Hindawi
Journal of Mathematics
Volume 2020, Article ID 8509726, 7 pages
https://doi.org/10.1155/2020/8509726

Research Article

Properties of Certain Subclasses of Meromorphically p-Valent Functions Associated with Certain Integral Operator

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Received 8 August 2020; Accepted 27 November 2020; Published 16 December 2020

Academic Editor: Elena Guardo

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The object of the this paper is to derive some interesting properties of certain subclasses of meromorphically p-valent functions which are defined by using an integral operator.

1. Introduction

Let $\sum_{p,m}$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=m}^{\infty} a_k z^k, \quad m > -p, p \in \mathbb{N} = \{1, 2, 3, \ldots\}, \tag{1}$$

which are analytic and p-valent in the punctured unit disc $U^* = U/\{0\}$, where $U = \{z: z \in \mathbb{C}, |z| < 1\}$. For convenience, we write $\sum_{p,1-p} = \sum_p$.

For functions $f(z) \in \sum_{p,m}$ given by (1) and $g(z) \in \sum_{p,m}$ defined by

$$g(z) = \frac{1}{z^p} + \sum_{k=m}^{\infty} b_k z^k, \quad m > -p, p \in \mathbb{N}, \tag{2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$\begin{aligned}(f * g)(z) &= \frac{1}{z^p} + \sum_{k=m}^{\infty} a_k b_k z^k \\ &= (g * f)(z).\end{aligned} \tag{3}$$

Let f and g be analytic in U. The function f is said to be subordinate to g, or g is superordinate to f, written $f \prec g$ $(z \in U)$, if there exists a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$, $(z \in U)$. If $g(z)$ is univalent in U, then the equivalence (cf., e.g., [1, 2])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0), f(U) \subset g(U). \tag{4}$$

For $0 \le \mu, \alpha \le 1$, $m > -p$, $p \in \mathbb{N}$, and $f \in \sum_{p,m}$, Saleh et al. [3] introduced the p-valent Rafid operator $S_{\mu,p}^{\alpha}: \sum_{p,m} \longrightarrow \sum_{p,m}$ as follows:

$$\begin{aligned}S_{\mu,p}^{\alpha} f(z) &= \frac{1}{(1-\mu)^{\alpha+1}\Gamma(\alpha+1)} \int_0^{\infty} t^{\alpha+p} e^{-(t/(1-\mu))} f(zt) \mathrm{d}t \\ &= \frac{1}{z^p} + \sum_{k=m}^{\infty} (1-\mu)^{k+p} (\alpha+1)_{k+p} a_k z^k,\end{aligned} \tag{5}$$

where $(\nu)_k$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$\begin{aligned}(\nu)_k &= \frac{\Gamma(\nu+k)}{\Gamma(\nu)} \\ &= \begin{cases} 1, & (\text{if } k = 0, \nu \in \mathbb{C}/\{0\}), \\ \nu(\nu+1), \cdots, (\nu+k-1), & (\text{if } k \in \mathbb{N}, \nu \in \mathbb{C}). \end{cases}\end{aligned} \tag{6}$$

Note that $S_{\mu,1}^{\alpha} f(z) = S_{\mu}^{\alpha} f(z)$ (see [4]).

It follows from (5) that

$$z\left(S_{\mu,p}^{\alpha} f(z)\right)' = (\alpha+1) S_{\mu,p}^{\alpha+1} f(z) - (p+\alpha+1) S_{\mu,p}^{\alpha} f(z), \quad (0 \le \mu, \alpha \le 1). \tag{7}$$

By using the integral operator $S_{\mu,p}^{\alpha} f(z)$, we define a subclasses of $\sum_{p,m}$ as follows.

Definition 1. For fixed parameters A and B, we say that a function $f(z) \in \sum_{p,m}$ is in the class $\sum_{p,m}^{S}(\alpha, \mu, \lambda, A, B)$ if it satisfies the following condition:

$$-\frac{z^{p+1}}{p}\left\{(1-\lambda)\left(S_{\mu,p}^{\alpha} f(z)\right)' + \lambda\left(S_{\mu,p}^{\alpha+1} f(z)\right)'\right\} \prec \frac{1+Az}{1+Bz},$$
$$(z \in U), p \in \mathbb{N}, 0 \le \mu, \alpha \le 1, \lambda \ge 0 \text{ and } -1 \le B < A \le 1. \tag{8}$$

There are many papers about some subclasses of meromorphic functions associated with several families of linear operators (see, for example, [5–11]). In this paper, we obtain some properties of the class $\sum_{p,m}^{S}(\alpha, \mu, \lambda, A, B)$.

2. Preliminary Lemmas

To establish our main results, in this paper, we shall need the following lemmas.

Lemma 1 (see [12] and [2]). *Suppose that the function $h(z)$ is analytic and convex (univalent) in U with $h(0) = 1$ and $\phi(z)$ given by*

$$\phi(z) = 1 + c_{p+m} z^{p+m} + c_{p+m+1} z^{p+m+1} + \cdots. \tag{9}$$

If

$$\phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z), \quad (\Re(\gamma) \ge 0, \gamma \ne 0, z \in U), \tag{10}$$

then

$$\phi(z) \prec \psi(z) = \frac{\gamma}{p+m} z^{-\frac{\gamma}{p+m}} \int_0^z t^{\frac{\gamma}{p+m}-1} h(t)\mathrm{d}t \prec h(z), \quad (z \in U), \tag{11}$$

and $\psi(z)$ is the best dominant of (10).

Let $P(\gamma)$ be the class of analytic in U of the form

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + \cdots, \tag{12}$$

which satisfies the following inequality:

$$\Re(\varphi(z)) > \gamma, \quad (0 \le \gamma < 1, z \in U). \tag{13}$$

Lemma 2 (see [13]). *Let the function $\varphi(z)$, given by (12), be in the class $P(\gamma)$. Then,*

$$\Re(\varphi(z)) \ge 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|}, \quad (0 \le \gamma < 1, z \in U). \tag{14}$$

Lemma 3 (see [14]). *If $\varphi_j \in P(\gamma_j)$, $(0 \le \gamma_j < 1,\ j = 1, 2)$, then*

$$\varphi_1 * \varphi_2 \in P(\gamma_3), \quad \gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2). \tag{15}$$

The result is the best possible.

Let a, b, and c be any real or complex numbers with $c \notin Z_0^- = \{0, -1, -2, \ldots\}$, and consider the function given by

$${}_2F_1(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}. \tag{16}$$

This function, called the Gauss hypergeometric function, is analytic and converges absolutely for $z \in U$ (see [15]).

Lemma 4 (see [15]). *Let a, b, and c any real or complex numbers with $c \notin Z_0^-$. Then,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b, c, z), \quad \Re(c) > \Re(b) > 0, \tag{17}$$

$${}_2F_1(a, b, c, z) = (1-z)^{-a} {}_2F_1\left(a, c-b, c, \frac{z}{z-1}\right), \tag{18}$$

$${}_2F_1\left(a, b, \frac{a+b+1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma((a+b+1)/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)}, \tag{19}$$

$${}_2F_1\left(1, 1, 2, \frac{z}{z+1}\right) = \frac{z+1}{z}\ln(1+z), \quad z \ne 0. \tag{20}$$

3. Main Results

Unless otherwise mentioned, we shall assume throughout the sequel that

$$m > -p, p \in \mathbb{N}, 0 \le \mu, \alpha \le 1, \lambda > 0, z \in U \text{ and } -1 \le B < A \le 1. \tag{21}$$

Theorem 1. *If $f \in \sum_{p,m}^{S}(\alpha, \mu, \lambda, A, B)$, then*

$$-\frac{z^{p+1}\left(S_{\mu,p}^{\alpha} f(z)\right)'}{p} \prec q_1(z) \prec \frac{1+Az}{1+Bz}, \tag{22}$$

where the function $q_1(z)$ given by

$$q_1(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1, 1, \dfrac{\alpha+1}{\lambda(p+m)} + 1, \dfrac{Bz}{Bz+1}\right), (B \neq 0), \\ 1 + \dfrac{(\alpha+1)A}{\lambda(p+m)+\alpha+1} z, (B = 0), \end{cases} \tag{23}$$

is the best dominant of (22). Furthermore,

$$\Re\left(-\frac{z^{p+1}\left(S_{\mu,p}^{\alpha} f(z)\right)'}{p}\right) > \rho, \tag{24}$$

where

$$\rho = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1, 1, \dfrac{\alpha+1}{\lambda(p+m)} + 1, \dfrac{B}{B-1}\right), (B \neq 0), \\ 1 - \dfrac{\alpha+1}{\lambda(p+m)+\alpha+1} A, (B = 0). \end{cases} \tag{25}$$

The result is the best possible.

Proof. Set

$$\phi(z) = -\frac{z^{p+1}\left(S_{\mu,p}^{\alpha} f(z)\right)'}{p}. \tag{26}$$

Then, the function $\phi(z)$ is of form (9) and is analytic in U. Differentiating (26) and with the aid of identity (7), we obtain

$$\begin{aligned} \phi(z) + \frac{\lambda z \phi'(z)}{\alpha+1} = -\frac{z^{p+1}}{p}\Big\{(1-\lambda)\left(S_{\mu,p}^{\alpha} f(z)\right)' \\ + \lambda\left(S_{\mu,p}^{\alpha+1} f(z)\right)'\Big\} \prec \frac{1+Az}{1+Bz}. \end{aligned} \tag{27}$$

Now, by using Lemma 1 for $\gamma = (\alpha+1)/\lambda$, we deduce that

$$\phi(z) \prec q_1(z) \prec \frac{1+Az}{1+Bz}, \tag{28}$$

where $q_1(z)$ is the best dominant of (22) given by

$$\begin{aligned} q_1(z) &= \frac{\alpha+1}{\lambda(p+m)} z^{-(\alpha+1/\lambda(p+m))} \int_0^z t^{(\alpha+1/\lambda(p+m))-1}\left(\frac{1+At}{1+Bt}\right) dt \\ &= \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1, 1, \dfrac{\alpha+1}{\lambda(p+m)} + 1, \dfrac{Bz}{Bz+1}\right), (B \neq 0), \\ 1 + \dfrac{\alpha+1}{\lambda(p+m)+\alpha+1} Az, (B = 0), \end{cases} \end{aligned} \tag{29}$$

by change of variables followed by the use of identities (17) and (18) (with $a = 1, b = (\alpha+1)/(\lambda(p+m))$, and $c = b+1$). This proves assertion (22) of Theorem 1. Next, in order to prove assertion (24) of Theorem 1, it suffices to show that

$$\inf_{|z|<1}\{\Re(q_1(z))\} = q_1(-1). \tag{30}$$

Indeed, for $|z| \le r < 1$,

$$\Re\left(\frac{1+Az}{1+Bz}\right) \ge \frac{1-Ar}{1-Br}. \tag{31}$$

Setting

$$G(s,z)=\frac{1+Asz}{1+Bsz},$$
$$\mathrm{d}\mu(s)=\frac{\alpha+1}{\lambda(p+m)}s^{\frac{\alpha+1}{\lambda(p+m)}-1}\mathrm{d}s,\quad (0\le s\le 1), \tag{32}$$

which is a positive measure on [0, 1], we obtain

$$q_1(z)=\int_0^1 G(s,z)\mathrm{d}\mu(s) \tag{33}$$

so that

$$\Re(q_1(z))\ge\int_0^1\frac{1-Asr}{1-Bsr}\mathrm{d}\mu(s)=q_1(-r),\quad (|z|\le r<1). \tag{34}$$

Letting $r\longrightarrow 1^-$ in the above inequality, we obtain assertion (30). The result in (24) is best possible as the function $q_1(z)$ is the best dominant of (22).

Putting $\lambda=(\sigma(\alpha+1))/(1-\sigma(p+1))$ $(0<\sigma<(1/(p+1)))$ in Theorem 1, we obtain the following corollary. □

Corollary 1. *If $f(z)\in\sum_{p,m}$ satisfies*

$$\frac{-z^{p+1}\left[\left(S_{\mu,p}^{\alpha}f(z)\right)'+\sigma z\left(S_{\mu,p}^{\alpha}f(z)\right)''\right]}{p[1-\sigma(p+1)]}\prec\frac{1+Az}{1+Bz}, \tag{35}$$

then

$$-\frac{z^{p+1}\left(S_{\mu,p}^{\alpha}f(z)\right)'}{p}\prec q_2(z)\prec\frac{1+Az}{1+Bz}, \tag{36}$$

where the function $q_2(z)$ given by

$$q_2(z)=\left\{\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1,\frac{1-\sigma(1-m)}{\sigma(p+m)},\frac{Bz}{Bz+1}\right),(B\ne 0),1+\frac{1-\sigma(p+1)}{1-\sigma(1-m)}Az,(B=0),\right. \tag{37}$$

is the best dominant of (36). Furthermore,

$$\Re\left(-\frac{z^{p+1}\left(S_{\mu,p}^{\alpha}f(z)\right)'}{p}\right)>\rho, \tag{38}$$

where

$$\rho=\left\{\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1,\frac{1-\sigma(10m)}{\sigma(p+m)},\frac{B}{B-1}\right),(B\ne 0),1-\frac{1-\sigma(p+1)}{1-\sigma(1-m)}A,(B=0).\right. \tag{39}$$

The result is the best possible.

Remark 1. For $m=0$ and $p=1$, the result (asserted by Corollary 1) was also obtained by Patel and Sahoo [16] and Lashin [17].

Applying Theorem 1 with $A=1-(2\delta/p)$, $(0\le\delta<p)$, $B=-1$, $m=2-p$, and $\lambda=\alpha+1$ $(\alpha>-1)$ and making use of (19), we obtain the following corollary.

Corollary 2. *If $f(z)\in\sum_{p,2-p}$ satisfies the following inequality*

$$\Re\left\{-z^{p+1}\left[(p+2)\left(S_{\mu,p}^{\alpha}f(z)\right)'+z\left(S_{\mu,p}^{\alpha}f(z)\right)''\right]\right\}>\delta,\quad (0\le\delta<p), \tag{40}$$

then

$$\Re\left\{-z^{p+1}\left(S_{\mu,p}^{\alpha}f(z)\right)'\right\}>\delta+(p-\delta)\left(\frac{\pi}{2}-1\right). \tag{41}$$

The result is the best possible.

Remark 2. The result (asserted by Corollary 2) was also obtained by Srivastava and Patel [18].

Taking $\delta=-(p(\pi-2))/(4-\pi)$ in Corollary 2, we have the following corollary.

Corollary 3. *If $f(z)\in\sum_{p,2-p}$ satisfies the following inequality*

$$\Re\left\{-z^{p+1}\left[(p+2)\left(S_{\mu,p}^{\alpha}f(z)\right)'+z\left(S_{\mu,p}^{\alpha}f(z)\right)''\right]\right\}>-\frac{p(\pi-2)}{4-\pi}, \tag{42}$$

by Replacing $S_{\mu,p}^{\alpha}f(z)$ by $f(z)$, then

$$\Re\left\{-z^{p+1}\left(S_{\mu,p}^{\alpha}f(z)\right)'\right\}>0. \tag{43}$$

The result is the best possible.

Remark 3. The result (asserted by Corollary 3) was also obtained by Pap [19].

Applying Theorem 1 with $A = 1 - (2\delta/p)$, $(0 \le \delta < p)$, $B = -1$, $m = 1 - p$, and $\lambda = \alpha + 1$ and making use of (20), we obtain the following corollary.

Corollary 4. *If $f(z) \in \sum_p$ satisfies the following inequality*

$$\Re\left\{-z^{p+1}\left[(p+2)\left(S_{\mu,p}^{\alpha} f(z)\right)' + z\left(S_{\mu,p}^{\alpha} f(z)\right)''\right]\right\} > \delta, \quad (0 \le \delta < p), \tag{44}$$

then

$$\Re\left\{-z^{p+1}\left(S_{\mu,p}^{\alpha} f(z)\right)'\right\} > p + 2(p-\delta)(\ln 2 - 1). \tag{45}$$

The result is the best possible.

Replacing $\phi(z)$ by $z^p S_{\mu,p}^{\alpha} f(z)$ in (26) and applying the same method and technique as the proof of Theorem 1, we can prove the following result.

Theorem 2. *If $f \in \sum_{p,m}$ satisfies*

$$z^p\left\{(1-\lambda) S_{\mu,p}^{\alpha} f(z) + \lambda S_{\mu,p}^{\alpha+1} f(z)\right\} \prec \frac{1+Az}{1+Bz}, \tag{46}$$

then

$$z^p S_{\mu,p}^{\alpha} f(z) \prec q_1(z) \prec \frac{1+Az}{1+Bz}, \tag{47}$$

$$\Re\left(z^p S_{\mu,p}^{\alpha} f(z)\right) > \rho, \tag{48}$$

where q_1 and ρ are given as in Theorem 1. The result is the best possible.

Theorem 3. *Let $-1 \le B_j < A_j \le 1$ $(j = 1, 2)$. If each of the functions $f_j(z) \in \sum_p$ satisfies the following subordination condition*

$$z^p\left\{(1-\lambda) S_{\mu,p}^{\alpha} f_j(z) + \lambda S_{\mu,p}^{\alpha+1} f_j(z)\right\} \prec \frac{1+A_j z}{1+B_j z}, \quad (j = 1, 2), \tag{49}$$

then

$$z^p\left\{(1-\lambda) S_{\mu,p}^{\alpha} H(z) + \lambda S_{\mu,p}^{\alpha+1} H(z)\right\} \prec \frac{1+(1-(2\gamma/p))z}{1-z}, \tag{50}$$

where

$$H(z) = S_{\mu,p}^{\alpha}(f_1 * f_2)(z), \tag{51}$$

$$\gamma = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}\,{}_2F_1\left(1, 1, \frac{\alpha+1}{\lambda} + 1, \frac{1}{2}\right)\right]. \tag{52}$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. If we let

$$\varphi_j(z) = z^p\left\{(1-\lambda) S_{\mu,p}^{\alpha}\left(f_j(z)\right) + \lambda S_{\mu,p}^{\alpha+1}\left(f_j(z)\right)\right\}, \quad (j = 1, 2), \tag{53}$$

then, by the hypothesis of Theorem 3, we have

$$\varphi_j(z) \in P(\gamma_j), \quad \left(\gamma_j = \frac{1-A_j}{1-B_j},\ j = 1, 2\right). \tag{54}$$

Using identity (7), (53) can be written as

$$S_{\mu,p}^{\alpha}\left(f_j(z)\right) = \frac{\alpha+1}{\lambda} z^{-\frac{\alpha+1}{\lambda} - p} \int_0^z t^{\frac{\alpha+1}{\lambda} - 1} \varphi_j(t)\,\mathrm{d}t, \quad (j = 1, 2). \tag{55}$$

From (51) and (55), we obtain

$$S_{\mu,p}^{\alpha} H(z) = \frac{\alpha+1}{\lambda} z^{-\frac{\alpha+1}{\lambda} - p} \int_0^z t^{\frac{\alpha+1}{\lambda} - 1} \varphi_0(t)\,\mathrm{d}t, \tag{56}$$

where

$$\begin{aligned} \varphi_0(z) &= z^p\left\{(1-\lambda) S_{\mu,p}^{\alpha} H(z) + \lambda S_{\mu,p}^{\alpha+1} H(z)\right\} \\ &= \frac{\alpha+1}{\lambda} z^{-\frac{\alpha+1}{\lambda}} \int_0^z t^{\frac{\alpha+1}{\lambda} - 1} (\varphi_1 * \varphi_2)(t)\,\mathrm{d}t. \end{aligned} \tag{57}$$

Since $\varphi_1(z) \in P(\gamma_1)$ and $\varphi_2(z) \in P(\gamma_2)$, it follows from Lemma 3 that

$$(\varphi_1 * \varphi_2)(z) \in P(\gamma_3), \quad (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)). \tag{58}$$

According to Lemma 2, we have

$$\Re\{(\varphi_1 * \varphi_2)(z)\} \ge 2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+|z|}. \tag{59}$$

Now, by using (59) in (57) and then appealing to Lemma 4, we obtain

$$\begin{aligned}\Re\{\varphi_o(z)\} &= \frac{\alpha+1}{\lambda}\int_0^1 u^{(\alpha+1/\lambda)-1}\Re\{(\varphi_1*\varphi_2)(uz)\}\mathrm{d}u \geq \frac{\alpha+1}{\lambda}\int_0^1 u^{(\alpha+1/\lambda)-1}\left(2\gamma_3-1+\frac{2(1-\gamma_3)}{1+u|z|}\right)\mathrm{d}u\\ &> \frac{\alpha+1}{\lambda}\int_0^1 u^{(\alpha+1/\lambda)-1}\left(2\gamma_3-1+\frac{2(1-\gamma_3)}{1+u}\right)\mathrm{d}u\\ &= 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{\alpha+1}{\lambda}\int_0^1 u^{(\alpha+1/\lambda)-1}(1+u)^{-1}\mathrm{d}u\right]\\ &= 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{1}{2}{}_2F_1\left(1,1,\frac{\alpha+1}{\lambda}+1,\frac{1}{2}\right)\right]\\ &= \gamma,\end{aligned} \tag{60}$$

which completes the proof of assertion (49).

When $B_1 = B_2 = -1$, we consider the functions $f_j(z) \in \sum_{p,m}$ $(j = 1, 2)$ defined by

$$S_{\mu,p}^{\alpha}\left(f_j(z)\right) = \frac{\alpha+1}{\lambda}z^{-\frac{\alpha+1}{\lambda}-p}\int_0^z t^{\frac{\alpha+1}{\lambda}-1}\left(\frac{1+A_jt}{1-t}\right)\mathrm{d}t, \quad (j = 1, 2). \tag{61}$$

Now, by using Lemma 4 and (57), we have

$$\begin{aligned}\varphi_o(z) &= \frac{\alpha+1}{\lambda}\int_0^1 u^{(\alpha+1/\lambda)-1}\left(1-(1+A_1)(1+A_2)+\frac{(1+A_1)(1+A_2)}{1-uz}\right)\mathrm{d}u\\ &= 1-(1+A_1)(1+A_2)+(1+A_1)(1+A_2)(1-z)^{-1}{}_2F_1\left(1,1,\frac{\alpha+1}{\lambda}+1,\frac{z}{z-1}\right) \longrightarrow 1\\ &\quad -(1+A_1)(1+A_2)+\frac{1}{2}(1+A_1)(1+A_2){}_2F_1\left(1,1,\frac{\alpha+1}{\lambda}+1,\frac{1}{2}\right),\end{aligned} \tag{62}$$

as $z \longrightarrow -1$, which ends the proof of Theorem 3.

Putting $A_j = 1-(2\eta_j/p)$, $(0 \leq \eta_j < p)$, $B_j = -1$ $(j = 1, 2)$, and $\lambda/(\alpha+1) = \tau$, in Theorem 3, we get the following result.

Corollary 5. *If $f \in \sum_p$ satisfies*

$$\Re z^p\left\{(1+p\tau)S_{\mu,p}^{\alpha}f_j(z)+\tau z\left(S_{\mu,p}^{\alpha}f_j(z)\right)'\right\} > \eta_j, \quad (j = 1, 2), \tag{63}$$

then

$$\Re z^p\left\{(1+p\tau)\left(S_{\mu,p}^{\alpha}f_1(z)*S_{\mu,p}^{\alpha}f_2(z)\right)+\tau z\left(S_{\mu,p}^{\alpha}f_1(z)*S_{\mu,p}^{\alpha}f_2(z)\right)'\right\} > \gamma, \tag{64}$$

where

$$\gamma = 1-4\left(1-\frac{\eta_1}{p}\right)\left(1-\frac{\eta_2}{p}\right)\left[1-\frac{1}{2}{}_2F_1\left(1,1,\frac{1}{\tau}+1,\frac{1}{2}\right)\right]. \tag{65}$$

Remark 4. For $p = 1$, the result (asserted by Corollary 5) was also obtained by Yang [20].

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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