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Research Article

On Discrete Time Wilson Systems

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In this paper, we define the discrete time Wilson frame (DTW frame) for $l^2(\mathbb{Z})$ and discuss some properties of discrete time Wilson frames. Also, we give an interplay between DTW frames and discrete time Gabor frames. Furthermore, a necessary and a sufficient condition for the DTW frame in terms of Zak transform are given. Moreover, the frame operator for the DTW frame is obtained. Finally, we discuss dual pair of frames for discrete time Wilson systems and give a sufficient condition for their existence.

1. Introduction

The idea of frame as a redundant peer of a basis was originated in 1952 by Duffin and Schaeffer [1]. It came to limelight only with the historic paper of Daubechies et al. [2]. A sequence of vectors $\{u_j\}_{j\in\mathbb{N}}\subseteq\mathscr{H}$ is termed as a *frame* (or *Hilbert frame*) for a separable Hilbert space $\mathscr{H}$ if there exist constants $\mathscr{A}_l, \mathscr{A}_u > 0$ such that

$$\mathscr{A}_l\|u\|^2 \le \sum_{j\in\mathbb{N}} \left|\langle u, u_j\rangle\right|^2 \le \mathscr{A}_u\|u\|^2, \quad \text{for all } u \in \mathscr{H}. \tag{1}$$

The positive numbers $\mathscr{A}_l$ and $\mathscr{A}_u$ are termed as *lower* and *upper frame bounds* of the frame, respectively. The bounds may not be unique. If $\mathscr{A}_l = \mathscr{A}_u$, then $\{u_j\}_{j\in\mathbb{N}}$ is called an $\mathscr{A}_l$*-tight frame*, and if $\mathscr{A}_l = \mathscr{A}_u = 1$, then $\{u_j\}_{j\in\mathbb{N}}$ is said to be a *Parseval frame*. The inequality in (1) is recognized as the *frame inequality* of the frame $\{u_j\}_{j\in\mathbb{N}}$.

A sequence of vectors $\{u_j\}_{j\in\mathbb{N}}\subseteq\mathscr{H}$ is called a *Riesz basis* if $\{u_j\}_{j\in\mathbb{N}}$ is complete and there are positive constants $\mathscr{A}_l$ and $\mathscr{A}_u$ such that

$$\mathscr{A}_l \sum_{j\in\mathbb{N}} \left|\alpha_j\right|^2 \le \left\|\sum_{j\in\mathbb{N}} \alpha_j u_j\right\|^2 \le \mathscr{A}_u \sum_{j\in\mathbb{N}} \left|\alpha_j\right|^2, \quad \text{for all } \{\alpha_j\}_{j\in\mathbb{N}} \in \ell^2(\mathbb{N}). \tag{2}$$

Gabor frame for $L^2(\mathbb{R})$ (which is a Riesz basis) has bad localization properties in either time or frequency. Thus, a system to replace Gabor systems which do not have bad localization properties in time and frequency was required. Wilson [3, 4] suggested a system of functions which are localized around the positive and negative frequency of the same order. The idea of Wilson was used by Daubechies et al. [5] to construct orthonormal "Wilson bases" which consist of functions given by

$$\psi_j^k(x) = \begin{cases} \varepsilon_k \cos(2k\pi x)\, w\left(x - \dfrac{j}{2}\right), & \text{if } j \text{ is even},\\[2mm] 2\sin(2(k+1)\pi x)\, w\left(x - \dfrac{j+1}{2}\right), & \text{if } j \text{ is odd}, \end{cases}$$

$$\varepsilon_k = \begin{cases} \sqrt{2}, & \text{if } k = 0,\\ 2, & \text{if } k \in \mathbb{N}, \end{cases} \tag{3}$$

with a smooth well-localized window function w. For such bases, the disadvantage described in the Balian–Low theorem is completely removed. Independent of the work of Daubechies et al. [5], orthonormal local trigonometric bases consisting of the functions $w_j \cos((k+(1/2))\pi(-j))$, $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, were introduced by Malvar [6], where window functions are assumed to be compactly supported, and only two immediately neighbouring windows are allowed to have overlapping support. Some generalizations of Malvar bases were studied in [7, 8]. To obtain more freedom for the choice of window functions, biorthogonal bases were investigated in [9]. A drawback of Malvar's construction is the restriction on the support of the window functions. Therefore, it was preferred to consider Wilson bases of Daubechies et al. [5].

Feichtinger et al. [10] proved that Wilson bases of exponential decay are not unconditional bases for all modulation spaces on $\mathbb{R}$ including the classical Bessel potential spaces and the Schwartz spaces. Also, Wilson bases are not unconditional bases for the ordinary L^p spaces for $p \neq 2$, as shown in [10]. Approximation properties of Wilson bases are studied by Bittner [11], and Wilson bases for general time-frequency lattices are studied by Kutyniok and Strohmer [12]. Generalizations of Wilson bases to nonrectangular lattices are discussed by Sullivan et al. [3], with motivation from wireless communication and cosines modulated filter banks. Wojdyllo [13] studied modified Wilson bases and discussed Wilson system for triple redundancy in [14]. Discrete time Wilson frames with general lattices are studied by Lian et al. [15]. Motivated by the fact that one has different trigonometric functions for odd and even indices, Bittner [11, 16] considered Wilson bases introduced by Daubechies et.al [5] with nonsymmetrical window functions for odd and even indices. This generalized system of Bittner was later studied extensively by Kaushik and Panwar [17–19] and Jarrah and Panwar [20].

In this article, we consider the system defined by Bittner [16] to define the discrete time Wilson frame (DTWF) and give examples for its existence. Some observations related to properties of discrete time Wilson frames are given. Also, a relationship between DTW frames and the discrete time Gabor frames is discussed. Furthermore, a necessary and a sufficient condition for the DTW frame in terms of Zak transform are obtained and the frame operator for the DTW frame is constructed. Finally, dual pair of frames for discrete time Wilson systems is defined and a sufficient condition for its existence is given.

The discrete time Wilson (DTW) system associated with $g_0, g_{-1} \in l^2(\mathbb{Z})$ is defined as

$$\psi_{\frac{m}{M}, kL} = \begin{cases} \left(E_{(m/M)}T_{(kL/2)} + E_{(-m/M)}T_{(kL/2)}\right)g_0, & \text{if } k \in 2\mathbb{Z},\ k \neq 0, \\ \frac{1}{i}\left(E_{(m+1/M)}T_{((k+1)L/2)} - E_{-(m+1/M)}T_{((k+1)L/2)}\right)g_{-1}, & \text{if } k \in 2\mathbb{Z}+1, \\ \frac{1}{\sqrt{2}}\left(E_{(m/M)} + E_{(-m/M)}\right)g_0, & \text{if } k = 0, \end{cases} \tag{4}$$

where $k \in \mathbb{Z}$, $L, M \in \mathbb{N}$ and $m = 0, 1, 2, \ldots, M-1$.

The DTW system given by (4) can be rewritten for any $n \in \mathbb{Z}$ as

$$\psi_{(m/M), kL}(n) = \begin{cases} \sqrt{2}\cos\left(\frac{2\pi mn}{M}\right)g_0(n), & \text{if } k = 0, \\ 2\cos\left(\frac{2\pi mn}{M}\right)g_0\left(n - \frac{kL}{2}\right), & \text{if } k \in 2\mathbb{Z},\ k \neq 0, \\ 2\sin\left(\frac{2\pi(m+1)n}{M}\right)g_{-1}\left(n - \frac{(k+1)L}{2}\right), & \text{if } k \in 2\mathbb{Z}+1. \end{cases} \tag{5}$$

Remark 1. For $g_0 = g_{-1} = g$, the DTW system has the form

$$\psi_{(m/M),kL}g = \begin{cases} \left(E_{(m/M)}T_{(kL/2)} + E_{(-m/M)}T_{(kL/2)}\right)g, & \text{if } k \in 2\mathbb{Z}, k \neq 0, \\ \frac{1}{i}\left(E_{(m+1/M)}T_{((k+1)L/2)} - E_{-(m+1/M)}T_{((k+1)L/2)}\right)g, & \text{if } k \in 2\mathbb{Z}+1, \\ \frac{1}{\sqrt{2}}\left(E_{(m/M)} + E_{(-m/M)}\right)g, & \text{if } k = 0, \end{cases} \tag{6}$$

where $k \in \mathbb{Z}$, $L, M \in \mathbb{N}$ and $m = 0, 1, 2, \ldots, M-1$.

2. Outline of the Paper

In this article, we define discrete time Wilson frames (DTW frames) and discuss various properties of DTW frames (see Observations (I) to (VIII)). An interplay between DTW frames and discrete time Gabor frames has been given in Theorem 1. Also, a necessary and a sufficient condition for the DTW frame in terms of Zak transform are given in Theorem 3 and 4, respectively. The construction of the frame operator for the DTW frame is discussed in Theorem 5. Finally, we discuss dual pair of frames for discrete time Wilson systems and give a sufficient condition for its existence. Various examples are given to illustrate the discussion.

3. Discrete Time Wilson Frames

In this section, we define the discrete time Wilson frame based on the Wilson system considered by Bittner [11, 16], explore their existence through examples, and investigate various properties including its relationship with discrete time Gabor systems. We begin with the following definition.

Definition 1. The discrete time Wilson system:

$$\left\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z}), k \in \mathbb{Z}, L, M \in \mathbb{N}, m = 0, 1, 2, \ldots, M-1\right\}, \tag{7}$$

where $\psi_{(m/M),kL}$ is as defined in (4) and is called a discrete time Wilson frame (DTWF) if there exist constants $0 < \mathcal{A}_l \leq \mathcal{A}_u < \infty$ satisfying

$$\mathcal{A}_l\|f\|^2 \leq \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 \leq \mathcal{A}_u\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{8}$$

The constants $\mathcal{A}_l$ and $\mathcal{A}_u$ are called lower and upper frame bounds for the DTWF $\left\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$. The supremum of all lower frame bounds and the infimum of all upper frame bounds are called optimal lower and optimal upper frames bounds, respectively.

In case the system $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z}), k \in \mathbb{Z}, L, M \in \mathbb{N}, m = 0, 1, 2, \ldots, M-1\}$ satisfy only the right-hand side of inequality (8), then the system is called a discrete time Wilson Bessel sequence for $l^2(\mathbb{Z})$.

In order to show the existence of discrete time Wilson Bessel sequences which are not DTWF for $l^2(\mathbb{Z})$, we give the following examples.

Example 1

(i) Let $\{g(n)\}_{n\in\mathbb{Z}} = e_n$, $n \in \mathbb{N}$. Then,

$$\begin{aligned} \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}|f(n+kL)|^2 \\ &= M\sum_{k\in\mathbb{Z}}|f(n+kL)|^2 \\ &\leq M\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \end{aligned} \tag{9}$$

Therefore, we obtain

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}g\rangle\right|^2 \leq 4M\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{10}$$

Hence, $\left\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\right\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a discrete time Bessel sequence for $l^2(\mathbb{Z})$ with Bessel bound $4M$.

However, it is a DTW frame if and only if $L = 1$.

(ii) Let $g(n) = \begin{cases} (1/n), n = 1, 2, 3, \ldots, A, \text{where} A < L < M, L \geq 2, \\ 0, \quad \text{otherwise}. \end{cases}$

Then, we have

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 \leq M\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{11}$$

Hence, $\left\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\right\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a discrete time Bessel sequence for $l^2(\mathbb{Z})$ with Bessel bound $4M$. Furthermore, it is not a frame as it does not satisfy the lower frame condition for $\{f(n)\}_{n\in\mathbb{Z}} = e_L \in l^2(\mathbb{Z})$. Moreover, note that

(1) If $A = L$, then $\left\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\right\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame with frame bounds $A = 2M$ and $B = 4M$.

(2) If $A = L = M$, then $\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame with frame bounds $A = M$ and $B = 4M$.

Next, we give examples of Wilson systems which are discrete time Wilson frames for $l^2(\mathbb{Z})$.

Example 2

(i) Let $g(n) = \begin{cases} (1/\sqrt{M}), & n = 0,1,2,\ldots,L-1, L<M,\\ 0, & \text{otherwise.}\end{cases}$

Then, using the fact that $\sum_{m=0}^{M-1} e^{2\pi i(m/M)(q-p)} = \begin{cases} M, & \text{if } q-p \in M\mathbb{Z},\\ 0, & \text{if } q-p \notin M\mathbb{Z},\end{cases}$ we have

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 = \frac{1}{M}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\sum_{p=0}^{L-1}|f(p+kL)|^2 = \|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}),$$

$$\sum_{m=0}^{M-1}\left|\langle f, \cos\left(\frac{2\pi m\cdot}{M}\right)g(\cdot)\rangle\right|^2 \le \frac{3}{2}\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{12}$$

Therefore, $\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a discrete time Wilson frame for $l^2(\mathbb{Z})$.

(ii) Let $g(n) = \begin{cases} (1/2^M), & n \in [0,L]\cap\mathbb{Z}, L<M,\\ 0, & \text{otherwise.}\end{cases}$

Note that $\sum_{m=0}^{M-1} e^{2\pi i(m/M)(q-p)} = \begin{cases} M, & \text{if } q-p \in M\mathbb{Z},\\ 0, & \text{if } q-p \notin M\mathbb{Z}.\end{cases}$

Therefore,

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 = \frac{M}{2^{2M}}\left(\|f\|^2 + \sum_{k\in\mathbb{Z}}|f((k+1)L)|^2\right),$$
$$2\sum_{m=0}^{M-1}\left|\langle f, \cos\left(\frac{2\pi m\cdot}{M}\right)g(\cdot)\rangle\right|^2 \le \frac{3M}{2^{2M}}\|f\|^2, \quad f \in l^2(\mathbb{Z}). \tag{13}$$

Hence, $\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame for $l^2(\mathbb{Z})$.

In view of the above discussion, we have the following observations in relation to DTW frames.

(I) Let $f, g \in l^2(\mathbb{Z})$ and let T_{kL} be the translation operator on $l^2(\mathbb{Z})$, where $k \in \mathbb{Z}$ and $L \in \mathbb{N}$. Then,

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\operatorname{Im}\left(\langle f, \cos\left(\frac{2\pi m(.)}{M}\right)T_{kL}g(.)\rangle \cdot \langle \overline{f}, \sin\left(\frac{2\pi m(.)}{M}\right)\overline{T_{kL}g(.)}\rangle\right) = 0. \tag{14}$$

Indeed, it follows from the fact that

$$\sum_{m=0}^{M-1} e^{2\pi i(m/M)(q-p)} = \begin{cases} M, & \text{if } q-p \in M\mathbb{Z},\\ 0, & \text{if } q-p \notin M\mathbb{Z}.\end{cases} \tag{15}$$

(II) Let $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a DTW system for $l^2(\mathbb{Z})$. Then,

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 = 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \cos\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}g_0(\cdot)\rangle\right|^2 + 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \sin\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}g_{-1}(\cdot)\rangle\right|^2$$
$$- 2\sum_{m=0}^{M-1}\left|\langle f, \cos\left(\frac{2\pi m(\cdot)}{M}\right)g_0(\cdot)\rangle\right|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{16}$$

Indeed, one can compute that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,\psi_{(m/M),kL}\rangle\right|^2 &= 4\sum_{\substack{k\in2\mathbb{Z}\\k\neq0}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m.}{M}\right)T_{(kL/2)}g_0(\cdot)\right\rangle\right|^2\\&+4\sum_{\substack{k\in2\mathbb{Z}+1\\k\neq0}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\left(\frac{2\pi(m+1).}{M}\right)T_{((k+1)L/2)}g_{-1}(\cdot)\right\rangle\right|^2+2\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m.}{M}\right)g_0(\cdot)\right\rangle\right|^2\\&=4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m.}{M}\right)T_{kL}g_0(\cdot)\right\rangle\right|^2-4\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m.}{M}\right)g_0(\cdot)\right\rangle\right|^2\\&+4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\left(\frac{2\pi m.}{M}\right)T_{kL}g_{-1}(\cdot)\right\rangle\right|^2+2\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m.}{M}\right)g_0(\cdot)\right\rangle\right|^2.\end{aligned}\tag{17}$$

In view of Observations (I) and (II), we obtain (III).

(III) Let $\left\{\psi_{(m/M),kL}\colon g_0,g_{-1}\in l^2(\mathbb{Z})\right\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a DTW system for $l^2(\mathbb{Z})$. Then,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,\psi_{(m/M),kL}\rangle\right|^2 &= 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_0(\cdot)\rangle\right|^2+4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_{-1}(\cdot)\rangle\right|^2\\&-4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\left(\frac{2\pi m.}{M}\right)T_{kL}g_0(\cdot)\right\rangle\right|^2-2\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m}{M}\right)g_0(\cdot)\right\rangle\right|^2\\&-4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m\cdot}{M}\right)T_{kL}g_{-1}(\cdot)\right\rangle\right|^2,\quad f\in l^2(\mathbb{Z}).\end{aligned}\tag{18}$$

Using Observations (II) and (III), we have (IV).

(IV) For all $f\in l^2(\mathbb{Z})$,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_0(\cdot)\rangle\right|^2+\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_{-1}(\cdot)\rangle\right|^2\\&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}g_0(\cdot)\right\rangle\right|^2+\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}g_0(\cdot)\right\rangle\right|^2\\&+\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}g_{-1}(\cdot)\right\rangle\right|^2+\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}g_{-1}(\cdot)\right\rangle\right|^2.\end{aligned}\tag{19}$$

(V) If $g_0 = g_{-1} = g$ for the DTW system $\{\psi_{(m/M),kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$, then for all $f \in l^2(\mathbb{Z})$, and we obtain

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 = 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g(.)\rangle\right|^2 - 2\sum_{m=0}^{M-1}\left|\left\langle f, \cos\left(\frac{2\pi m}{M}\right)g(.)\right\rangle\right|^2. \quad (20)$$

(VI) Let $\{E_{(m/M)}T_{kL}g_0\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{E_{(m/M)}T_{kL}g_{-1}\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be two DTG Bessel sequences with Bessel bounds B_1 and B_2, respectively. Then, the system $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence with Bessel bound $4(B_1 + B_2)$.

Indeed, using observation (III) and the hypothesis, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 &\le 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g_0(.)\rangle\right|^2 + 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g_{-1}(.)\rangle\right|^2 \\ &\le 4(B_1 + B_2)\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}).\end{aligned} \quad (21)$$

Remark 2. The converse of observation (VI) may not be true even if additionally we assume that the system $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a frame for $l^2(\mathbb{Z})$.

Example 3. Let $L = 2, M = 4,$ $L = 2, M = 4,$ and $\{g_{-1}(n)\}_{n\in\mathbb{Z}} = e_1$. Then,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g_0\rangle\right|^2 &= 4\sum_{k\in\mathbb{Z}}|f(2k)|^2, \quad \text{for all } f \in l^2(\mathbb{Z}), \\ \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g_{-1}\rangle\right|^2 &= 4\sum_{k\in\mathbb{Z}}|f(2k+1)|^2, \quad \text{for all } f \in l^2(\mathbb{Z}).\end{aligned} \quad (22)$$

Thus, the systems $\{E_{(m/M)}T_{kL}g_0\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{E_{(m/M)}T_{kL}g_{-1}\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ are not DTG frames for $l^2(\mathbb{Z})$. Now, using observation (III), we obtain

$$8\|f\|^2 \le \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}g\rangle\right|^2 \le 16\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \quad (23)$$

Hence, $\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame for $l^2(\mathbb{Z})$ with frame bounds 8 and 16.

(VII) Let $g_0, g_{-1} \in l^2(\mathbb{Z})$ be such that

$$\begin{aligned}B_1 &= M\sup_{n\in[0,L]\cap\mathbb{N}}\sum_{p\in\mathbb{Z}}\left|\sum_{k\in\mathbb{Z}}g_0(n-kL)\overline{g_0(n-kL-pM)}\right| < \infty, \\ B_2 &= M\sup_{n\in[0,L]\cap\mathbb{N}}\sum_{p\in\mathbb{Z}}\left|\sum_{k\in\mathbb{Z}}g_{-1}(n-kL)\overline{g_{-1}(n-kL-pM)}\right| < \infty.\end{aligned} \quad (24)$$

Then, the system $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence for $l^2(\mathbb{Z})$ with Bessel bound $4(B_1 + B_2)$.

(VIII) If $g_0, g_{-1} \in l^2(\mathbb{Z})$ are functions having bounded support, then the DTW system $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence for $l^2(\mathbb{Z})$.

Indeed, one may perceive that, since the functions g_0 and g_{-1} have bounded support, B_1 and B_2 as defined in observation (VII) are finite, and hence the DTW system $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence for $l^2(\mathbb{Z})$.

Now, we prove a result related to DTW systems for the particular case when $g_0 = g_{-1} = g$.

Lemma 1. *For $f, g \in l^2(\mathbb{Z})$, we have*

$$2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 \le \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 \le 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2. \tag{25}$$

Proof. Using observation (V), we obtain

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 = 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g(\cdot)\rangle\right|^2 - 2\sum_{m=0}^{M-1}\left|\left\langle f, \cos\left(\frac{2\pi m.}{M}\right)g(\cdot)\right\rangle\right|^2. \tag{26}$$

Hence, we compute

$$\begin{aligned}2\sum_{m=0}^{M-1}\left|\left\langle f, \cos\left(\frac{2\pi m.}{M}\right)g(.)\right\rangle\right|^2 &= 2\sum_{m=0}^{M-1}\left|\left\langle f, \frac{1}{2}\left(E_{(m/M)} + E_{(-m/M)}\right)g(.)\right\rangle\right|^2\\ &= \frac{1}{2}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}g\rangle + \langle f, E_{(-m/M)}g\rangle\right|^2\\ &\le 2\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}g\rangle\right|^2 \le 2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2.\end{aligned} \tag{27}$$

In the following result, we give an interplay between the DTW frame and DTG frame for $l^2(\mathbb{Z})$. □

Theorem 1. *The Wilson system $\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame for $l^2(\mathbb{Z})$ if and only if $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTG frame for $l^2(\mathbb{Z})$.*

Proof. Let $\mathscr{A}_l$ and $\mathscr{A}_u$ be the positive constants such that

$$\mathscr{A}_l\|f\|^2 \le \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f, \psi_{\frac{m}{M},kL}\right\rangle\right|^2 \le \mathscr{A}_u\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{28}$$

Then, using Lemma 1, it is easy to conclude that $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTG frame for $l^2(\mathbb{Z})$ with frame bounds $(\mathscr{A}_l/4)$ and $(\mathscr{A}_u/2)$.

Conversely, let $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a DTG frame for $l^2(\mathbb{Z})$. Then, there exist positive constants $\mathscr{B}_l$ and $\mathscr{B}_u$ such that

$$\mathscr{B}_l\|f\|^2 \le \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 \le \mathscr{B}_u\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{29}$$

Again, by utilizing Lemma 1, we deduce that $\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame with frame bounds $2\mathscr{B}_l$ and $4\mathscr{B}_u$.

Now, we define discrete time tight Wilson frame for $l^2(\mathbb{Z})$ and investigate their relationship with discrete time Gabor frame for $l^2(\mathbb{Z})$. □

Definition 2. The discrete time Wilson system $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ given by (4) is called a discrete time tight Wilson frame (DTTWF) if there exists a constant $\mathscr{C}_\infty \ge 0$ such that

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 = \mathscr{C}_1\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{30}$$

If $\mathscr{C}_\infty = 1$, then the frame $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}$ is called Discrete Time Parseval frame.

Next, we state two results whose proofs can be worked out using Lemma 1.

Proposition 1. *Let $\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a DTTW frame for $l^2(\mathbb{Z})$ with frame bound $\mathscr{C}_\infty$. Then, $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTG frame with frame bounds $(\mathscr{C}_\infty/4)$ and $(\mathscr{C}_\infty/2)$.*

Proposition 2. *Let* $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *be a DTTG frame for* $l^2(\mathbb{Z})$ *with frame bound* $\mathscr{C}_\infty$. *Then,* $\{\psi_{(m/M),kL}: g\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *is a DTW frame for* $l^2(\mathbb{Z})$ *with frame bounds* $2\mathscr{C}_\infty$ *and* $4\mathscr{C}_\infty$.

4. Discrete Zak Transform and Discrete Time Wilson Frames

Various properties of the Zak transform (continuous version) were studied by Janssen [21,22] and the discrete version is discussed by Heil [23] who gave the following definition of discrete Zak transform.

Definition 3 (see [23]). The discrete Zak transform of a sequence $f\in l^2(\mathbb{Z})$ is given by

$$Zf(n,x)=\sum_{j\in\mathbb{Z}} f(n+ja)e^{2\pi ijx},\quad \forall (n,x)\in\mathbb{Z}\times\widehat{\mathbb{R}},\tag{31}$$

where $a\in\mathbb{Z}^+$ is a fixed parameter and $\widehat{\mathbb{R}}$ is the dual group of $\mathbb{R}$.

Next, we state a result related to Zak transform proved by Heil [23].

Theorem 2 (see [23]). *Given a fixed* $g\in L^2(\mathbb{R})$ *and* $L\in\mathbb{Z}^+$. *If* $L=M$, *then the system* $\{E_{(m/M)}T_{kL}g\}$ *is a frame for* $l^2(\mathbb{Z})$ *with frame bounds* $\mathscr{D}_1$ *and* $\mathscr{D}_2 \Leftrightarrow 0<M^{-1}\mathscr{D}_1\le |Zg|^2\le M^{-1}\mathscr{D}_2<\infty a.e.$

Now, we give a necessary condition for DTW frame in terms of the discrete Zak transform.

Theorem 3. *Let* $L=M$. *If* $\{\psi_{(m/M),kL}: g\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *is a DTW frame for* $l^2(\mathbb{Z})$ *with frame bounds* $\mathscr{D}_l$ *and* $\mathscr{D}_u$, *then*

$$0<\frac{L^{-1}\mathscr{D}_l}{4}\le |Zg|^2\le \frac{L^{-1}\mathscr{D}_u}{2}<\infty \text{ a.e.}\tag{32}$$

Proof. Since $\{\psi_{(m/M),kL}: g\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame for $l^2(\mathbb{Z})$ with frame bounds $\mathscr{D}_l$ and $\mathscr{D}_u$, using Theorem 1, the system $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTG frame for $l^2(\mathbb{Z})$ with frame bounds $(\mathscr{D}_l/4)$ and $(\mathscr{D}_u/2)$.

Hence, the result follows using Theorem 2.

Towards, the converse of Theorem 3, we have the following result. □

Theorem 4. *Let* $L=M$. *If there exists* $\mathscr{D}_l>0$ *and* $\mathscr{D}_u>0$ *such that the following inequality holds*

$$0<\frac{L^{-1}\mathscr{D}_l}{2}\le |Zg|^2\le \frac{L^{-1}\mathscr{D}_u}{4}<\infty. \text{ a.e.},\tag{33}$$

then $\{\psi_{(m/M),kL}: g\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *is a DTW frame for* $l^2(\mathbb{Z})$ *with frame bounds* $\mathscr{D}_l$ *and* $\mathscr{D}_u$.

Proof. It can be worked out using Theorem 1 and Theorem 2. □

Remark 3. For $L>M$, the system $\{\psi_{(m/M),kL}: g\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is not a frame for $l^2(\mathbb{Z})$.

5. Discrete Time Wilson Frame Operator

The frame operator for a frame is constructed by the composition of two important operators, namely, the analysis operator and the synthesis operator. The frame operator is positive, bounded, invertible, and self-adjoint. It ensures the existence of a canonical dual frame of a given frame, i.e., if $\{f_n\}$ is a frame and S is the frame operator, then $\{S^{-1}f_n\}$ is a frame called the canonical dual of the frame $\{f_n\}$. It is known that the canonical tight frame leads to a perfect reconstruction when used for both analysis and synthesis. Keeping this in mind, we make an attempt to construct the frame operator for the discrete time Wilson frame. We begin with the following definition.

Definition 4. Let $\{\psi_{(m/M),kL}: g_0, g_{-1}\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a discrete time Bessel sequence for $l^2(\mathbb{Z})$. Then, DTWF operator $S: l^2(\mathbb{Z})\longrightarrow l^2(\mathbb{Z})$ is defined as

$$Sf=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,\psi_{(m/M),kL}\rangle\psi_{(m/M),kL},\quad \forall f\in l^2(\mathbb{Z}).\tag{34}$$

In the following result, we construct the frame operator for the discrete time Wilson frame with the help of the frame operators of the two associated discrete time Gabor Bessel sequences.

Theorem 5. *For* $g_0, g_{-1}\in l^2(\mathbb{Z})$, *let* $\{E_{(m/M)}T_{kL}g_0\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *and* $\{E_{(m/M)}T_{kL}g_{-1}\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *be DTG Bessel sequences with frame operators* S_1 *and* S_2, *respectively. Then, the frame operator* S *for DTW system* $\{\psi_{(m/M),kL}: g_0, g_{-1}\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *is given by* $S=2(S_1+S_2+P_1-P_2+R)$, *where*

$$\begin{aligned}
S_1f&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}g_0\rangle E_{(m/M)}T_{kL}g_0,\\
S_2f&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}g_{-1}\rangle E_{(m/M)}T_{kL}g_{-1},\\
P_1f&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}g_0\rangle E_{(-m/M)}T_{kL}g_0,\\
P_2f&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}g_{-1}\rangle E_{(-m/M)}T_{kL}g_{-1},\\
Rf&=\sum_{m=0}^{M-1}\langle f,E_{(m/M)}g_0\rangle\cos\left(2\pi\frac{m}{M}(\cdot)\right)g_0(\cdot).
\end{aligned}\tag{35}$$

Proof. By hypothesis, we have $S_1f=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}g_0\rangle E_{(m/M)}T_{kL}g_0$ and $S_2f=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}$

$g_{-1}\rangle E_{(m/M)}T_{kL}g_{-1}$. Since $\{E_{(m/M)}T_{kL}g_0\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{E_{(m/M)}T_{kL}g_{-1}\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ are DTG Bessel sequences, we obtain

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 = \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{-(m/M)}T_{kL}g\rangle\right|^2. \tag{36}$$

Also, using observation (VI), we deduce that the systems $\{E_{(-m/M)}T_{kL}g_0\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{E_{(-m/M)}T_{kL}g_{-1}\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ are DTG Bessel sequences and the system $\{\psi_{(m/M),kL}\colon g_0, g_{-1}\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence with their frame operators denoted by K_1, K_2, and S, respectively. Then, for all $f\in l^2(\mathbb{Z})$, we obtain

$$\begin{aligned}
K_1 f &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}T_{kL}g_0\rangle E_{-(m/M)}T_{kL}g_0,\\
K_2 f &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}T_{kL}g_{-1}\rangle E_{-(m/M)}T_{kL}g_{-1},\\
S f &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, \psi_{(m/M),kL}\rangle \psi_{(m/M),kL}\\
&= \sum_{m=0}^{M-1}\left\langle f, \frac{1}{\sqrt{2}}\left(E_{(m/M)} + E_{-(m/M)}\right)g_0\right\rangle \frac{1}{\sqrt{2}}\left(E_{(m/M)} + E_{-(m/M)}\right)g_0\\
&\quad + \sum_{k\in 2\mathbb{Z},k\neq 0}\sum_{m=0}^{M-1}\left\langle f, \left(E_{(m/M)}T_{(kL/2)} + E_{-(m/M)}T_{(kL/2)}\right)g_0\right\rangle\left(E_{(m/M)}T_{(kL/2)} + E_{-(m/M)}T_{(kL/2)}\right)g_0\\
&\quad + \sum_{k\in 2\mathbb{Z}+1}\sum_{m=0}^{M-1}\left\langle f, \frac{1}{i}\left(E_{(m+1/M)}T_{((k+1)L/2)} - E_{-(m+1/M)}T_{((k+1)L/2)}\right)g_{-1}\right\rangle \frac{1}{i}\left(E_{(m+1/M)}T_{((k+1)L/2)} - E_{-(m+1/M)}T_{((k+1)L/2)}\right)g_{-1}\\
&= \frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}g_0\rangle E_{(m/M)}g_0 + \frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}g_0\rangle E_{(m/M)}g_0 + \frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}g_0\rangle E_{-(m/M)}g_0\\
&\quad + \frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}g_0\rangle E_{-(m/M)}g_0 + \sum_{k\in 2\mathbb{Z},k\neq 0}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}T_{(kL/2)}g_0\rangle E_{(m/M)}T_{(kL/2)}g_0\\
&\quad + \sum_{k\in 2\mathbb{Z},k\neq 0}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}T_{(kL/2)}g_0\rangle E_{(m/M)}T_{(kL/2)}g_0 + \sum_{k\in 2\mathbb{Z},k\neq 0}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}T_{(kL/2)}g_0\rangle E_{-(m/M)}T_{(kL/2)}g_0\\
&\quad + \sum_{k\in 2\mathbb{Z},k\neq 0}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}T_{(kL/2)}g_0\rangle E_{-(m/M)}T_{(kL/2)}g_0 + \sum_{k\in 2\mathbb{Z}+1}\sum_{m=0}^{M-1}\langle f, E_{(m+1/M)}T_{((k+1)L/2)}g_{-1}\rangle E_{(m+1/M)}T_{((k+1)L/2)}g_{-1}\\
&\quad - \sum_{k\in 2\mathbb{Z}+1}\sum_{m=0}^{M-1}\langle f, E_{-(m+1/M)}T_{((k+1)L/2)}g_{-1}\rangle E_{(m+1/M)}T_{((k+1)L/2)}g_{-1}\\
&\quad - \sum_{k\in 2\mathbb{Z}+1}\sum_{m=0}^{M-1}\langle f, E_{(m+1/M)}T_{((k+1)L/2)}g_{-1}\rangle E_{-(m+1/M)}T_{((k+1)L/2)}g_{-1}\\
&\quad + \sum_{k\in 2\mathbb{Z}+1}\sum_{m=0}^{M-1}\langle f, E_{-(m+1/M)}T_{((k+1)L/2)}g_{-1}\rangle E_{-(m+1/M)}T_{((k+1)L/2)}g_{-1}nn,\\
&= S_1 f + S_2 f + K_1 f + K_2 f + P_1 f + T_1 f - P_2 f - T_2 f - R_1 f - R_2 f - R_3 f - R_4 f,
\end{aligned} \tag{37}$$

where

$$
\begin{aligned}
P_1 f &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}T_{kL}g_0\rangle E_{-(m/M)}T_{kL}g_0,\\
T_1 f &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}T_{kL}g_0\rangle E_{(m/M)}T_{kL}g_0,\\
P_2 f &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}T_{kL}g_{-1}\rangle E_{-(m/M)}T_{kL}g_{-1},\\
T_2 f &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}T_{kL}g_{-1}\rangle E_{(m/M)}T_{kL}g_{-1},\\
R_1 f &= \frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}g_0\rangle E_{(m/M)}g_0,\\
R_2 f &= \frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}g_0\rangle E_{(m/M)}g_0,\\
R_3 f &= \frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}g_0\rangle E_{-(m/M)}g_0,\\
R_4 f &= \frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}g_0\rangle E_{-(m/M)}g_0.
\end{aligned}
\tag{38}
$$

Now, for all $h \in l^2(\mathbb{Z})$, we compute

$$
\begin{aligned}
\langle T_1 f, h\rangle &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, E_{(-m/M)}T_{kL}g_0\rangle\langle E_{(m/M)}T_{kL}g_0, h\rangle\\
&= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} f(p)\overline{g_0(p-kL)}g_0(q-kL)\overline{h(q)}\exp^{2\pi i(m/M)(p+q)}\\
&= M\sum_{k\in\mathbb{Z}}\sum_{p,q\in\mathbb{Z},p+q\in M\mathbb{Z}} f(p)\overline{g_0(p-kL)}g_0(q-kL)\overline{h(q)},\\
\langle P_1 f, h\rangle &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}T_{kL}g_0\rangle\langle E_{(-m/M)}T_{kL}g_0, h\rangle\\
&= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} f(p)\overline{g_0(p-kL)}g_0(q-kL)\overline{h(q)}\exp^{-2\pi i(m/M)(p+q)}\\
&= M\sum_{k\in\mathbb{Z}}\sum_{p,q\in\mathbb{Z},p+q\in M\mathbb{Z}} f(p)\overline{g_0(p-kL)}g_0(q-kL)\overline{h(q)}.
\end{aligned}
\tag{39}
$$

Thus, $T_1 f = P_1 f$, for all $f \in l^2(\mathbb{Z})$. Similarly, it can be proved that $T_2 f = P_2 f$, $S_1 f = K_1 f$, $S_2 f = K_2 f$, $R_1 f = R_4 f$, and $R_2 f = R_3 f$, for all $f \in l^2(\mathbb{Z})$.

Hence, we conclude that $S = 2(S_1 + S_2 + P_1 - P_2 + R)$. □

6. Dual Pair of Frames for Discrete Time Wilson Systems

In this section, we study dual pair of frames and obtain a sufficient condition for the existence of a dual pair of discrete time Wilson systems. First, we state the definition of a dual pair of frames discussed by Christensen [24, 25].

Definition 5 (see [25]). Let H be a Hilbert space and let $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$, $\{p_j\}_{j\in J}$, and $\{q_i\}_{j\in J}$ be Bessel sequences. Then, $F = \{f_i\}_{i\in I}\cup\{p_j\}_{j\in J}$ and $G = \{g_i\}_{i\in I}\cup\{q_j\}_{j\in J}$ are dual pair of frames if

$$
f = \sum_{i\in I}\langle f, f_i\rangle g_i + \sum_{j\in J}\langle f, p_j\rangle q_j, \quad \text{for all } f \in H. \tag{40}
$$

In the following result, we give a sufficient condition for the existence of a dual pair of discrete time Wilson systems.

Theorem 6. *Let $L \leq M$ and let $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{\xi_{(m/M),kL}\colon w_0, w_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be two DTW Bessel sequences for $l^2(\mathbb{Z})$. Then, there exist DTW Bessel sequences $\{P_{(m/M),kL}\colon p_0, p_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{Q_{(m/M),kL}\colon q_0, q_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ such that $P = \{\psi_{(m/M),kL}\} \cup \{P_{(m/M),kL}\}$ and $Q = \{\xi_{(m/M),kL}\} \cup \{Q_{(m/M),kL}\}$ are dual pair of frames for $l^2(\mathbb{Z})$.*

Proof. Let T and U be the preframe operators for the DTW Bessel sequences $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{\xi_{(m/M),kL}\colon w_0, w_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$, respectively. Then, $T\colon l^2(\mathbb{Z}\times\mathbb{Z}_M) \longrightarrow l^2(\mathbb{Z})$ and $U\colon l^2(\mathbb{Z}\times\mathbb{Z}_M) \longrightarrow l^2(\mathbb{Z})$ are given by

$$
\begin{aligned}
T\left(\{c_{m,k}\}\right) &= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} c_{m,k}\psi_{(m/M),kL},\\
U\left(\{c_{m,k}\}\right) &= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} c_{m,k}\xi_{(m/M),kL},
\end{aligned}
\tag{41}
$$

where $\mathbb{Z}_M = \{0, 1, 2, \ldots, M-1\}$. Then,

$$
UT^* f = \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \langle f, \psi_{(m/M),kL}\rangle \xi_{(m/M),kL}, \quad \text{for all } f \in l^2(\mathbb{Z}).
\tag{42}
$$

Also, the operator $\Phi = I - UT^*$ is bounded on $l^2(\mathbb{Z})$. Furthermore, $\Phi^* = I - TU^*$. Let $R = \{R_{(m/M),kL}\colon r_0, r_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $S = \{S_{(m/M),kL}\colon s_0, s_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a pair of DTW dual frames for $l^2(\mathbb{Z})$. Then, using Proposition 2.1 of [25], we compute

$$
\begin{aligned}
f = &\sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \langle f, \psi_{(m/M),kL}\rangle \xi_{(m/M),kL}\\
&+ \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \langle f, \Phi^* R_{(m/M),kL}\rangle S_{(m/M),kL}.
\end{aligned}
\tag{43}
$$

Also, using Lemma 6.3.2 of [24], we deduce that $R = \{\psi_{(m/M),kL}\} \cup \{\Phi^* R_{(m/M),kL}\}$ and $S = \{\xi_{(m/M),kL}\} \cup \{S_{(m/M),kL}\}$ form a dual pair of frames for $l^2(\mathbb{Z})$ if $\{\Phi^* R_{(m/M),kL}\}$ is a DTW Bessel sequence. Now, observe that $\{\Phi^* R_{(m/M),kL}\}$ is a DTW system given by $\{\Phi^* R_{(m/M),kL}\} = \{R^*_{(m/M),kL}\} = \{R_{(m/M),kL}\colon \Phi^* r_0, \Phi^* r_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$. Since $R = \{R_{(m/M),kL}\colon r_0, r_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence and Φ is a bounded operator, $\{\Phi^* R_{(m/M),kL}\}$ is a DTW Bessel sequence.

Finally, we prove a result related to compact support of functions generating DTW Bessel sequences. □

Theorem 7. *Let $L \leq M$, and let $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{\xi_{(m/M),kL}\colon w_0, w_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be two DTW Bessel sequences for $l^2(\mathbb{Z})$. If the functions g_0, g_{-1}, w_0, and w_{-1} are compactly supported, then the functions p_0, p_{-1}, q_0, and q_{-1} are also compactly supported.*

Proof. Suppose that $R = \{R_{(m/M),kL}\colon r_0, r_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $S = \{S_{(m/M),kL}\colon s_0, s_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be such that r_0, r_{-1}, s_0, and s_{-1} be compactly supported. Then, in view of the proof of Theorem 6, one can conclude that the functions p_0, p_{-1}, q_0, and q_{-1} are compactly supported if $\Phi^* r_0$ and $\Phi^* r_{-1}$ are compactly supported.

By assumption, $g_0, g_{-1}, w_0, w_{-1}, r_0, r_{-1}, s_0$, and s_{-1} are all compactly supported. Therefore, there exists an $N \in \mathbb{N}$ such that

$$
\begin{aligned}
g_0(n) &= g_{-1}(n) = w_0(n) = w_{-1}(n) = s_0(n)\\
&= s_{-1}(n) = r_0(n) = r_{-1}(n) = 0, \quad \text{for all } n \notin [-N, N].
\end{aligned}
\tag{44}
$$

Since

$$
\Phi^* r_0 = (I - TU^*) r_0 = r_0 - \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \langle r_0, \xi_{(m/M),kL}\rangle \psi_{(m/M),kL},
\tag{45}
$$

$\Phi^* r_0$ is compactly supported. Similarly, $\Phi^* r_{-1}$ is compactly supported. □

7. Conclusion

Gabor frame for $L^2(\mathbb{R})$ (which is a Riesz basis) has bad localization properties in either time or frequency. Wilson [3, 4] suggested a system of functions which are localized around the positive and negative frequency of the same order. Based on the Wilson systems, Wilson frames for $L^2(\mathbb{R})$ were introduced and studied in [17–20]. In this article, discrete time Wilson frames (DTWF) are defined and their relationship with discrete time Gabor frames is investigated. Also, frame operator for the DTWF has been constructed. Finally, keeping duality in mind, dual pair of frames for the discrete time Wilson systems have been studied.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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