3}\,i - 1) + 2/6$, or $2\sqrt[3]{-2} + \sqrt[3]{4}\ (-\sqrt{3}\,i - 1) + 2/6$. It implies that G_{59} and G_{516} are not A_α-cospectral when $\alpha \in (1/2, t1)$.

(iii) Similarly, by Lemma 4 (iv) and Table 1, we have $c_{\alpha 3}(G_{510}) - c_{\alpha 3}(G_{524}) = 6\alpha^3 - 6\alpha^2 + 6\alpha - 2$. By the roots of equation (15), we know that G_{510} and G_{524} are not A_α-cospectral, where $\alpha \in (1/2, t1)$.

(iv) Analogously, by Lemma 4 (iv) and (v) and Tables 1–3, we obtain that

$$\begin{aligned}
c_{\alpha 4}(G_{53}) - c_{\alpha 4}(G_{57}) &= 4(\alpha-1)^4 + 2\alpha^2(\alpha-1)^2 \\
&\quad - 8\alpha(\alpha-1)^3, \\
c_{\alpha 3}(G_{53}) - c_{\alpha 3}(G_{513}) &= 4\alpha^3 - 6\alpha^2 + 6\alpha - 2, \\
c_{\alpha 3}(G_{53}) - c_{\alpha 3}(G_{519}) &= 2\alpha^3, \\
c_{\alpha 3}(G_{53}) - c_{\alpha 3}(G_{522}) &= 2\alpha^3, \\
c_{\alpha 3}(G_{57}) - c_{\alpha 3}(G_{513}) &= 4\alpha^3 - 6\alpha^2 + 6\alpha - 2, \\
c_{\alpha 3}(G_{57}) - c_{\alpha 3}(G_{519}) &= 2\alpha^3, \\
c_{\alpha 3}(G_{57}) - c_{\alpha 3}(G_{522}) &= 2\alpha^3, \\
c_{\alpha 3}(G_{513}) - c_{\alpha 3}(G_{519}) &= -2\alpha^3 + 6\alpha^2 - 6\alpha + 2, \\
c_{\alpha 3}(G_{513}) - c_{\alpha 3}(G_{522}) &= -2\alpha^3 + 6\alpha^2 - 6\alpha + 2, \\
c_{\alpha 4}(G_{519}) - c_{\alpha 4}(G_{522}) &= \alpha^2(\alpha-1)^2 - 4\alpha(\alpha-1)^3, \\
c_{\alpha 4}(G_{517}) - c_{\alpha 4}(G_{511}) &= 2(\alpha-1)^4 + \alpha^2(\alpha-1)^2 \\
&\quad - 4\alpha(\alpha-1)^3.
\end{aligned} \tag{16}$$

Solving equation

$$\alpha^2(\alpha-1)^2 - 4\alpha(\alpha-1)^3 = 0, \tag{17}$$

we obtain $\alpha = 0, 1, 1$, or $4/3$. By the roots of equations (8), (9), (11), (13), and (17), we obtain that G_{53}, G_{57}, G_{513}, G_{519}, and G_{522} are not pairwise A_α − cospectral when $\alpha \in (0,1/2) \cup (1/2, t1)$.

(v) Similarly, by Lemma 4 (iv) and (v) and Tables 1–3, we obtain that

$$\begin{aligned}
c_{\alpha 3}(G_{55}) - c_{\alpha 3}(G_{518}) &= 2\alpha^3 - 6\alpha^2 + 6\alpha - 2, \\
c_{\alpha 3}(G_{55}) - c_{\alpha 3}(G_{520}) &= 2\alpha^3, \\
c_{\alpha 4}(G_{55}) - c_{\alpha 4}(G_{523}) &= -2(\alpha-1)^4 - \alpha^2(\alpha-1)^2 \\
&\quad + 4\alpha(\alpha-1)^3, \\
c_{\alpha 3}(G_{518}) - c_{\alpha 3}(G_{520}) &= 6\alpha^2 - 6\alpha + 2, \\
c_{\alpha 3}(G_{518}) - c_{\alpha 3}(G_{523}) &= -2\alpha^3 + 6\alpha^2 - 6\alpha + 2, \\
c_{\alpha 3}(G_{520}) - c_{\alpha 3}(G_{523}) &= -2\alpha^3.
\end{aligned} \tag{18}$$

Solving equation

$$6\alpha^2 - 6\alpha + 2 = 0, \tag{19}$$

we obtain $\alpha = 3 + \sqrt{3}i/6$ or $3 - \sqrt{3}i/6$. By the roots of equations (9), (11), (13), and (19), we obtain that G_{55}, G_{518}, G_{520}, and G_{523} are not pairwise A_α − cospectral when $\alpha \in (0,1/2) \cup (1/2, t1)$.

(vi) By Lemma 4 (iv) and (v) and Tables 1–3, we have

$$\begin{aligned}
c_{\alpha 3}(G_{56}) - c_{\alpha 3}(G_{512}) &= 8\alpha^3 - 6\alpha^2 + 6\alpha - 2, \\
c_{\alpha 3}(G_{56}) - c_{\alpha 3}(G_{515}) &= 6\alpha^3, \\
c_{\alpha 4}(G_{56}) - c_{\alpha 4}(G_{525}) &= 4\alpha^3, \\
c_{\alpha 3}(G_{512}) - c_{\alpha 3}(G_{515}) &= -2\alpha^3 + 6\alpha^2 - 6\alpha + 2, \\
c_{\alpha 3}(G_{512}) - c_{\alpha 3}(G_{525}) &= -4\alpha^3 + 6\alpha^2 - 6\alpha + 2, \\
c_{\alpha 3}(G_{515}) - c_{\alpha 3}(G_{525}) &= -2\alpha^3.
\end{aligned} \tag{20}$$

Solving equation

$$8\alpha^3 - 6\alpha^2 + 6\alpha - 2 = 0, \tag{21}$$

we obtain $\alpha = \sqrt[3]{9} + \sqrt[3]{-3} + 1/4 < 1/2$, $2\sqrt[3]{9} + \sqrt[3]{-3}(-\sqrt{3}\,i - 1) + 2/8$, or $2\sqrt[3]{-3} + \sqrt[3]{9}(-\sqrt{3}\,i - 1) + 2/8$. By the roots of equations (8), (11), (13), and (21), we obtain that G_{56}, G_{512}, G_{515}, and G_{525} are not A_α-cospectral when $\alpha \in (1/2, t1)$.

(vii) Finally, by Lemma 4 (iv) and (v) and Tables 1–3, we have

$$
\begin{aligned}
c_{\alpha 3}(G_{58}) - c_{\alpha 3}(G_{511}) &= 2\alpha^3,\\
c_{\alpha 3}(G_{58}) - c_{\alpha 3}(G_{514}) &= 4\alpha^3 - 6\alpha^2 + 6\alpha - 2,\\
c_{\alpha 3}(G_{58}) - c_{\alpha 3}(G_{517}) &= 2\alpha^3,\\
c_{\alpha 3}(G_{511}) - c_{\alpha 3}(G_{514}) &= 2\alpha^3 - 6\alpha^2 + 6\alpha - 2,\\
c_{\alpha 4}(G_{511}) - c_{\alpha 4}(G_{517}) &= -2(\alpha-1)^4 - \alpha^2(\alpha-1)^2\\
&\quad + 4\alpha(\alpha-1)^3,\\
c_{\alpha 3}(G_{514}) - c_{\alpha 3}(G_{517}) &= -2\alpha^3 + 6\alpha^2 - 6\alpha + 2.
\end{aligned}
\tag{22}
$$

By the roots of equations (8), (9), (11), and (13), we obtain that G_{58}, G_{511}, G_{514}, and G_{517} are not A_α-cospectral when $\alpha \in (0, 1/2) \cup (1/2, t1)$. □

Theorem 5. *Graphs G_{50}, G_{51}, G_{52}, G_{53}, G_{54}, G_{55}, G_{56}, G_{57}, G_{58}, G_{59}, G_{510}, G_{511}, G_{512}, G_{513}, G_{514}, G_{515}, G_{516}, G_{517}, G_{518}, G_{519}, G_{520}, G_{522}, G_{523}, G_{524}, and G_{525} are, respectively, determined by their A_α-spectra, where when $\alpha \in (1/2, t1)$.*

Proof. By simple computations, we have that $\sum_{i=1}^n d_i^2(G_{50}) = \sum_{i=1}^n d_i^2(G_{51}) = n^3 - 2n^2 - 19n + 34$, $\sum_{i=1}^n d_i^2(G_{52}) = n^3 - 2n^2 - 19n + 32$, $\sum_{i=1}^n d_i^2(G_{53}) = \sum_{i=1}^n d_i^2(G_{57}) = \sum_{i=1}^n d_i^2(G_{513}) = \sum_{i=1}^n d_i^2(G_{519}) = \sum_{i=1}^n d_i^2(G_{522}) = n^3 - 2n^2 - 19n + 38$, $\sum_{i=1}^n d_i^2(G_{54}) = n^3 - 2n^2 - 19n + 50$, $\sum_{i=1}^n d_i^2(G_{55}) = \sum_{i=1}^n d_i^2(G_{518}) = \sum_{i=1}^n d_i^2(G_{520}) = \sum_{i=1}^n d_i^2(G_{523}) = n^3 - 2n^2 - 19n + 40$, $\sum_{i=1}^n d_i^2(G_{56}) = \sum_{i=1}^n d_i^2(G_{512}) = \sum_{i=1}^n d_i^2(G_{515}) = \sum_{i=1}^n d_i^2(G_{525}) = n^3 - 2n^2 - 19n + 42$, $\sum_{i=1}^n d_i^2(G_{58}) = \sum_{i=1}^n d_i^2(G_{511}) = \sum_{i=1}^n d_i^2(G_{514}) = \sum_{i=1}^n d_i^2(G_{517}) = n^3 - 2n^2 - 19n + 36$, $\sum_{i=1}^n d_i^2(G_{59}) = \sum_{i=1}^n d_i^2(G_{516}) = n^3 - 2n^2 - 19n + 46$, and $\sum_{i=1}^n d_i^2(G_{510}) = \sum_{i=1}^n d_i^2(G_{524}) = n^3 - 2n^2 - 19n + 44$.

By Corollaries 1 and 3 (iii) and (iv) and Lemmas 5 and 11, graphs G_{50}, G_{51}, G_{52}, G_{53}, G_{54}, G_{55}, G_{56}, G_{57}, G_{58}, G_{59}, G_{510}, G_{511}, G_{512}, G_{513}, G_{514}, G_{515}, G_{516}, G_{517}, G_{518}, G_{519}, G_{520}, G_{521}, G_{522}, G_{523}, G_{524}, and G_{525} are determined by their A_α-DS graphs, respectively, where $\alpha \in (1/2, t1)$.

By Corollary 1 and Theorems 2–5, directly yields the following result. □

Theorem 6. *Let $G \in \mathrm{G}_n$ be a graph with $n(\neq 7)$ vertices. G is determined by its A_α-spectrum, where $\alpha \in (1/2, t1)$.*

Remark 3. By Theorems 2–4, we know that almost complete graphs are determined by their A_α-spectra, where $\alpha \in (0, 1/2) \cup (1/2, t1)$, each G_{ij} is a join. Thus, these results is a solution of Problem 1. Motivated by these results, we pose the following two questions.

Question 1. Prove or disprove that G_{59} and G_{516} are A_α-cospectral, where $\alpha \in (0, 1/2)$.

Question 2. Prove or disprove that G_{510} and G_{524} are A_α-cospectral, where $\alpha \in (0, 1/2)$.

Data Availability

Data from previous studies were used to support this study. They are cited at relevant places within the text as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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