Research Article

Properties of Certain Subclasses of Meromorphically\(p\)-Valent Functions Associated with Certain Integral Operator

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Received 8 August 2020; Accepted 27 November 2020; Published 16 December 2020

Academic Editor: Elena Guardo

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The object of the this paper is to derive some interesting properties of certain subclasses of meromorphically\(p\)-valent functions which are defined by using an integral operator.

1. Introduction

Let \(\Sigma_{p,m}\) denote the class of functions of the form
\[
f(z) = \frac{1}{z^p} + \sum_{k=m}^{\infty} a_k z^k, \quad m > -p, \quad p \in \mathbb{N} = \{1, 2, 3, \ldots\},
\]
which are analytic and \(p\)-valent in the punctured unit disc \(U^* = U/\{0\},\) where \(U = \{ z : z \in \mathbb{C}, |z| < 1 \}.\) For convenience, we write \(\Sigma_{p,-p} = \Sigma_p.\)

For functions \(f(z) \in \Sigma_{p,m}\) given by (1) and \(g(z) \in \Sigma_{p,m}\) defined by
\[
g(z) = \frac{1}{z^p} + \sum_{k=m}^{\infty} b_k z^k, \quad m > -p, \quad p \in \mathbb{N},
\]
the Hadamard product (or convolution) of \(f(z)\) and \(g(z)\) is given by
\[
(f \ast g)(z) = \frac{1}{z^p} + \sum_{k=m}^{\infty} a_k b_k z^k
\]
\[= (g \ast f)(z).
\]

Let \(f\) and \(g\) be analytic in \(U.\) The function \(f\) is said to be subordinate to \(g,\) or \(g\) is superordinate to \(f,\) written \(f \prec g\) \((z \in U),\) if there exists a Schwarz function \(w(z)\) in \(U\) with \(w(0) = 0\) and \(|w(z)| < 1\) \((z \in U),\) such that
\[
f(z) = g(w(z)), \quad (z \in U).\]
If \(g(z)\) is univalent in \(U,\) then the equivalence (cf., e.g., [1, 2])
\[
f(z) \prec g(z) \iff f(0) = g(0), f(U) \subset g(U).
\]

For \(0 \leq \mu, \alpha \leq 1, m > -p, p \in \mathbb{N},\) and \(f \in \Sigma_{p,m},\) Saleh et al. [3] introduced the \(p\)-valent Rafid operator 
\[
S_{p,p}^{\alpha} f(z) = \frac{1}{(1 - \mu)^{\alpha+1}} \Gamma(\alpha + 1) \int_{0}^{\infty} t^{\alpha+1} e^{-t/(1-\mu)} f(zt) dt
\]
\[= \frac{1}{z^p} + \sum_{k=m}^{\infty} (1 - \mu)^{k+p} (\alpha + 1)_{k+p} a_k z^k,
\]
where \((\nu)_k\) is the Pochhammer symbol defined, in terms of the Gamma function \(\Gamma,\) by
\[
(\nu)_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)}
\]
\[= \begin{cases} 1, & \text{if } k = 0, \nu \in \mathbb{C}/\{0\}, \\
\nu, & \text{if } k \in \mathbb{N}, \nu \in \mathbb{C}.
\end{cases}
\]
Note that \( S_{\mu, f}^a f(z) = S_{\mu, f}^a f(z) \) (see [4]).
It follows from (5) that
\[
z(S_{\mu, f}^a f(z)) = (a + 1)S_{\mu, f}^{a+1} f(z) - (p + a + 1)S_{\mu, f}^a f(z),
\]
\((0 \leq \mu, a \leq 1). \quad (7)

By using the integral operator \( S_{\mu, f}^a f(z) \), we define a subclass of \( \sum_{p, m} \) as follows.

**Definition 1.** For fixed parameters \( A \) and \( B \), we say that a function \( \phi(z) \in \sum_{p, m} \) is in the class \( \sum_{p, m}(\alpha, \lambda, A, B) \) if it satisfies the following condition:
\[
\frac{z^{p+1}}{p} \{(1 - \lambda)\left(S_{\mu, f}^a f(z)\right)^\prime + (1 + Bz)\left(S_{\mu, f}^a f(z)\right)^\prime\} < \frac{1 + Az}{1 + Bz},
\]
\((z \in U), \ p \in \mathbb{N}, 0 \leq \mu, \alpha \leq 1, \lambda \geq 0 \) and \(-1 \leq B < A \leq 1). \quad (8)

There are many papers about some subclasses of meromorphic functions associated with several families of linear operators (see, for example, [5–11]). In this paper, we obtain some properties of the class \( \sum_{p, m}(\alpha, \mu, \lambda, A, B) \).

### 2. Preliminary Lemmas

To establish our main results, in this paper, we shall need the following lemmas.

**Lemma 1** (see [12] and [2]). Suppose that the function \( h(z) \) is analytic and convex (univalent) in \( U \) with \( h(0) = 1 \) and \( \phi(z) \) given by
\[
\phi(z) = 1 + c_{p+m}z^{p+m} + c_{p+m+1}z^{p+m+1} + \cdots. \quad (9)
\]
If
\[
\frac{z^\phi(z)}{y} \prec h(z), \quad (\Re(y) \geq 0, y \neq 0, z \in U), \quad (10)
\]
then
\[
\phi(z) < \psi(z) = \frac{y}{p + m} - \frac{y}{p + m} \int_0^1 \frac{h(t) - h(z)}{t^{p + m}} \ dt < h(z), \quad (z \in U), \quad (11)
\]
and \( \psi(z) \) is the best dominant of (10).

Let \( P(\gamma) \) be the class of analytic in \( U \) of the form
\[
\varphi(z) = 1 + b_1z + b_2z^2 + \cdots, \quad (12)
\]
which satisfies the following inequality:
\[
\Re(\varphi(z)) > \gamma, \quad (0 \leq \gamma < 1, z \in U). \quad (13)
\]

**Lemma 2** (see [13]). Let the function \( \varphi(z) \), given by (12), be in the class \( P(\gamma) \). Then,
\[
\Re(\varphi(z)) \geq 2\gamma - 1 + \frac{2(1 - \gamma)}{1 + |z|}, \quad (0 \leq \gamma < 1, z \in U). \quad (14)
\]

**Lemma 3** (see [14]). If \( \varphi_j \in P(\gamma_j), \quad (0 \leq \gamma_j < 1, j = 1, 2), \) then
\[
\varphi_1 \ast \varphi_2 \in P(\gamma_3), \quad \gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2). \quad (15)
\]

The result is the best possible.

Let \( a, b, \) and \( c \) be any real or complex numbers with \( c \notin Z_0 = \{0, -1, -2, \ldots\} \), and consider the function given by
\[
_{2}F_1(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}. \quad (16)
\]

This function, called the Gauss hypergeometric function, is analytic and converges absolutely for \( z \in U \) (see [15]).

**Lemma 4** (see [15]). Let \( a, b, \) and \( c \) any real or complex numbers with \( c \notin Z_0 \). Then,
\[
_{2}F_1(a, b, c, z) = \begin{cases} 1 & \Re(c) > \Re(b) > 0, \\ (1 - z)^{-a} & \Re(c) = \Re(b) > 0, \\ \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \int_0^1 \int_0^{b-1-t} (1 - t)^{c-b-1} (1 - zt)^{-a} dt, \quad \Re(c) < \Re(b) > 0, \end{cases} \quad (17)
\]
\[
_{2}F_1(a, b, c, z) = (1 - z)^{-a} _{2}F_1\left(a, c - b, c, \frac{z}{z - 1}\right). \quad (18)
\]
\[
_{2}F_1\left(a, b + b + 1, c, \frac{z}{2}ight) = \frac{\sqrt{\pi}\Gamma((a + b + 1)/2)}{\Gamma((a + b + 1)/2)} \Gamma((b + 1)/2). \quad (19)
\]
\[
_{2}F_1\left(1, 1, 2, \frac{z}{z + 1}\right) = \frac{z + 1}{z} \ln(1 + z), \quad z \neq 0. \quad (20)
\]
3. Main Results

Unless otherwise mentioned, we shall assume throughout the sequel that

$$m > -p, \, p \in \mathbb{N}, 0 \leq \mu, \alpha \leq 1, \lambda$$

$$> 0, \, z \in U \text{ and } -1 < B < A \leq 1.$$  \hspace{1cm} (21)

For

$$q_1(z) = \left[ \frac{A}{B} + \left( 1 - \frac{A}{B} \right) (1 + Bz)^{-1} \right] F_1 \left( 1, 1, \frac{\alpha + 1}{\lambda(p + m)} + 1, \frac{Bz}{Bz + 1} \right), (B \neq 0), 1 - \frac{\alpha + 1}{\lambda(p + m) + \alpha + 1} A, (B = 0).$$

is the best dominant of (22). Furthermore,

$$\Re \left( \frac{z^{p+1}(S^a_{\mu p f}(z))'}{p} \right) > \rho,$$  \hspace{1cm} (24)

where

$$\rho = \left\{ \frac{A}{B} + \left( 1 - \frac{A}{B} \right) (1 - B)^{-1} \right\} F_1 \left( 1, 1, \frac{\alpha + 1}{\lambda(p + m)} + 1, \frac{B}{B - 1} \right), (B \neq 0), 1 - \frac{\alpha + 1}{\lambda(p + m) + \alpha + 1} A, (B = 0).$$  \hspace{1cm} (25)

The result is the best possible.

**Proof.** Set

$$\phi(z) = -\frac{z^{p+1}(S^a_{\mu p f}(z))'}{p}.$$  \hspace{1cm} (26)

Then, the function $\phi(z)$ is of form (9) and is analytic in $U$. Differentiating (26) and with the aid of identity (7), we obtain

$$\phi(z) + \frac{\lambda z \phi'(z)}{\alpha + 1} = -\frac{z^{p+1}}{p} \left\{ (1 - \lambda)(S^a_{\mu p f}(z))' \right\}$$

$$+ \lambda \left( S^a_{\mu p f}(z) \right)' < \frac{1 + Az}{1 + Bz}$$  \hspace{1cm} (27)

$$q_1(z) = \frac{\alpha + 1}{\lambda (p + m)} z^{-\frac{1}{\lambda(p + m)}} \int_0^z t^{\frac{\alpha + 1}{\lambda(p + m)} - 1} \left( \frac{1 + At}{1 + Bt} \right) dt$$

$$= \left\{ \frac{A}{B} + \left( 1 - \frac{A}{B} \right) (1 + Bz)^{-1} \right\} F_1 \left( 1, 1, \frac{\alpha + 1}{\lambda(p + m)} + 1, \frac{Bz}{Bz + 1} \right), (B \neq 0), 1 - \frac{\alpha + 1}{\lambda(p + m) + \alpha + 1} A, (B = 0).$$  \hspace{1cm} (29)

by change of variables followed by the use of identities (17) and (18) (with $a = 1, b = (\alpha + 1)/\lambda (p + m)$), and $c = b + 1$.

This proves assertion (22) of Theorem 1. Next, in order to prove assertion (24) of Theorem 1, it suffices to show that

$$\inf_{|z| < 1} \Re \left( q_1(z) \right) = q_1(-1).$$  \hspace{1cm} (30)

Indeed, for $|z| \leq r < 1,$

$$\Re \left( \frac{1 + Az}{1 + Bz} \right) \geq \frac{1 - Ar}{1 - Br}.$$  \hspace{1cm} (31)

Setting

Theorem 1. If $f \in \sum_{p,m} (a, \mu, \lambda, A, B),$ then

$$-\frac{z^{p+1}(S^a_{\mu p f}(z))'}{p} < q_1(z) < \frac{1 + Az}{1 + Bz}$$  \hspace{1cm} (22)

where the function $q_1(z)$ given by

$$\Re \left( \frac{1 + Az}{1 + Bz} \right) \geq \frac{1 - Ar}{1 - Br}.$$  \hspace{1cm} (31)


Remark 1. For \( L \) in [17], Corollary 1) was also obtained by Patel and Sahoo [16] and obtained by Srivastava and Patel [18].

The result is the best possible.

Remark 2. The result (asserted by Corollary 1) was also obtained by Srivastava and Patel [18].

Taking \( \delta = - (p(\pi - 2))/(4 - \pi) \) in Corollary 2, we have the following corollary.

Letting \( r \rightarrow 1^- \) in the above inequality, we obtain assertion (30). The result in (24) is best possible as the function \( q_1(z) \) is the best dominant of (22).

Putting \( \lambda = (\sigma(\alpha + 1))/(1 - \sigma(p + 1)) \) in Theorem 1, we obtain the following corollary.

**Corollary 1.** If \( f(z) \in \sum_{\rho,m} \) satisfies

\[
-\frac{z^{\pi+1}[F_{\rho,m}^\alpha (z)]'}{p} < q_1(z) < \frac{1 + Az}{1 + Bz}
\]

then

\[
-\frac{z^{\pi+1}[F_{\rho,m}^\alpha (z)]'}{p} < q_1(z) < \frac{1 + Az}{1 + Bz}
\]

where the function \( q_1(z) \) given by

\[
q_1(z) = \left\{ \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1} \right\}^{-1} F_1 \left( 1, 1, 1 - \frac{\sigma(1 - m)}{\sigma(1 - m)} \right) \frac{Bz}{Bz + 1}, \quad (B \neq 0), 1 + \frac{1 - \sigma(p + 1)}{1 - \sigma(1 - m)} A, \quad (B = 0),
\]

where

\[
\rho = \left\{ \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 - B)^{-1} \right\}^{-1} F_1 \left( 1, 1, 1 - \frac{\sigma(10m)}{\sigma(1 - m)} \right) \frac{B}{B - 1}, \quad (B \neq 0), 1 - \frac{1 - \sigma(p + 1)}{1 - \sigma(1 - m)} A, \quad (B = 0).
\]
The result is the best possible.

Remark 3. The result (asserted by Corollary 3) was also obtained by Pap [19]. Applying Theorem 1 with $A = 1 - (2\delta/p)$, $B = -1$, $m = 1 - \rho$, and $\lambda = \alpha + 1$ and making use of (20), we obtain the following corollary.

Corollary 4. If $f(z) \in \sum_p$ satisfies the following inequality
\[
\Re \left\{ -z^{p+1} \left[ (p+2)(S_{p,p}^\alpha(z) + z(S_{p,p}^\alpha(z))^\nu) \right] \right\} > \delta, \quad (0 \leq \delta < p),
\]
then
\[
\Re \left\{ -z^{p+1}(S_{p,p}^\alpha(f(z))^\nu) \right\} > p + 2(p - \delta)(\ln 2 - 1).
\]

The result is the best possible.
Replacing $\phi(z)$ by $z^pS_{p,p}^\alpha(f(z))$ in (26) and applying the same method and technique as the proof of Theorem 1, we can prove the following result.

Theorem 2. If $f(z) \in \sum_{p,m}$ satisfies
\[
z^p \left\{ (1 - \lambda)S_{p,p}^\alpha(z) + \lambda S_{p,p}^{\alpha+1}(f(z)) \right\} < \frac{1 + Az}{1 + Bz}
\]
then
\[
z^p S_{p,p}^\alpha(f(z)) < q_1(z) < \frac{1 + Az}{1 + Bz}
\]
where $q_1$ and $\rho$ are given as in Theorem 1. The result is the best possible.

Theorem 3. Let $-1 \leq B_j < A_j \leq 1 \quad (j = 1, 2)$. If each of the functions $f_j(z) \in \sum_p$ satisfies the following subordination condition
\[
z^p \left\{ (1 - \lambda)S_{p,p}^\alpha(z) + \lambda S_{p,p}^{\alpha+1}(f_j(z)) \right\} < \frac{1 + A_j z}{1 + B_j z}, \quad (j = 1, 2),
\]
then
\[
z^p \left\{ (1 - \lambda)S_{p,p}^\alpha(H(z) + \lambda S_{p,p}^{\alpha+1}(H(z)) \right\} < 1 + (1 - 2y/p)z, \quad (1 - z)
\]
where
\[
H(z) = S_{p,p}^\alpha(f_1 \ast f_2)(z),
\]
\[
y = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[ 1 - \frac{1}{2} F_1 \left( 1, 1, \frac{\alpha + 1}{\lambda}, \frac{1}{2} \right) \right].
\]

The result is the best possible when $B_1 = B_2 = -1$.

Proof. If we let
\[
\phi_j(z) = z^p \left\{ (1 - \lambda)S_{p,p}^\alpha(f_j(z)) + \lambda S_{p,p}^{\alpha+1}(f_j(z)) \right\}, \quad (j = 1, 2),
\]
then, by the hypothesis of Theorem 3, we have
\[
\phi_j(z) \in P(\gamma_j), \quad \left( \gamma_j = \frac{1 - A_j}{1 - B_j}, j = 1, 2 \right).
\]

Using identity (7), (53) can be written as
\[
S_{p,p}^\alpha(f_j(z)) = \frac{\alpha + 1}{\lambda} z - \frac{\alpha + 1}{\lambda} p \int_0^z t \frac{\alpha + 1}{\lambda} - 1 \phi_j(t) dt,
\]
\[
\lambda = (1, 2).
\]

From (51) and (55), we obtain
\[
S_{p,p}^\alpha(H(z)) = \frac{\alpha + 1}{\lambda} - \frac{\alpha + 1}{\lambda} p \int_0^z t \frac{\alpha + 1}{\lambda} - 1 \phi_0(t) dt,
\]
where
\[
\phi_0(z) = z^p \left\{ (1 - \lambda)S_{p,p}^\alpha(H(z) + \lambda S_{p,p}^{\alpha+1}(H(z)) \right\}
\]
\[
= \frac{\alpha + 1}{\lambda} z \int_0^z t \frac{\alpha + 1}{\lambda} - 1 \phi_1(t) \phi_2(t) dt.
\]

Since $\phi_1(z) \in P(\gamma_1)$ and $\phi_2(z) \in P(\gamma_2)$, it follows from Lemma 3 that
\[
(\phi_1 \ast \phi_2)(z) \in P(\gamma_3), \quad \gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2).
\]

According to Lemma 2, we have
\[
\Re (\phi_1 \ast \phi_2)(z) \geq 2\gamma_3 - 1 + \frac{2(1 - \gamma_1)}{1 + |z|}.
\]

Now, by using (59) in (57) and then appealing to Lemma 4, we obtain
\[ \Re \{ \varphi_o(z) \} = \frac{\alpha + 1}{\lambda} \int_0^1 u^{(\alpha+1)/\lambda-1} \Re \{(\varphi_1 + \varphi_2)(uz)\} du \geq \frac{\alpha + 1}{\lambda} \int_0^1 u^{(\alpha+1)/\lambda-1} \left( 2y_3 - 1 + \frac{2(1 - y_3)}{1 + u|z|} \right) du \]

\[ > \frac{\alpha + 1}{\lambda} \int_0^1 u^{(\alpha+1)/\lambda-1} \left( 2y_3 - 1 + \frac{2(1 - y_3)}{1 + u} \right) du \]

\[ = 1 - 4 \frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left( 1 - \frac{\alpha + 1}{\lambda} \int_0^1 u^{(\alpha+1)/\lambda-1} (1 + u)^{-1} du \right) \]

\[ = 1 - 4 \frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[ 1 - \frac{1}{2} \Gamma_1 \left( 1, 1, \frac{\alpha + 1}{\lambda} + 1, \frac{1}{2} - \frac{1}{2} \mu \right) \right] \]

which completes the proof of assertion (49).

When \( B_1 = B_2 = -1 \), we consider the functions \( f_j(z) \in \sum_{\rho,m} (j = 1, 2) \) defined by

\[ S^\rho_{\mu,p}(f_j(z)) = \frac{\alpha + 1}{\lambda} - \frac{\alpha + 1}{\lambda} - p \int_0^z \frac{\alpha + 1}{\lambda} - 1 \left( 1 + A_1 \right) \frac{1}{1 - t} dt, \quad (j = 1, 2). \]

Now, by using Lemma 4 and (57), we have

\[ \varphi_o(z) = \frac{\alpha + 1}{\lambda} \int_0^1 u^{(\alpha+1)/\lambda-1} \left( 1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - uz} \right) du \]

\[ = 1 - (1 + A_1)(1 + A_2) + (1 + A_1)(1 + A_2)(1 - z)^{-1} \frac{1}{2} \Gamma_1 \left( 1, 1, \frac{\alpha + 1}{\lambda} + 1, \frac{z}{z - 1} \right) \]

\[ - (1 + A_1)(1 + A_2) + \frac{1}{2} (1 + A_1)(1 + A_2) \frac{1}{2} \Gamma_1 \left( 1, 1, \frac{\alpha + 1}{\lambda} + 1, \frac{1}{2} \right), \]

as \( z \xrightarrow{} -1 \), which ends the proof of Theorem 3.

Putting \( A_j = 1 - (2 \eta_j / p) \), \( 0 \leq \eta_j < p \), \( B_j = -1 \) \( (j = 1, 2) \), and \( \lambda / (\alpha + 1) = \tau \), in Theorem 3, we get the following result.

**Corollary 5.** If \( f \in \sum_p \) satisfies

\[ \Re \{ (1 + pr)S^\rho_{\mu,p} f_j(z) + rz \left( S^\rho_{\mu,p} f_j(z) \right)^r \} \geq \eta_j, \]

\( (j = 1, 2) \),

then

\[ \Re \{ (1 + pr) \left( S^\rho_{\mu,p} f_j(z) \right)^r \} \geq \gamma, \]

where

\[ \gamma = 1 - 4 \left( 1 - \frac{\eta_1}{p} \right)^2 \left( 1 - \frac{\eta_2}{p} \right)^2 \left[ 1 - \frac{1}{2} \frac{1}{2} \Gamma_1 \left( 1, 1, \frac{\alpha + 1}{\lambda} + 1, \frac{1}{2} \right) \right]. \]

**Remark 4.** For \( p = 1 \), the result (asserted by Corollary 5) was also obtained by Yang [20].

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares no conflicts of interest.

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