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Research Article

Multiple Dependent State Repetitive Sampling-Based Control Chart for Birnbaum–Saunders Distribution

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This paper proposes a new control chart for the Birnbaum–Saunders distribution based on multiple dependent state repetitive sampling (MDSRS). The proposed control chart is a generalization of the control charts based on single sampling, repetitive sampling, and multiple dependent state sampling. Its sensitivity is evaluated in terms of the average run length (ARL) using both exact formulae and simulations. A comprehensive comparison between the Birnbaum–Saunders distribution control chart based on the MDSRS method and other existing competing methods is provided using a simulation study as well as a real-life illustration. The results reveal that the proposed chart outperforms the existing charts considered in this study by having better shift detection ability.

1. Introduction

Attribute and variable control charts have been widely used in manufacturing as well as in industrial and nonindustrial service operations to monitor the flaws in the process in order to improve the quality of the outputs and/or services. The variable chart is more informative but not simple in the application as compared to the attribute one. In addition, the variable control charts are employed to estimate the variation in a process where the measurement is a variable measured on a continuous scale. Both control charts are designed to detect shifts in the process. The operational process of both control charts is the same, whereas their corresponding charting statistics are technically different based on their properties. In both cases, the decision on whether the process is “in a state of statistical control” (or simply in control) is taken by plotting the charting statistic (i.e., sample point) on the chart. For the basic chart, if the sample point plots within the upper control limit (UCL) and lower control limit (UCL), the process is declared to be in control; otherwise, the process is considered to be out of control.

Control charts are a popular vital apparatus which are used to maintain the process at the given target. Control charts are expected to detect any slight change in the process as soon as possible in order to help industrial engineers to fix the problem before it is too late. A more detailed account on the difference between attributes and variable control charts can be found in the study by Montgomery [1] and Aslam et al. [2]. Hussain et al. [3] developed a variable exponential weighted moving average (EWMA) control chart using the auxiliary-information-based feature. Zhou et al. [4] proposed an optimal design of the attribute control chart in the presence of autocorrelation for monitoring the process mean. Quinino et al. [5] proposed an attribute chart to simultaneously monitor the average and dispersion in the process. The applications of attribute and variable control charts can be found in [6–12].

According to Aslam et al. [13], "*Designing a variable control chart for a non-normal distribution may be difficult because the exact distribution of the associated statistic may not be known. Therefore, designing of an attribute control chart for some non-normal distribution based on a life test has attracted the attention of researchers. Particularly, a time-truncated life test is popularly employed to save the experiment time.*" Aslam and Jun [14] and Aslam et al. [15] developed a preassigned lifetime termination control chart based on Weibull distribution and the Pareto distribution of the 2nd kind, respectively. Birnbaum and Saunders [16] proposed a distribution for the fatigue process called Birnbaum–Saunders (BS) distribution. Lio and Park [17] proposed the chart for BS distribution using the bootstrap approach. Saulo et al. [18] proposed the chart for BS distribution using the single sampling. Marchant et al. [19] used the BS distribution in a multivariate control chart. Bourguignon et al. [20] proposed a control chart for BS distribution using the median parameter. A variety of applications for BS distribution are available in [18, 21–23].

Statistical process control (SPC) literature shows that repetitive sampling (RS) and multiple dependent state sampling (MDSS) methods are mostly used to construct professional control charts. The control charts using these sampling schemes are efficient in decreasing the value of the out-of-control average run length (ARL). The ARL is the number of rational samples to be plotted on the chart before it signals for the first time. These sampling schemes are applied if an engineer is not able to take a decision on the state of the production process based on the first sample. Aslam et al. [13] proposed a control chart based on BS distribution with RS. Shaheen et al. [24] developed the lognormal distribution scheme using the RS method. Aslam et al. [25] proposed a control chart when the quality characteristics follow a gamma distribution using the MDSS method. For more details, readers are referred to Jeyadurga et al. [26] and Riaz et al. [27].

Aldosari et al. [28] introduced the attribute control chart using the multiple dependent state repetitive sampling (MDSRS) method. The MDSRS is introduced by combining the RS and MDSS methods. The MDSRS-based scheme is shown to be more efficient than RS-based and MDSS-based schemes in terms of the ARL values in [28]. More recently, Aldosari et al. [29] developed a control chart based on MDSRS when the quality characteristics follow a multivariate Poisson distribution. More details on the use of MDSRS-based schemes are documented by Krishnan and Deepa [30]. Up to now, the MDSRS method has not yet been used to improve the existing charts when the quality characteristics follow a BS distribution. Therefore, the present article deals with the development of the control chart based on the BS distribution using the MDSRS method.

The rest of the manuscript is organized as follows: the proposed control chart is proposed in Section 2. In addition, the in-control and out-of-control ARL expressions are also derived. In Section 3, the proposed control chart is compared to the existing BS control chart based on RS and MDSS methods in terms of the ARL values. In Section 4, a real-life application of the proposed chart in the industry is provided, and the concluding remarks are provided in Section 5.

2. The Proposed Control Chart

In this section, the design of the MDSRS BS control chart is introduced in Section 2.1, with its operational procedure shown in Section 2.2 and the closed-form expressions of the in-control and out-of-control ARL in Sections 2.3 and 2.4, respectively. Finally, the sensitivity analysis of the new control chart is conducted in Section 2.5.

2.1. Design of the Proposed MDSRS BS Control Chart. Let T denote the lifetime/failure time of items that follows a BS distribution with an unknown shape parameter. Let b and σ denote the shape and scale parameters of the BS distribution, respectively, i.e., $T \sim BS(b, \sigma)$. Thus, the cumulative distribution function (cdf) of the BS $(b\sigma)$ distribution is given by

$$F(t; b, \sigma) = \Phi\left(\frac{1}{b}\xi\left(\frac{t}{\sigma}\right)\right), \quad t > 0, \tag{1}$$

where $\Phi(\cdot)$ is the cdf of the standard normal variable and $\xi(t) = \sqrt{t} - \sqrt{(1/t)}$. The average lifetime, say μ, of a product under the BS distribution is defined by

$$\mu = E[T] = \sigma\left(1 + \frac{1}{2}b^2\right). \tag{2}$$

Assuming that $t_0 = a\mu_0$ be the termination time of the lifetime experiment, where a is the termination constant and μ_0 is the prespecified average life. Therefore, the target mean can be written as

$$\mu_0 = \sigma_0\left(1 + \frac{1}{2}b_0^2\right). \tag{3}$$

Hence, the probability of failure, say p_0 under $t_0 = a\mu_0$, is expressed as

$$\begin{aligned} p_0 &= P\left\{T \le t_0 \mid \sigma_0, b_0\right\} = \Phi\left(\frac{1}{b_0}\xi\left(\frac{t_0}{\sigma_0}\right)\right) \\ &= \Phi\left(\frac{1}{b_0}\xi\left(a\left(1 + \left(\frac{b_0^2}{2}\right)\right)\right)\right). \end{aligned} \tag{4}$$

Let D denote the number of failed items before preassigned test time, t_0. One of the main steps in the design of the proposed chart is the determination of the control limits. The MDSRS BS chart for the number of failed items has two pairs of control limits named as outer and inner control limits denoted as $(\mathrm{LCL}_1, \mathrm{UCL}_1)$ and $(\mathrm{LCL}_2, \mathrm{UCL}_2)$, respectively, where $\mathrm{LCL}_1 < \mathrm{LCL}_2 < \mathrm{UCL}_2 < \mathrm{UCL}_1$. When the parameters are known, the control limits are defined by

$$\mathrm{UCL}_1 = np_0 + k_1\sqrt{np_0(1 - p_0)}, \tag{5a}$$

$$\mathrm{LCL}_1 = \max\left[0, np_0 - k_1\sqrt{np_0(1 - p_0)}\right], \tag{5b}$$

$$\mathrm{UCL}_2 = np_0 + k_2\sqrt{np_0(1-p_0)}, \tag{6a}$$

$$\mathrm{LCL}_2 = np_0 - k_2\sqrt{np_0(1-p_0)}, \tag{6b}$$

where k_1 and k_2 (with $k_1 > k_2$) are the control limit coefficients that are set such that the control chart yields a prespecified in-control ARL value such as 370 and 500. Thus, the process is declared in control if $\mathrm{LCL}_2 \le D \le \mathrm{UCL}_2$ and out of control if $D > \mathrm{UCL}_1$ or $D < \mathrm{LCL}_1$.

When p_0 is not known to industrial engineers, as it is usually the case in real-life applications, the following control limits based on the average number of failed items can be used to estimate the control limits given in (5a), (5b), (6a), and (6b):

$$\mathrm{UCL}_1 = \overline{D} + k_1\sqrt{\overline{D}\left(1-\left(\frac{\overline{D}}{n}\right)\right)}, \tag{7a}$$

$$\mathrm{LCL}_1 = \max\left[0, \overline{D} - k_1\sqrt{\overline{D}\left(1-\left(\frac{\overline{D}}{n}\right)\right)}\right], \tag{7b}$$

$$\mathrm{UCL}_2 = \overline{D} + k_2\sqrt{\overline{D}\left(1-\left(\frac{\overline{D}}{n}\right)\right)}, \tag{8a}$$

$$\mathrm{LCL}_2 = \overline{D} - k_2\sqrt{\overline{D}\left(1-\left(\frac{\overline{D}}{n}\right)\right)}, \tag{8b}$$

where $\overline{D}$ denotes the estimated average number of failed items observed from the preliminary subgroups taken from in-control process. The charting statistic of the proposed chart follows the binomial distribution with parameters of n and p_0, which is more efficient as compared to the one of the BS-based attribute control chart. Different control charts for the BS-based attribute chart are particular cases for the proposed chart. The latter turns to a single sampling BS-based attribute control chart as soon as $k_1 = k_2$. The proposed chart is a BS attribute chart using the MDSS method when the probability of repetition is zero; however, it becomes the BS attribute control chart based on the repetitive method when $i = 0$.

2.2. Operational Procedure of the Proposed Control Chart. The step-by-step procedure for the design of the proposed control chart using the MDSRS method is given as follows:

Step 1: choosing the sample size (n) and fixing the preassigned test time t_0.

Step 2: setting k_1 and k_2 to some nonzero positive values and calculating the control limits using (7) and (8). The process is considered as in control if $\mathrm{LCL}_2 \le \overline{D} \le \mathrm{UCL}_2$. The process is to be out of control if $\overline{D} > \mathrm{UCL}_1$ or $\overline{D} < \mathrm{LCL}_1$. Otherwise, go to step 3.

Step 3: the process can be stated as in control if i proceeding subgroups statistic D satisfies $\mathrm{UCL}_2 \le \overline{D} \le \mathrm{UCL}_1$ or $\mathrm{LCL}_1 \le \overline{D} \le \mathrm{LCL}_2$. Otherwise, repeat Step 1.

2.3. In-Control ARL of the MDSRS BS Control Chart. For the proposed control chart, the probability under the MDSRS method while the process is actually in control is given by

$$\begin{aligned} P_{\mathrm{in},1}^0 = {} & P\left(\mathrm{LCL}_2 \le D \le \mathrm{UCL}_2 \mid p_0\right) \\ & + \left\{P\left(\mathrm{LCL}_1 \le D \le \mathrm{LCL}_2 \mid p_0\right) \right. \\ & \left. + P\left(\mathrm{UCL}_2 \le D \le \mathrm{UCL}_1 \mid p_0\right)\right\} \\ & \cdot \left\{P\left(\mathrm{LCL}_2 \le D \le \mathrm{UCL}_2 \mid p_0\right)\right\}^i. \end{aligned} \tag{9}$$

Since the plotting statistic follows a binomial distribution with parameters of n and p_0, then (9) can be written as

$$\begin{aligned} P_{\mathrm{in},1}^0 = {} & \sum_{d=\lfloor \mathrm{LCL}_2\rfloor+1}^{\lfloor \mathrm{UCL}_2\rfloor} \binom{n}{d} p_0^d (1-p_0)^{n-d} + \left\{ \sum_{d=\lfloor \mathrm{LCL}_1\rfloor+1}^{\lfloor \mathrm{LCL}_2\rfloor} \binom{n}{d} p_0^d (1-p_0)^{n-d} + \sum_{d=\lfloor \mathrm{UCL}_2\rfloor+1}^{\lfloor \mathrm{UCL}_1\rfloor} \binom{n}{d} p_0^d (1-p_0)^{n-d} \right\} \\ & \cdot \left\{ \sum_{d=\lfloor \mathrm{LCL}_2\rfloor+1}^{\lfloor \mathrm{UCL}_2\rfloor} \binom{n}{d} p_0^d (1-p_0)^{n-d} \right\}^i. \end{aligned} \tag{10}$$

If the researcher is in the state of uncertainty after plotting the statistic, based on step 3, the process should be repeated. Suppose, P_{rep}^0 represents the probability of the repeated state. Thus, the latter is mathematically defined by

$$P_{\mathrm{rep}}^0 = \left\{P\left(\mathrm{LCL}_1 < \overline{D} < \mathrm{LCL}_2 \mid p_0\right) + P\left(\mathrm{UCL}_2 < \overline{D} < \mathrm{UCL}_1 \mid p_0\right)\right\}\left(1 - \left[P\left\{\mathrm{LCL}_2 \le \overline{D} \le \mathrm{UCL}_2 \mid p_0\right\}\right]^i\right). \tag{11}$$

The probability in (11) can be written as

$$P_{\text{rep}}^{0} = \left\{ \sum_{d=\lfloor \text{LCL}_1 \rfloor + 1}^{\lfloor \text{LCL}_2 \rfloor} \binom{n}{d} p_0^d (1 - p_0)^{n-d} + \sum_{d=\lfloor \text{UCL}_2 \rfloor + 1}^{\lfloor \text{UCL}_1 \rfloor} \binom{n}{d} p_0^d (1 - p_0)^{n-d} \right\} \left(1 - \left[\sum_{d=\lfloor \text{LCL}_2 \rfloor + 1}^{\lfloor \text{UCL}_2 \rfloor} \binom{n}{d} p_0^d (1 - p_0)^{n-d} \right]^i \right). \tag{12}$$

Finally, the in-control probability of the BS control chart under the MDSRS method is given by

$$P_{\text{in}}^{0} = \frac{P_{\text{in},1}^{0}}{1 - P_{\text{rep}}^{0}}. \tag{13}$$

Therefore, the in-control ARL is then defined by

$$\text{ARL}_0 = \frac{1}{1 - P_{\text{in}}^{0}}. \tag{14}$$

2.4. ARL When Process Is Shifted for the MDSRS BS Control Chart. Suppose that due to controllable factors, there is a change (or shift) in the parameters. The new values are $\sigma_1 = f\sigma_0$ and $b_1 = gb_0$ where f is the shift in the scale parameter and g is the shift in the shape parameter. When $f \neq 1$ and/or $g \neq 1$, the process is out of control. The probability that an item fails by t_0 is given by

$$\begin{aligned} p_1 &= P\{T \le t_0 \mid \sigma_1, b_1\} = \Phi\left(\frac{1}{b_1}\xi\left(\frac{t_0}{\sigma_1}\right)\right) \\ &= \Phi\left(\frac{1}{gb_0}\xi\left(\frac{a\left(1 + (1/2)b_0^2\right)}{f}\right)\right), \end{aligned} \tag{15}$$

where the shift constants f and g are typically taken to be positive between 0 and 1.

The probability of declaring the process is out of control for the MDSRS BS chart when in fact it is shifted (denoted as $P_{\text{in},1}^{1}$) is given by

$$P_{\text{in},1}^{1} = P\left(\text{LCL}_2 \le \overline{D} \le \text{UCL}_2 \mid p_1\right) + \left\{P\left(\text{LCL}_1 \le \overline{D} \le \text{LCL}_2 \mid p_1\right) + P\left(\text{UCL}_2 \le \overline{D} \le \text{UCL}_1 \mid p_1\right)\right\}\left\{P\left(\text{LCL}_2 \le \overline{D} \le \text{UCL}_2 \mid p_1\right)\right\}^i. \tag{16}$$

Note that (16) can be written as

$$\begin{aligned} P_{\text{in},1}^{1} = {} & \sum_{d=\lfloor \text{LCL}_2 \rfloor + 1}^{\lfloor \text{UCL}_2 \rfloor} \binom{n}{d} p_1^d (1 - p_1)^{n-d} + \left\{ \sum_{d=\lfloor \text{LCL}_1 \rfloor + 1}^{\lfloor \text{LCL}_2 \rfloor} \binom{n}{d} p_1^d (1 - p_1)^{n-d} + \sum_{d=\lfloor \text{UCL}_2 \rfloor + 1}^{\lfloor \text{UCL}_1 \rfloor} \binom{n}{d} p_1^d (1 - p_1)^{n-d} \right\} \\ & \cdot \left\{ \sum_{d=\lfloor \text{LCL}_2 \rfloor + 1}^{\lfloor \text{UCL}_2 \rfloor} \binom{n}{d} p_1^d (1 - p_1)^{n-d} \right\}^i. \end{aligned} \tag{17}$$

When the process parameter has shifted, the repeated state probability, i.e., P_{rep}^{1} at p_1, is as follows:

$$P_{\text{rep}}^{1} = \{P(\text{LCL}_1 < \overline{D} < \text{LCL}_2) + P(\text{UCL}_2 < \overline{D} < \text{UCL}_1)\}\left(1 - [P\{\text{LCL}_2 \le \overline{D} \le \text{UCL}_2\}]^i\right). \tag{18}$$

Note that (18) can be written as

$$P_{\text{rep}}^{1} = \left\{ \sum_{d=\lfloor \text{LCL}_1 \rfloor + 1}^{\lfloor \text{LCL}_2 \rfloor} \binom{n}{d} p_1^d (1 - p_1)^{n-d} + \sum_{d=\lfloor \text{UCL}_2 \rfloor + 1}^{\lfloor \text{UCL}_1 \rfloor} \binom{n}{d} p_1^d (1 - p_1)^{n-d} \right\} \left(1 - \left[\sum_{d=\lfloor \text{LCL}_2 \rfloor + 1}^{\lfloor \text{UCL}_2 \rfloor} \binom{n}{d} p_1^d (1 - p_1)^{n-d} \right]^i \right). \tag{19}$$

Therefore, the in-control probability of the proposed chart under the MDSRS method, when the process is shifted, is given by

$$P_{\text{in}}^{1} = \frac{P_{\text{in},1}^{1}}{1 - P_{\text{rep}}^{1}}. \tag{20}$$

Hence, the in-control ARL when the process has shifted is defined by

$$\text{ARL}_1 = \frac{1}{1 - P_{\text{in}}^{1}}. \tag{21}$$

2.5. Empirical Analysis. Tables 1–6 display the ARL profile of the proposed MDSRS BS chart for different values of f and g for a prespecified in-control ARL value of 370. The results in Tables 1–6 can be summarized as follows:

(1) When the values of b and n are kept fixed, the values of ARL increase as the values of i increase.

(2) For the proposed BS chart, the values of ARL increase as the values of n increase.

(3) When the shift in the scale parameter $f < 0.6$, regardless of the design parameter and the shift in the shape parameter, the proposed chart gives a signal on the first sample (i.e., $\text{ARL} = 1$).

(4) When b and i are kept fixed, for $f = 1$, when $0.1 < g \leq 0.4$, the sensitivity of the proposed chart is higher for large sample sizes and lower for small sample sizes. However, when $0.4 \leq g < 1$, the sensitivity of the chart is lower for large values of n and higher for small values of n. For $f = 0.9$, the larger the sample size, the more sensitive the proposed chart. For $f < 0.9$, sensitivity is similar regardless of the sample size.

(5) The sensitivity of the proposed chart decreases for small values of b. Moreover, for small values of b, the larger the sample, the more sensitive the chart.

The four-step procedure below is used to construct Tables 1–6.

Usually, control charts are developed in favor of the industrial purpose so that in-control ARL must be as close as the particular prespecified ARL value, denoted as r_0. The proposed chart comprises seven parameters. Five parameters of shift-invariant f, sample size n, shape and scale parameters (b, σ), and i $(i \leq 5)$ are prefixed. But, the control chart parameters k_1 and k_2 are obtained such that inf $\{\text{ARL}_0 \mid \text{ARL}_0 \geq r_0\}$. The best combination is the one in which in-control ARL is very close to the particular ARL. The control chart coefficient values are calculated by means of the following steps:

Step 1: assigning values of b, a, n, and i.

Step 2: determining the chart parameter k_1 and k_2 values which satisfy the condition inf $\{\text{ARL}_0 \mid \text{ARL}_0 \geq r_0\}$ and $k_1 > k_2$. Then, 10,000 possible values for control chart coefficients ranging from 2 to 3.5 for k_1 and 2 to 3 for k_2 are generated and the ARL values are calculated. Then, the arrangement of parameters is chosen, which satisfied the condition inf $\{\text{ARL}_0 \mid \text{ARL}_0 \geq r_0\}$.

Step 3: therefore, step 2 is repeated 10000 times, and the most suitable mixture of parameters is selected such that ARL_0 is nearest to r_0.

Step 4: after selecting the combination of parameters while the process is in control, the out-of-control ARL values are calculated for various shifts of f and g where both range from 1 to 0.1.

3. Comparative Studies Using the ARLs

In this section, the BS control chart using the MDSRS method is compared to the ones using the single sampling (by [18]), RS (by [13]), and MDSS methods. Note that the BS control chart based on the MDSS method does not exist yet in the SPM literature but is added here for comparison purpose. The same combinations of parameters are used to evaluate the performance of the competing projected charts.

The values of ARLs of the proposed chart with three other competing charts when $b = 0.31$ and 1 and $n = 20$ are given in Table 7. From Table 7, it is quite clear that the proposed control chart is superior to Saulo et al. [18] and Aslam et al. [13] control charts. The proposed control chart cuts down the ARL values significantly as compared to the competing control charts. For instance, if $f = 1$ and $g = 0.9$, the ARL is 312, 351, 358, and 366 for the BS control chart based on the MDSRS, RS, MDSS, and single sampling methods, respectively. From this study, it can be observed that for very small shifts in the parameters, the BS control chart based on the MDSRS method is very sensitive to the changes in the manufacturing process as compared to the other methods. In Table 7, it is observed that the BS control chart provides the smaller values of ARL at each different combinations of f and g.

4. Application Examples

4.1. Simulation Study. In this section, the sensitivity of the proposed control chart as well as those of the two BS control charts based on RS and MDSS is evaluated based on simulated data. The following procedure is used in the simulations:

(1) Assuming that the in-control process parameters of BS distribution are given by $b_0 = 1$ and $\sigma_0 = 1.5$.

(2) Assuming that the shifted process as $f = 0.9$ and $g = 0.9$, which means that $\sigma_1 = 0.9 \times 1.5 = 1.35$ and $b_1 = 0.9$.

(3) Generating the first 20 random subgroups from the BS distribution each of size 30 (i.e., $n = 30$) with the in-control process, and subsequently 10 random test subgroups each of size 30 are generated from the shifted process.

(4) Considering $a = 0.9902$ and computing the four control limits of the proposed control chart.

Table 1: The values of ARLs when $b = 0.31$, $n = 20$, and $i = 3$.

f	$k_1 = 2.66049$; $k_2 = 2.2704$; $a = 0.91459$									
	g									
	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
	ARL									
1	370.21	312.20	279.31	195.65	108.97	94.11	76.59	52.29	25.68	1.11
0.9	10.21	8.80	7.35	5.89	4.47	3.17	2.09	1.34	1.02	1.00
0.8	1.64	1.41	1.23	1.11	1.03	1.00	1.00	1.00	1.00	1.00
0.7	1.02	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 2: The values of ARLs when $b = 0.31$, $n = 30$, and $i = 3$.

f	$k_1 = 2.93654$; $k_2 = 2.37526$; $a = 1.0025$									
	g									
	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
	ARL									
1	370.45	340.86	290.14	222.18	148.65	83.91	37.74	12.19	2.50	1.00
0.9	15.62	10.22	6.35	3.75	2.15	1.32	1.03	1.00	1.00	1.00
0.8	1.80	1.38	1.14	1.02	1.00	1.00	1.00	1.00	1.00	1.00
0.7	1.02	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 3: The values of ARLs when $b = 0.31$, $n = 30$, and $i = 4$.

f	$k_1 = 3.03226639$; $k_2 = 1.6412647$; $a = 0.9131$									
	g									
	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
	ARLs									
1	370.37	379.94	373.26	337.61	263.46	161.32	69.00	18.42	3.21	1.01
0.9	5.85	4.74	3.75	2.91	2.21	1.67	1.29	1.06	1.00	1.00
0.8	1.21	1.12	1.06	1.02	1.00	1.00	1.00	1.00	1.00	1.00
0.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

(5) The plots of the charting statistic (i.e., the number of defectives in a subgroup of 30 items) for the MDSRS, RS, and MDSS methods using the BS control charts are displayed in Figures 1–3, respectively.

From Figures 1–3, the following is observed:

(1) Figure 1 shows that the BS control chart using the MDSRS method detects the shift at the 14$^{\text{th}}$ subgroup

(2) In Figure 2, the BS control chart using the RS method (see [13]) detects the shift at the 23$^{\text{rd}}$ subgroup

(3) In Figure 3, the BS control chart using the MDSS method detects the out-of-control shift at the 26$^{\text{th}}$ subgroup

Thus, we conclude that the BS chart using the MDSRS method is more powerful as compared to the existing charts.

Table 4: The values of ARLs when $b = 0.31$, $n = 20$, and $i = 4$.

f	$k_1 = 2.910$; $k_2 = 1.347$; $a = 0.9939$									
	g									
	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
	ARL									
1	371.38	347.20	309.23	256.52	190.77	120.18	59.03	20.07	4.29	1.09
0.9	21.74	17.01	12.74	9.06	6.06	3.81	2.30	1.43	1.04	1.00
0.8	1.97	1.65	1.40	1.22	1.08	1.02	1.00	1.00	1.00	1.00
0.7	1.07	1.03	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 5: The values of ARLs when $b = 1$, $n = 20$, and $i = 2$.

f	$k_1 = 2.8520574$; $k_2 = 0.6578287$; $a = 0.9752$									
	g									
	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
	ARL									
1	370.92	352.14	245.72	132.68	120.46	87.67	73.33	29.93	23.87	1.00
0.9	152.87	141.26	110.91	83.76	40.24	25.69	19.65	8.23	1.34	1.00
0.8	56.17	45.06	28.34	12.87	6.73	5.34	3.58	2.23	1.02	1.00
0.7	16.45	8.29	4.87	2.01	1.00	1.00	1.00	1.00	1.00	1.00
0.6	3.67	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.5	1.74	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.4	1.14	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.3	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 6: The values of ARLs when $b = 1$, $n = 30$, and $i = 2$.

f	$k_1 = 2.9624$; $k_2 = 1.846159$; $a = 0.7798$									
	g									
	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
	ARL									
1	370.46	346.15	314.70	214.12	115.27	40.91	12.45	2.38	1.02	1.00
0.9	173.59	116.23	62.67	36.10	12.86	3.91	1.51	1.02	1.00	1.00
0.8	36.47	15.98	12.99	5.76	2.41	1.26	1.02	1.00	1.00	1.00
0.7	8.93	5.23	2.68	1.51	1.10	1.01	1.00	1.00	1.00	1.00
0.6	2.41	1.55	1.17	1.03	1.00	1.00	1.00	1.00	1.00	1.00
0.5	1.15	1.04	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.4	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

4.2. Application of the Proposed Chart in Industry. The application of the proposed control chart is illustrated based on coupon data, see [13]. The scheming statistics to apply the proposed control chart are displayed in Table 8. For this data, it is identified to tag on the BS distribution with shape parameter $b = 0.31$ and presume that $a = 0.9939$ and $n = 20$. The parameters of the BS chart using the MDSRS method are given by $k_1 = 2.910$ and $k_2 = 1.347$ with $n = 20$ and $p_0 = 0.55$, while the control parameters for the RS method are given by $k_1 = 2.9527$ and $k_2 = 1.5404$, and for the MDSS method, the control parameters are given by $k_1 = 2.735$ and $k_2 = 2.507$. Figures 4–6 present the BS control chart using the MDSRS (proposed here), RS (by Aslam et al. [13]), and MDSS methods, respectively. From Figures 4–6, it can be seen that for the real data, the MDSRS-based control chart shows that two points are beyond the lower control limit and several

Table 7: Comparison between the BS control chart using single sampling (SS), repetitive sampling (RS), multiple dependent state sampling (MDSS), and multiple dependent state repetitive sampling (MDSRS) methods.

		$b = 0.31$, $n = 20$				$b = 1$, $n = 20$			
f	g	SS	RS	MDSS	MDSRS	SS	RS	MDSS	MDSRS
1.0	1.0	370	370	370.32	370.21	370.09	370.06	370.17	370.92
	0.9	366.77	351.96	358.54	312.20	368.01	367.51	367.62	352.14
	0.8	361.46	314.95	323.43	279.31	366.53	365.51	365.45	245.72
	0.7	353.96	257.12	289.78	195.65	352.79	333.81	345.32	132.68
	0.6	342.94	184.15	210.54	108.97	341.43	272.38	296.34	120.46
0.9	1.0	38.85	24.84	30.45	10.21	284.98	241.33	245.76	152.87
	0.9	29.83	20.72	26.43	8.80	267.63	191.90	198.56	141.26
	0.8	21.82	16.59	19.65	7.37	246.30	139.88	140.86	110.91
	0.7	15.02	12.57	14.36	5.89	220.07	92.33	110.31	83.76
	0.6	9.58	8.80	9.34	4.47	188.09	53.16	92.92	40.24
0.8	1.0	4.11	1.71	3.52	1.65	103.64	79.35	80.34	56.17
	0.9	3.06	1.41	2.76	1.21	85.66	54.05	65.89	45.06
	0.8	2.26	1.20	1.98	1.13	67.85	33.80	50.34	28.34
	0.7	1.68	1.08	1.32	1.07	50.78	18.90	30.53	12.87
	0.6	1.30	1.02	1.07	1.03	35.18	9.14	22.43	6.73
0.7	1.0	1.34	1.01	1.03	1.00	33.91	22.16	30.78	16.45
	0.9	1.18	1.0	1.02	1.00	25.83	13.68	18.56	8.29
	0.8	1.07	1.0	1.00	1.00	18.75	7.76	11.34	4.87
	0.7	1.02	1.0	1.00	1.00	12.82	4.05	7.52	2.01
	0.6	1.0	1.0	1.00	1.00	8.14	2.07	4.32	1.23
0.6	1.0	1.03	1.0	1.00	1.00	11.62	5.80	8.12	3.67
	0.9	1.0	1.0	1.00	1.00	8.51	3.54	6.87	1.00
	0.8	1.0	1.0	1.00	1.00	6.00	2.13	4.21	1.00
	0.7	1.0	1.0	1.00	1.00	4.07	1.40	2.54	1.00
	0.6	1.0	1.0	1.00	1.00	2.67	1.10	1.32	1.00

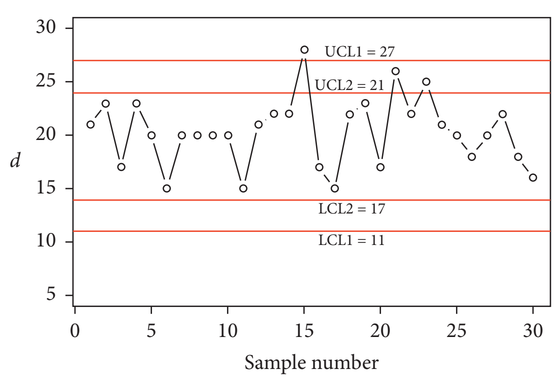

Figure 1: The Birnbaum–Saunders chart based on the MDSRS method for simulated data.

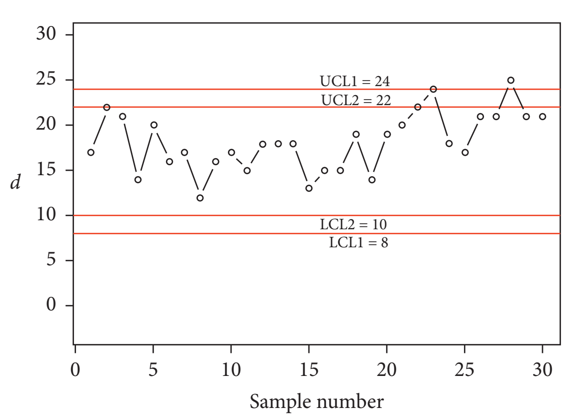

Figure 2: The Birnbaum–Saunders chart based on the RS method for simulated data.

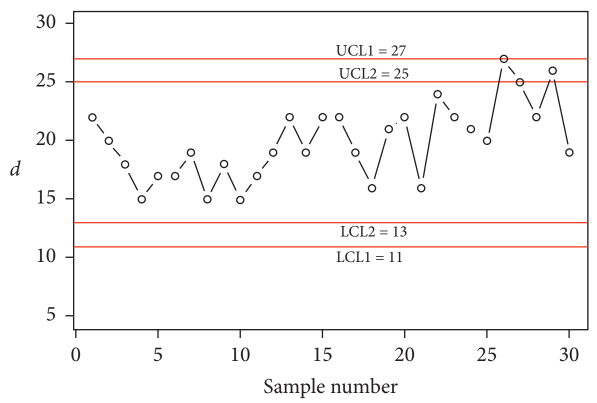

Figure 3: The Birnbaum–Saunders chart based on the MDSS method for simulated data.

Table 8: The plotting statistic for the coupon data.

Subgroup no.	d	Subgroup no.	d
1	9	16	9
2	9	17	11
3	8	18	7
4	9	19	10
5	8	20	11
6	6	21	9
7	12	22	11
8	10	23	4
9	10	24	12
10	10	25	10
11	8	26	14
12	8	27	7
13	6	28	7
14	10	29	10
15	8	30	3

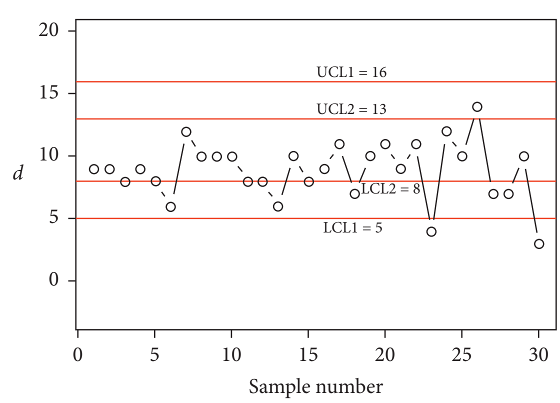

Figure 4: The Birnbaum–Saunders chart based on the MDSRS method for the coupon data.

points are within the repetitive areas. These points clearly indicate some issues in the coupon built-up process. From another point of view, the RS-based control chart proposed by Aslam et al., [13] indicates no point plotted beyond the control limits. The MDSS-based control chart shows that one point lies outside the lower control limit. From this comparison, it is important to note that the proposed control

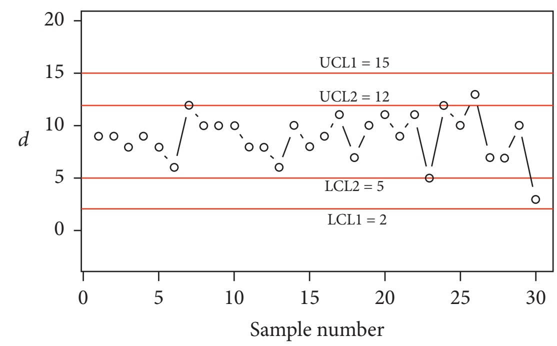

Figure 5: The Birnbaum–Saunders chart based on the RS method for the coupon data.

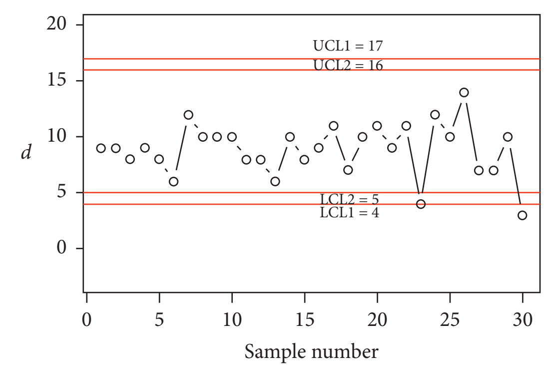

Figure 6: The Birnbaum–Saunders chart using the MDSS method for the coupon data.

chart is reasonably suitable to employ for supervising of the coupon production.

5. Concluding Remarks

This paper proposed a new control chart based on the BS distribution using the MDSRS method. The comparison of the proposed control chart with three existing control charts revealed its superiority. The comparative study and application using real data showed that the proposed chart is efficient in reducing the ARLs significantly as compared to the competing control charts considered in this study. Based on the findings, the proposed chart is recommended when industrial engineers or any other operators are interested to monitor the number of nonconforming items in the industry. The economic design of the proposed control chart for univariate and multivariate processes can be considered in the future.

Appendix

A. *R* Code

nl = 30; $b = 0.31$; ssize = 100; $a = 0.9952$; $f = 1$; $g = 1$; im = 3

$x = (a * (1 + b^{2/2}))/f$

fx = sqrt(x)-sqrt($1/x$)

$Pl0$ = pnorm((1/(g * b)) * fx); $Pl0$

```
kl1 = 3.095; kl1
kl2 = 0.8388; kl2
LCLl1 = as.integer(nl * Pl0-kl1 * sqrt(nl * Pl0 * (1-Pl0)));
LCLl1
LCLl2 = as.integer(nl * Pl0-kl2 * sqrt(nl * Pl0 * (1-Pl0)));
LCLl2
UCLl1 = as.integer(nl * Pl0+kl1 * sqrt(nl * Pl0 * (1-Pl0)));
UCLl1
UCLl2 = as. integer(nl * Pl0+kl2 * sqrt(nl * Pl0 * (1-Pl0)));
UCLl2
wLCLl2 < -which(LCLl2 < -0)
LCLl2[wLCLl2] = 0; #print(cbind(LCLl1, UCLl1))
l < -length(LCLl2); Poutl1 < -c()
for(i in 1 : l){
  if(LCLl2[i] = = 0){Poutl1[i] < -sum(dbinom(LCLl2
[i]:UCLl2[i], nl, Pl0, FALSE))}
  else {Poutl1[i] < -sum(dbinom((LCLl2[i] + 1):UCLl2
[i], nl, Pl0, FALSE))}
}
Poutl1
wUCLl2 < -which(UCLl2 < 0)
UCLl2[wUCLl2] = 0; #print(cbind(UCLl2, UCLl1))
l < -length(UCLl2); Poutl2 < -c()
for(i in 1 : l){
  if(UCLl2[i] = = 0){Poutl2[i] < -sum(dbinom(UCLl2
[i]:UCLl1[i], nl, Pl0, FALSE))}
  else {Poutl2[i] < -sum(dbinom((UCLl2[i] + 1):
UCLl1[i], nl, Pl0, FALSE))}
}
Poutl2
wLCLl1 < -which(LCLl1 < 0)
LCLl1[wLCLl1] = 0; #print(cbind(LCLl1, LCLl2))
l < -length(LCLl1); Poutl3 < -c()
for(i in 1 : l){
if(LCLl1[i] = = 0){Poutl3[i] < -sum(dbinom(LCLl1[i]:
LCLl2[i], nl, Pl0, FALSE))}
else {Poutl3[i] < -sum(dbinom((LCLl1[i] + 1):LCLl2[i],
nl, Pl0, FALSE))}
}
Poutl3; Pout < -c()
Pout < -Poutl1 + (Poutl2 + Poutl3) * Poutl1^im
Prep < -(Poutl2 + Poutl3) * (1-Poutl1^im); Prep
Pin < -Pout/(1-Prep); Pin
ARL0<-1/(1-Pin); ARL0
```

Data Availability

The data are given in the paper.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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References

[1] D. C. Montgomery, *Statistical Quality Control: A Modern Introduction*, John Wiley & Sons, Hoboken, NJ, USA, 7th edition, 2013.

[2] M. Aslam, A. Saghir, and L. Ahmad, *Introduction to Statistical Process Control*, John Wiley & Sons, Hoboken, NJ, USA, 1st edition, 2020.

[3] S. Hussain, L. Song, S. Ahmad, and M. Riaz, "On auxiliary information based improved EWMA median control charts," *Scientia Iranica*, vol. 25, no. 2, pp. 954–982, 2018.

[4] W. Zhou, C. Cheng, and Z. Zheng, "Optimal design of an attribute control chart for monitoring the mean of autocorrelated processes," *Computers & Industrial Engineering*, vol. 137, Article ID 106081, 2019.

[5] R. Quinino, F. R. B. Cruz, and L. L. Ho, "Attribute inspection control charts for the joint monitoring of mean and variance," *Computers & Industrial Engineering*, vol. 139, Article ID 106131, 2020.

[6] E. K. Epprecht, A. F. B. Costa, and F. C. T. Mendes, "Adaptive control charts for attributes," *IIE Transactions*, vol. 35, no. 6, pp. 567–582, 2003.

[7] L.-F. Hsu, "Note on 'Design of double- and triple-sampling *X*-bar control charts using genetic algorithms'," *International Journal of Production Research*, vol. 42, no. 5, pp. 1043–1047, 2004.

[8] Z. Wu and Q. Wang, "An NP control chart using double inspections," *Journal of Applied Statistics*, vol. 34, no. 7, pp. 843–855, 2007.

[9] A. A. De Araújo Rodrigues, E. K. Epprecht, and M. S. De Magalhães, "Double-sampling control charts for attributes," *Journal of Applied Statistics*, vol. 38, no. 1, pp. 87–112, 2011.

[10] S. Joekes and E. Pimentel Barbosa, "An improved attribute control chart for monitoring non-conforming proportion in high quality processes," *Control Engineering Practice*, vol. 21, no. 4, pp. 407–412, 2013.

[11] P. Jeyadurga and S. Balamurali, "Evaluation of an np control chart under truncated life test using quick switching sampling system," *Transactions of the Institute of Measurement and Control*, vol. 40, no. 12, pp. 3407–3414, 2018.

[12] O. A. Adeoti, "On control chart for monitoring exponentially distributed quality characteristic," *Transactions of the Institute of Measurement and Control*, vol. 42, no. 2, pp. 295–305, 2020.

[13] M. Aslam, O. H. Arif, and C.-H. Jun, "An attribute control chart based on the Birnbaum-Saunders distribution using repetitive sampling," *IEEE Access*, vol. 5, pp. 9350–9360, 2016.

[14] M. Aslam and C.-H. Jun, "Attribute control charts for the Weibull distribution under truncated life tests," *Quality Engineering*, vol. 27, no. 3, pp. 283–288, 2015.

[15] M. Aslam, N. Khan, and C.-H. Jun, “A control chart for time truncated life tests using Pareto distribution of second kind,” *Journal of Statistical Computation and Simulation*, vol. 86, no. 11, pp. 2113–2122, 2016.

[16] Z. W. Birnbaum and S. C. Saunders, “A new family of life distributions,” *Journal of Applied Probability*, vol. 6, no. 2, pp. 319–327, 1969.

[17] Y. L. Lio and C. Park, “A bootstrap control chart for Birnbaum-Saunders percentiles,” *Quality and Reliability Engineering International*, vol. 24, no. 5, pp. 585–600, 2008.

[18] H. Saulo, V. Leiva, and F. Ruggeri, “Monitoring environmental risk by a methodology based oncontrol charts,” in *Theory and Practice of Risk Assessment*, C. P. Kitsos, T. A. Oliveira, A. Rigas, and S. Gulati, Eds., pp. 177–197, Springer, Berlin, Germany, 2013.

[19] C. Marchant, V. Leiva, F. J. A. Cysneiros, and S. Liu, “Robust multivariate control charts based on Birnbaum-Saunders distributions,” *Journal of Statistical Computation and Simulation*, vol. 88, no. 1, pp. 182–202, 2018.

[20] M. Bourguignon, L. Lee Ho, and F. H. Fernandes, “Control charts for monitoring the median parameter of Birnbaum-Saunders distribution,” *Quality and Reliability Engineering International*, vol. 36, no. 4, pp. 1333–1363, 2020.

[21] M. Aslam, C.-H. Jun, and M. Ahmad, “New acceptance sampling plans based on life tests for Birnbaum-Saunders distributions,” *Journal of Statistical Computation and Simulation*, vol. 81, no. 4, pp. 461–470, 2011.

[22] V. Leiva, M. Barros, G. A. Paula, and A. Sanhueza, “Generalized Birnbaum-Saunders distributions applied to air pollutant concentration,” *Environmetrics*, vol. 19, no. 3, pp. 235–249, 2008.

[23] V. Leiva, M. G. Ponce, C. Marchant, and O. Bustos, “Fatigue statistical distributions useful for modeling diameter and mortality of trees,” *Revista Colombiana de Estadística*, vol. 35, no. 3, pp. 349–367, 2012.

[24] U. Shaheen, M. Azam, and M. Aslam, “A control chart for monitoring the lognormal process variation using repetitive sampling,” *Quality and Reliability Engineering International*, vol. 36, no. 3, pp. 1028–1047, 2020.

[25] M. Aslam, O.-H. Arif, and C.-H. Jun, “A control chart for gamma distribution using multiple dependent state sampling,” *Industrial Engineering and Management Systems*, vol. 16, no. 1, pp. 109–117, 2017.

[26] P. Jeyadurga, S. Balamurali, and M. Aslam, “Design of an attribute np control chart for process monitoring based on repetitive group sampling under truncated life tests,” *Communications in Statistics-Theory and Methods*, vol. 47, no. 24, pp. 5934–5955, 2017.

[27] M. Riaz, M. Abid, H. Z. Nazir, and S. A. Abbasi, “An enhanced nonparametric EWMA sign control chart using sequential mechanism,” *PLoS One*, vol. 14, no. 11, Article ID e0225330, 2019.

[28] M. S. Aldosari, M. Aslam, and C.-H. Jun, “A new attribute control chart using multiple dependent state repetitive sampling,” *IEEE Access*, vol. 5, pp. 6192–6197, 2017.

[29] M. S. Aldosari, M. Aslam, G. S. Rao, and C.-H. Jun, “An attribute control chart for multivariate Poisson distribution using multiple dependent state repetitive sampling,” *Quality and Reliability Engineering International*, vol. 35, no. 2, pp. 627–643, 2019.

[30] S. M. Krishnan and O. S. Deepa, “Control charts for multiple dependent state repetitive sampling plan using fuzzy Poisson distribution,” *International Journal of Civil Engineering and Technology*, vol. 10, no. 1, pp. 509–519, 2019.