

Research Article

Uniform Energy Decay Rates for the Fuzzy Viscoelastic Model with a Nonlinear Source

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This paper considers the fuzzy viscoelastic model with a nonlinear source $u_{tt} + Lu + \int_0^t g(t - \zeta)\Delta u(\zeta)d\zeta - |u|^\gamma u - \eta\Delta u_t = 0$ in a bounded field Ω . Under weak assumptions of the function $g(t)$, with the aid of Mathematica software, the computational technique is used to construct the auxiliary functionals and precise priori estimates. As time goes to infinity, we prove that the solution is global and energy decays to zero in two different ways: the exponential form and the polynomial form.

1. Introduction

In this paper, we take the following fuzzy viscoelastic model into account:

$$\begin{cases} u_{tt} + Lu + \int_0^t g(t - \zeta)\Delta u(\zeta)d\zeta - |u|^\gamma u - \eta\Delta u_t = 0, & x \in \Omega, t \in (0, \infty), \\ u(x, t) = 0, & (x, t) \in \Gamma \times (0, \infty), \\ u(x, t)|_{t=0} = u_0(x), u_t(x, t)|_{t=0} = u_1(x), & x \in \Omega, \end{cases} \quad (1)$$

where the fuzzy number $\eta \in (1, 1000)$, $\gamma > 0$:

$$Lu(t) = -\Delta u(t) + c^2 u(t). \quad (2)$$

In R^N ($N \geq 1$), Ω is a domain which is well bounded. Besides, the boundary of Ω is smooth perfectly and expressed as $\Gamma: = \partial\Omega$. Meanwhile, the memory kernel $g(t)$ is positive and some assumptions will be given in detail.

This type of problems has been observed in many areas of scientific and engineering fields. For example, time analitycity for the viscoelastic equation is studied as follows [1]:

$$u_{tt} - \Delta u + \int_0^t g(t - \zeta)\Delta u(\zeta)d\zeta = |u|^\gamma u. \quad (3)$$

S. Berrimi and S. A. Messaoudi applied weak conditions on the memory kernel g . Meanwhile, considering the

condition that the energy is positive and relatively small, they obtained the existence of global solutions.

Taking

$$u_{tt} - \Delta u + \int_0^t g(t - \zeta)\Delta u(\zeta)d\zeta + a(x)u_t + |u|^\gamma u = 0, \quad (4)$$

into account, they also obtained a decay rate exponentially in [2]. M.M. Cavalcanti and H.P. Oquendo improved this latter result in [3]. In their work, two situations, the internal dissipation and the viscoelastic dissipation, are considered to act on respective part separately. The authors in [3] expanded the internal dissipation to nonlinear cases as much as possible. Simultaneously, the system is well stabilized through the dissipation which is induced by the integral term.

As we know, the viscoelastic terms attract many mathematicians; for instance, the authors in [4] studied the energy decay rate for the global solution of a quasilinear viscoelastic model, and the authors in [5–9] considered time analysis of solutions for some viscoelastic models. Moreover, F. Y. Zhang et al. considered a nonlinear viscoelastic equation in [10]:

$$u_{tt} + Lu + \int_0^t g(t - \zeta)\Delta u(\zeta)d\zeta = 0. \quad (5)$$

The solution is perfectly stabilized through the dissipation, which is induced by the viscoelastic term. The modified energy functional in [10] has been used to prove the energy decay through two different ways: the exponential form and the polynomial form. Additionally, the construction of auxiliary functions is organized by the computational technique of undetermined coefficients. Besides, the authors in [11, 12] discussed the adaptive fuzzy control of nonlinear systems, and the discussion of fuzzy coefficients involved in these papers is quite interesting.

Inspired by these works, we consider (1) in this paper, the two optimal decay rates, exponential decay and polynomial decay, are easily and directly established through the application of Mathematica software. The specific arrangement of this work is as follows: in Section 2, we present some notations and necessary materials; in Section 3, in view of the fuzzy number η , we give the whole decay result, and our choice of the ‘‘Lyapunov’’ functional shows the extensive applicability and practical significance of the computational technique.

2. Preliminaries

In this section, $L^p(\Omega)$ and $H_0^1(\Omega)$ are understood and applied in their usual senses. We impose the following hypotheses and preliminaries on the memory kernel $g(t)$. In addition, the definition of energy function plays a significant role to our main result:

H_1 : as a bounded C^1 -function, $g(t): R_+ \rightarrow R_+$ satisfies

$$1 - \int_0^\infty g(\zeta)d\zeta = \ell, \tag{6}$$

where both $g(0)$ and ℓ are positive.

H_2 : the existence of a positive constant ξ makes the following formula hold:

$$g'(t) \leq -\xi \cdot g^p(t), \quad t \geq 0, p \in \left[1, \frac{3}{2}\right). \tag{7}$$

Remark 1. From the assumption above, if $p = 1$, we have

$$g(t) \leq ce^{-\xi t}. \tag{8}$$

If $1 < p < (3/2)$, we have

$$g(t) \leq \frac{1}{[g^{1-p}(0) + (p-1)\xi t]^{(1/p-1)}} \triangleq \frac{1}{[d_1 t + d_2]^{(1/p-1)}}, \tag{9}$$

where $d_1 = (p-1)\xi > 0$ and $d_2 = g^{1-p}(0)$.

Indeed, the condition $p < (3/2)$ plays an important role to ensure that

$$\int_0^\infty g^{2-p}(\zeta)d\zeta < \infty. \tag{10}$$

Theorem 1. Assuming that $(u_0, u_1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$, then $u(t)$ can be found as a unique solution to model (1) with

$$\begin{aligned} u &\in L_{loc}^\infty(0, \infty; H_0^1(\Omega) \cap H^2(\Omega)), \\ u' &\in L_{loc}^\infty(0, \infty; H_0^1(\Omega)), u'' \in L_{loc}^\infty(0, \infty; L^2(\Omega)). \end{aligned} \tag{11}$$

Simultaneously, we get

$$u \in C([0, \infty); H_0^1(\Omega)), u' \in C^1([0, \infty); L^2(\Omega)). \tag{12}$$

Taking the initial data, less regularity, and the priori estimate into consideration, the theorem above guarantees the existence of solutions for model (1) as in [7, 13]. In addition, the Galerkin approximation method can be applied to accomplish the proof of the theorem above.

Our primary assignment is to find out the energy function $\varepsilon(t)$. Combining the multiplier method, the integral subsection integration, and (H_1, H_2) , the calculation is provided:

$$\begin{aligned} 0 &= \int_\Omega \left(u_{tt} - \Delta u + c^2 u + \int_0^t g(t-\zeta)\Delta u(\zeta)d\zeta - |u|^\gamma u - \eta \Delta u_t \right) u_t dx \\ &= \frac{1}{2} \frac{d}{dt} \left(\int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^2 dx + c^2 \int_\Omega |u|^2 dx \right) + \int_0^t g(t-\zeta) \int_\Omega \Delta u(\zeta) u_t dx d\zeta \\ &\quad - \int_\Omega |u|^\gamma u u_t dx + \eta \int_\Omega |\nabla u_t|^2 dx = \frac{1}{2} \frac{d}{dt} \left(\int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^2 dx + c^2 \int_\Omega |u|^2 dx - \frac{1}{1+(\gamma/2)} \int_\Omega |u|^{\gamma+2} dx \right) \\ &\quad - \int_0^t g(t-\zeta) \int_\Omega \nabla u(\zeta) \nabla u_t dx d\zeta + \eta \int_\Omega |\nabla u_t|^2 dx, \end{aligned} \tag{13}$$

$$\begin{aligned}
 & - \int_0^t g(t-\zeta) \int_{\Omega} \nabla u(\zeta) \nabla u_t dx d\zeta \\
 & = - \int_{\Omega} \nabla u_t \int_0^t g(t-\zeta) \nabla u(\zeta) d\zeta dx \\
 & = - \int_{\Omega} \nabla u_t \int_0^t g(t-\zeta) (\nabla u(\zeta) - \nabla u(t)) d\zeta dx - \int_{\Omega} \nabla u_t \nabla u(t) \int_0^t g(t-\zeta) d\zeta dx \\
 & = \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} \int_0^t g(t-\zeta) |\nabla u(\zeta) - \nabla u(t)|^2 d\zeta dx \right) \\
 & \quad - \frac{1}{2} \int_{\Omega} \int_0^t g'(t-\zeta) |\nabla u(\zeta) - \nabla u(t)|^2 d\zeta dx - \int_{\Omega} \nabla u_t \nabla u(t) \int_0^t g(t-\zeta) d\zeta dx \\
 & = \frac{1}{2} \frac{d}{dt} \left(g \circ \nabla u - \int_0^t g(\zeta) d\zeta \int_{\Omega} |\nabla u(t)|^2 dx \right) - \frac{1}{2} (g' \circ \nabla u) + \frac{g(t)}{2} \int_{\Omega} |\nabla u(t)|^2 dx.
 \end{aligned} \tag{14}$$

Now, (13) yields

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} |u_t|^2 dx + \left(1 - \int_0^t g(\zeta) d\zeta \right) \int_{\Omega} |\nabla u(t)|^2 dx \right. \\
 & \quad \left. + c^2 \int_{\Omega} |u|^2 dx + g \circ \nabla u - \frac{\int_{\Omega} |u(t)|^{\gamma+2} dx}{1 + (\gamma/2)} \right) \\
 & - \frac{1}{2} (g' \circ \nabla u) + \frac{g(t)}{2} \int_{\Omega} |\nabla u|^2 dx + \eta \int_{\Omega} |\nabla u_t|^2 dx = 0.
 \end{aligned} \tag{15}$$

Now, it is obvious that the energy of model (1) can be given directly:

$$\begin{aligned}
 \varepsilon(t) & = \frac{1}{2} \int_{\Omega} |u_t|^2 dx + \frac{1}{2} \left(1 - \int_0^t g(\zeta) d\zeta \right) \int_{\Omega} |\nabla u(t)|^2 dx \\
 & \quad + \frac{c^2}{2} \int_{\Omega} |u|^2 dx + \frac{1}{2} g \circ \nabla u - \frac{\int_{\Omega} |u(t)|^{\gamma+2} dx}{\gamma + 2},
 \end{aligned} \tag{16}$$

where

$$g \circ f = \int_{\Omega} \int_0^t g(t-\zeta) |f(t) - f(\zeta)|^2 d\zeta dx. \tag{17}$$

Thereby, we can find

$$\begin{aligned}
 \varepsilon'(t) & = \frac{d}{dt} \varepsilon(t) \\
 & = \frac{1}{2} \left((g' \circ \nabla u) - \int_{\Omega} |\nabla u|^2 dx \right) - \eta \int_{\Omega} |\nabla u_t|^2 dx \\
 & \leq \frac{1}{2} \left((g' \circ \nabla u) - \int_{\Omega} |\nabla u|^2 dx \right) \\
 & \leq \frac{1}{2} \left(-\xi (g^p \circ \nabla u) - g(t) \int_{\Omega} |\nabla u|^2 dx \right) \\
 & \leq 0,
 \end{aligned} \tag{18}$$

which means $\varepsilon(t)$ is decreasing.

3. Main Results

In this chapter, the main result of this work is put forward directly and clearly. We introduce the following auxiliary functionals firstly:

$$\begin{aligned}
 \varphi(t) & = \int_{\Omega} u_t u dx + \int_{\Omega} |\nabla u|^2 dx, \\
 \chi(t) & = \int_{\Omega} (u + \alpha \Delta u - \beta u_t) \int_0^t g(t-\zeta) (u(t) - u(\zeta)) d\zeta dx,
 \end{aligned} \tag{19}$$

where the coefficients α and β will be determined later.

Lemma 1. *The existence of a positive constant C makes the following conclusion:*

$$|\varphi(t)| \leq C\varepsilon(t), \quad \forall t \geq 0. \tag{20}$$

Proof. Recalling the inequalities of Cauchy and Poincaré, the following estimates can be arrived:

$$\begin{aligned}
 |\varphi(t)| & \leq \frac{\int_{\Omega} |u_t|^2 dx + \int_{\Omega} |u|^2 dx}{2} + \int_{\Omega} |\nabla u|^2 dx \\
 & \leq \frac{C_p + 1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} |u_t|^2 dx,
 \end{aligned} \tag{21}$$

where the symbol $C_p > 0$ satisfies $\|u\|^2 \leq C_p \|\nabla u\|^2$ for all $u \in H_0^1(\Omega)$.

Here, it is necessary to indicate that the positive constants C and C_p later in this article represent different meanings in different places.

From the representation of $\varepsilon(t)$, it is naturally for us to get

$$|\varphi(t)| \leq C\varepsilon(t), \quad \forall t \geq 0. \tag{22}$$

Thus, (20) follows. \square

Lemma 2. *For any $t \geq 0$, the following estimate holds:*

$$\begin{aligned} \varphi'(t) \leq & \int_{\Omega} |u_t|^2 dx + \left(\frac{1}{4\varepsilon_1} + \varepsilon_1 \left(1 + \frac{1}{2\varepsilon_2} \right) (1 - \ell)^2 \right. \\ & + \frac{C_p^{2\gamma+2}}{2} \left(\frac{(2\gamma + 4)E(0)}{\gamma\ell} \right)^\gamma - \eta \left. \right) \int_{\Omega} |\nabla u|^2 dx \\ & + \left(\frac{1}{2} - c^2 \right) \int_{\Omega} |u|^2 dx + \varepsilon_1 (1 \\ & + 2\varepsilon_2) \left(\int_0^t g^{2-p}(\zeta) d\zeta \right) g^p \circ \nabla u + \frac{2-\eta}{2} \int_{\Omega} |\nabla u_t|^2 dx, \end{aligned} \tag{23}$$

where ε_1 and ε_2 are selected appropriately.

Proof. Deriving function $\varphi(t)$ with respect to time, the following calculation will be given clearly:

$$\begin{aligned} \frac{d}{dt} \varphi(t) &= \int_{\Omega} |u_t|^2 dx + \int_{\Omega} u_{tt} u dx + 2 \int_{\Omega} \nabla u \nabla u_t dx \\ &= \int_{\Omega} |u_t|^2 dx - \int_{\Omega} |\nabla u|^2 dx - c^2 \int_{\Omega} |u|^2 dx \\ &+ \int_{\Omega} \int_0^t g(t - \zeta) \nabla u(t) \nabla u(\zeta) d\zeta dx \\ &+ \int_{\Omega} |u|^\gamma u^2 dx + (2 - \eta) \int_{\Omega} \nabla u \nabla u_t dx. \end{aligned} \tag{24}$$

Considering the fourth item of (23), we adopt the following estimate from [10] without proof:

$$\begin{aligned} \int_{\Omega} \int_0^t g(t - \zeta) \nabla u(t) \nabla u(\zeta) d\zeta dx \\ \leq \left(\frac{1}{4\varepsilon_1} + \varepsilon_1 \left(1 + \frac{1}{2\varepsilon_2} \right) (1 - \ell)^2 \right) \int_{\Omega} |\nabla u|^2 dx + \varepsilon_1 (1 + 2\varepsilon_2) \left(\int_0^t g^{2-p}(\zeta) d\zeta \right) g^p \circ \nabla u. \end{aligned} \tag{25}$$

Taking the sixth item of (23) into account, we adopt the following estimate from [1] as follows:

$$\begin{aligned} \int_{\Omega} |u|^\gamma u^2 dx &= \int_{\Omega} |u|^{\gamma+1} u dx \\ &\leq \frac{1}{2} \int_{\Omega} |u|^{2(\gamma+1)} dx + \frac{1}{2} \int_{\Omega} |u|^2 dx \\ &\leq \frac{C_p^{2\gamma+2}}{2} \left(\frac{(2\gamma + 4)E(0)}{\gamma\ell} \right)^\gamma \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} |u|^2 dx. \end{aligned} \tag{26}$$

For the last item of (23), we have

$$(2 - \eta) \int_{\Omega} \nabla u \nabla u_t dx \leq \left(1 - \frac{\eta}{2} \right) \left(\int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla u_t|^2 dx \right). \tag{27}$$

Thus, (21) follows. \square

Lemma 3. For $t \geq 0$, the following estimate holds:

$$\begin{aligned} \chi'(t) \leq & (\beta c^2 \varepsilon_6 + \varepsilon_9 + (1 - \ell) \varepsilon_{12}) \int_{\Omega} |u|^2 dx + \left(\varepsilon_3 + \beta \varepsilon_{11} + \left(\frac{1}{4\varepsilon_{12}} - \beta \right) (1 - \ell) \right) \int_{\Omega} |u_t|^2 dx \\ & + \left(\frac{C_p}{4\varepsilon_3} + \frac{\beta\eta - \alpha}{4\varepsilon_4} + \frac{c^2\beta}{4\varepsilon_6} + \beta \left(\varepsilon_7 + \frac{1}{4\varepsilon_7} \right) + \frac{\beta}{4\varepsilon_8} \right) \left(\int_0^t g^{2-p}(\zeta) d\zeta \right) g^p \circ \nabla u \\ & + g(0) \left(\frac{C_p}{4\varepsilon_9} + \frac{\alpha}{4\varepsilon_{10}} + \frac{\beta C_p}{4\varepsilon_{11}} \right) (-g' \circ \nabla u) + \left(\beta \varepsilon_5 + \beta \varepsilon_7 (1 - \ell)^2 + \beta \varepsilon_8 C_p^{2\gamma+2} \left(\frac{(2\gamma + 4)E(0)}{\gamma\ell} \right)^\gamma + \alpha \varepsilon_{10} + \alpha \varepsilon_{13} (1 - \ell) \right) \int_{\Omega} |\nabla u|^2 dx \\ & + \left(\varepsilon_4 (\beta\eta - \alpha) + \frac{\alpha}{4\varepsilon_{13}} \right) \int_{\Omega} |\nabla u_t|^2 dx, \end{aligned} \tag{28}$$

where $\varepsilon_i > 0$, ($5 \leq i \leq 13$), are selected appropriately.

Proof. Deriving function $\chi(t)$ with respect to time, the expression will be given distinctly:

$$\begin{aligned}
 & \frac{d}{dt}\chi(t) \\
 &= \int_{\Omega} (u_t + \alpha\Delta u_t - \beta u_{tt}) \int_0^t g(t-\zeta)(u(t) - u(\zeta))d\zeta dx \\
 &+ \int_{\Omega} (u + \alpha\Delta u - \beta u_t) \int_0^t g'(t-\zeta)(u(t) - u(\zeta))d\zeta dx + \int_{\Omega} (u + \alpha\Delta u - \beta u_t) \int_0^t g(t-\zeta)u_t d\zeta dx \\
 &= \int_{\Omega} \left(u_t + \alpha\Delta u_t - \beta \left(-\Delta u + c^2 u + \int_0^t g(t-\zeta)\Delta u(\zeta)d\zeta - |u|^\gamma u - \eta\Delta u_t \right) \right) \left(\int_0^t g(t-\zeta)(u(t) - u(\zeta))d\zeta \right) dx \\
 &+ \int_{\Omega} (u + \alpha\Delta u - \beta u_t) \int_0^t g'(t-\zeta)(u(t) - u(\zeta))d\zeta dx + \left(\int_0^t g(\zeta)d\zeta \right) \int_{\Omega} (u + \alpha\Delta u - \beta u_t)u_t dx \\
 &= \int_{\Omega} \left(u_t + (\alpha - \beta\eta)\Delta u_t - \beta\Delta u + \beta c^2 u + \beta \int_0^t g(t-\zeta)\Delta u(\zeta)d\zeta - \beta|u|^\gamma u \right) \left(\int_0^t g(t-\zeta)(u(t) - u(\zeta))d\zeta \right) dx \\
 &+ \int_{\Omega} (u + \alpha\Delta u - \beta u_t) \int_0^t g'(t-\zeta)(u(t) - u(\zeta))d\zeta dx + \left(\int_0^t g(\zeta)d\zeta \right) \int_{\Omega} (u + \alpha\Delta u - \beta u_t)u_t dx \\
 &= \int_{\Omega} u_t \left(\int_0^t g(t-\zeta)(u(t) - u(\zeta))d\zeta \right) dx \\
 &+ \int_{\Omega} \nabla u_t(t)(\beta\eta - \alpha) \int_0^t g(t-\zeta)(\nabla u(t) - \nabla u(\zeta))d\zeta dx \\
 &+ \beta \int_{\Omega} \nabla u \int_0^t g(t-\zeta)(\nabla u(t) - \nabla u(\zeta))d\zeta dx \\
 &+ \beta c^2 \int_{\Omega} u \int_0^t g(t-\zeta)(u(t) - u(\zeta))d\zeta dx \\
 &+ \beta \left(\int_0^t g(t-\zeta)\Delta u(\zeta)d\zeta \right) \left(\int_0^t g(t-\zeta)(u(t) - u(\zeta))d\zeta \right) dx \\
 &- \beta \int_{\Omega} |u|^{1+\gamma} \int_0^t g(t-\zeta)(u(t) - u(\zeta))d\zeta dx \\
 &+ \int_{\Omega} (u + \alpha\Delta u - \beta u_t) \int_0^t g'(t-\zeta)(u(t) - u(\zeta))d\zeta dx + \left(\int_0^t g(\zeta)d\zeta \right) \int_{\Omega} (uu_t + \alpha u_t \Delta u - \beta u_t^2) dx.
 \end{aligned}
 \tag{29}$$

For any $\alpha, \beta \geq 0$, we will estimate every item of the right hand of (29):

$$\begin{aligned} & \int_{\Omega} u_t \int_0^t g(t-\zeta)(u(t)-u(\zeta))d\zeta dx \\ & \leq \varepsilon_3 \int_{\Omega} |u_t|^2 dx + \frac{1}{4\varepsilon_3} \int_{\Omega} \left| \int_0^t g(t-\zeta)(u(t)-u(\zeta))d\zeta \right|^2 dx \\ & = \varepsilon_3 \int_{\Omega} |u_t|^2 dx + \frac{1}{4\varepsilon_3} \int_{\Omega} \left| \int_0^t g^{1-(p/2)}(t-\zeta)g^{(p/2)}(t-\zeta)(u(t)-u(\zeta))d\zeta \right|^2 dx \\ & \leq \varepsilon_3 \int_{\Omega} |u_t|^2 dx + \frac{C_p}{4\varepsilon_3} \left(\int_0^t g^{2-p}(\zeta)d\zeta \right) (g^p \circ \nabla u), \end{aligned} \quad (30)$$

$$\begin{aligned} & (\beta\eta - \alpha) \int_{\Omega} \nabla u_t(t) \int_0^t g(t-\zeta)(\nabla u(t) - \nabla u(\zeta))d\zeta dx \\ & \leq (\beta\eta - \alpha) \left(\varepsilon_4 \int_{\Omega} |\nabla u_t|^2 dx + \frac{1}{4\varepsilon_4} \int_{\Omega} \left(\int_0^t g(t-\zeta)|\nabla u(t) - \nabla u(\zeta)|d\zeta \right)^2 dx \right) \\ & \leq (\beta\eta - \alpha)\varepsilon_4 \int_{\Omega} |\nabla u_t|^2 dx + \frac{\beta\eta - \alpha}{4\varepsilon_4} \left(\int_0^t g^{2-p}(\zeta)d\zeta \right) (g^p \circ \nabla u), \end{aligned} \quad (31)$$

$$\beta \int_{\Omega} \nabla u \int_0^t g(t-\zeta)(\nabla u(t) - \nabla u(\zeta))d\zeta dx \leq \beta\varepsilon_5 \int_{\Omega} |\nabla u|^2 dx + \frac{\beta}{4\varepsilon_5} \left(\int_0^t g^{2-p}(\zeta)d\zeta \right) (g^p \circ \nabla u). \quad (32)$$

Similar to the first term, considering the third item of (29), similar estimate will be given:

$$\begin{aligned} & \beta c^2 \int_{\Omega} u \int_0^t g(t-\zeta)(\nabla u(t) - \nabla u(\zeta))d\zeta dx \\ & \leq \beta c^2 \varepsilon_6 \int_{\Omega} |u|^2 dx + \frac{\beta c^2}{4\varepsilon_6} \left(\int_0^t g^{2-p}(\zeta)d\zeta \right) (g^p \circ \nabla u) \end{aligned} \quad (33)$$

Applying Young's inequality properly, the following estimates can be obtained from [10]:

$$\begin{aligned} & \beta \left(\int_0^t g(t-\zeta)\Delta u(\zeta)d\zeta \right) \left(\int_0^t g(t-\zeta)(u(t)-u(\zeta))d\zeta \right) dx \\ & = \beta \left(\int_0^t g(t-\zeta)\nabla u(\zeta)d\zeta \right) \left(\int_0^t g(t-\zeta)(\nabla u(\zeta) - \nabla u(t))d\zeta \right) dx \\ & \leq \beta(1-\ell)^2 \varepsilon_7 \int_{\Omega} |\nabla u|^2 dx + \beta \left(\varepsilon_7 + \frac{1}{4\varepsilon_7} \right) \left(\int_0^t g^{2-p}(\zeta)d\zeta \right) (g^p \circ \nabla u), \end{aligned} \quad (34)$$

$$\begin{aligned}
 & -\beta \int_{\Omega} |u|^{\gamma} u \int_0^t g(t-\zeta)(u(t)-u(\zeta))d\zeta dx \\
 & \leq \beta \varepsilon_8 \int_{\Omega} |u|^{2\gamma+2} dx + \frac{\beta}{4\varepsilon_8} \int_{\Omega} \left(\int_0^t g(t-\zeta)|u(t)-u(\zeta)|^2 d\zeta \right) dx \\
 & \leq \beta \varepsilon_8 C_p^{2\gamma+2} \left(\frac{(2\gamma+4)E(0)}{\gamma \ell} \right)^{\gamma} \int_{\Omega} |\nabla u|^2 dx + \frac{\beta}{4\varepsilon_8} \left(\int_0^t g^{2-p}(\zeta)d\zeta \right) (g^p \circ \nabla u),
 \end{aligned} \tag{35}$$

$$\begin{aligned}
 & \int_{\Omega} u \int_0^t g'(t-\zeta)(u(t)-u(\zeta))d\zeta dx \\
 & \leq \varepsilon_9 \int_{\Omega} |u|^2 dx + \frac{g(0)C_p}{4\varepsilon_9} \int_{\Omega} \left(\int_0^t -g'(t-\zeta)|\nabla u(t)-\nabla u(\zeta)|^2 d\zeta \right) dx \\
 & = \varepsilon_9 \int_{\Omega} |u|^2 dx + \frac{g(0)C_p}{4\varepsilon_9} (-g' \circ \nabla u),
 \end{aligned} \tag{36}$$

$$\begin{aligned}
 & \alpha \int_{\Omega} \Delta u \int_0^t g'(t-\zeta)(u(t)-u(\zeta))d\zeta dx \\
 & \leq \alpha \varepsilon_{10} \int_{\Omega} |\nabla u|^2 dx + \frac{\alpha g(0)}{4\varepsilon_{10}} \int_{\Omega} \left(\int_0^t -g'(t-\zeta)|\nabla u(t)-\nabla u(\zeta)|^2 d\zeta \right) dx \\
 & = \alpha \varepsilon_{10} \int_{\Omega} |\nabla u|^2 dx + \frac{\alpha g(0)}{4\varepsilon_{10}} (-g' \circ \nabla u).
 \end{aligned} \tag{37}$$

Similarly, we have

$$\begin{aligned}
 & -\beta \int_{\Omega} u_t \int_0^t g'(t-\zeta)(u(t)-u(\zeta))d\zeta dx \\
 & \leq \beta \varepsilon_{11} \int_{\Omega} |u_t|^2 dx + \frac{\beta g(0)C_p}{4\varepsilon_{11}} (-g' \circ \nabla u).
 \end{aligned} \tag{38}$$

Furthermore,

$$\begin{aligned}
 & \left(\int_0^t g(\zeta)d\zeta \right) \int_{\Omega} (uu_t + \alpha u_t \Delta u - \beta u_t^2) dx \\
 & \leq \left(\int_0^t g(\zeta)d\zeta \right) \left(\varepsilon_{12} \int_{\Omega} |u|^2 dx + \left(\frac{1}{4\varepsilon_{12}} - \beta \right) \int_{\Omega} |u_t|^2 dx + \alpha \varepsilon_{13} \int_{\Omega} |\nabla u|^2 dx + \frac{\alpha}{4\varepsilon_{13}} \int_{\Omega} |\nabla u_t|^2 dx \right) \\
 & = (1-\ell) \left(\varepsilon_{12} \int_{\Omega} |u|^2 dx + \left(\frac{1}{4\varepsilon_{12}} - \beta \right) \int_{\Omega} |u_t|^2 dx + \alpha \varepsilon_{13} \int_{\Omega} |\nabla u|^2 dx + \frac{\alpha}{4\varepsilon_{13}} \int_{\Omega} |\nabla u_t|^2 dx \right).
 \end{aligned} \tag{39}$$

Taking (29)–(39) and H_2 into account, for all $t \geq 0$, we get (28). \square

$$\begin{aligned}
 \varepsilon(t) & \leq \frac{K_1 \varepsilon(0)}{e^{kt}}, \quad p = 1, \\
 \varepsilon(t) & \leq \frac{K_2}{(t+1)^{(1/2)(p-1)}}, \quad p > 1.
 \end{aligned} \tag{40}$$

Theorem 2. Let every pair $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$, H_1 and H_2 hold, and $t_0 \in (0, \infty)$ For every $t \in [t_0, \infty)$, there must be some positive constants K_1, K_2 , and k , which would enable the solution of the model (1) to satisfy the following:

Proof. We start the proof by selecting an appropriate auxiliary functional

$$L(t) = M\varepsilon(t) + a\varphi(t) + b\chi(t), \quad (41)$$

where the positive constants M , a , and b will be chosen in the sequel.

Applying Lemma 1 and the hypothesis $g'(t) \leq -\xi \cdot g^p(t)$, the equivalence between $\varepsilon(t)$ and $L(t)$ is achieved. To this point, firstly, a simple calculation shows that

$$\begin{aligned} & |L(t) - M\varepsilon(t)| \\ &= |a\varphi(t) + b\chi(t)| \\ &= \left| a \int_{\Omega} u_t u dx + a \int_{\Omega} |\nabla u|^2 dx + b \int_{\Omega} (u + \alpha \Delta u - \beta u_t) \int_0^t g(t-\zeta) (u(t) - u(\zeta)) d\zeta dx \right| \\ &= \left| a \int_{\Omega} u_t u dx + a \int_{\Omega} |\nabla u|^2 dx + b \int_{\Omega} (u - \beta u_t) \int_0^t g(t-\zeta) (u(t) - u(\zeta)) d\zeta dx - b\alpha \int_{\Omega} \nabla u \int_0^t g(t-\zeta) (u(t) - u(\zeta)) d\zeta dx \right| \\ &\leq \frac{|a|}{2} \int_{\Omega} |u_t|^2 dx + \frac{|a|C_p + 2a}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{|b|}{2} \int_{\Omega} (u - \beta u_t)^2 dx + \frac{|b\alpha|}{2} \int_{\Omega} |\nabla u|^2 dx \\ &\quad + \frac{|b|}{2} \int_{\Omega} \left(\int_0^t g(t-\zeta) (u(t) - u(\zeta)) d\zeta \right)^2 dx + \frac{|b\alpha|}{2} \int_{\Omega} \left(\int_0^t g(t-\zeta) (\nabla u(t) - \nabla u(\zeta)) d\zeta \right)^2 dx \\ &\leq \frac{|a|}{2} \int_{\Omega} |u_t|^2 dx + \frac{|a|C_p + 2a + |b\alpha|}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{|b|}{2} \int_{\Omega} |u|^2 dx \\ &\quad + \frac{\beta^2 |b|}{2} \int_{\Omega} |u_t|^2 dx - \beta |b| \int_{\Omega} u_t u dx + \frac{|b|}{2} \left(\frac{1}{2} \int_{\Omega} \int_0^t g(t-\zeta) d\zeta dx + \frac{1}{2} \int_{\Omega} \int_0^t g(t-\zeta) |u(t) - u(\zeta)|^2 d\zeta dx \right) \\ &\quad + \frac{|b\alpha|}{2} \left(\frac{1}{2} \int_{\Omega} \int_0^t g(t-\zeta) d\zeta dx + \frac{1}{2} \int_{\Omega} \int_0^t g(t-\zeta) |\nabla u(t) - \nabla u(\zeta)|^2 d\zeta dx \right) \\ &\leq \frac{|a| + |b|\beta^2}{2} \int_{\Omega} |u_t|^2 dx + \frac{|a|C_p + 2a + |b\alpha|}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{|b|}{2} \int_{\Omega} |u|^2 dx + \frac{|b\alpha| + |b|C_p}{4} g \circ \nabla u \\ &\quad + \frac{|b|(1+|\alpha|)(1-\ell)}{4} \int_{\Omega} dx \\ &\leq C_1 \varepsilon(t). \end{aligned} \quad (42)$$

By (25), we get

$$\begin{aligned} & |L(t) - M\varepsilon(t)| + \frac{1}{\gamma+2} \int_{\Omega} |u|^{\gamma+2} dx \\ &\leq \frac{|a| + |b|\beta^2}{2} \int_{\Omega} |u_t|^2 dx + \frac{1}{2} \left(|a|C_p + 2a + |b\alpha| + C_p^{2\gamma+2} \left(\frac{(2\gamma+4)E(0)}{\gamma\ell} \right)^{\gamma} \right) \int_{\Omega} |\nabla u|^2 dx \\ &\quad + \frac{|b|+1}{2} \int_{\Omega} |u|^2 dx + \frac{|b\alpha| + |b|C_p}{4} g \circ \nabla u + \frac{|b|(1+|\alpha|)(1-\ell)}{4} \int_{\Omega} dx \\ &\leq C_1 \left(\varepsilon(t) + \frac{1}{\gamma+2} \int_{\Omega} |u|^{\gamma+2} dx \right). \end{aligned} \quad (43)$$

Thus, the equivalence between $L(t)$ and $\varepsilon(t)$ is complete.

Secondly, differentiating (41) with respect to t , considering the assumption H_2 and Lemmas 1 and 2, the estimate of $L'(t)$ is given naturally:

$$\begin{aligned}
 L'(t) &= M\varepsilon'(t) + a\varphi'(t) + b\chi'(t) \\
 &\leq M \left[-\frac{\xi}{2} (g^p \circ \nabla u) - \frac{g(t)}{2} \int_{\Omega} |\nabla u|^2 dx \right] + a\varphi'(t) + b\chi'(t) \\
 &\leq c_1 \int_{\Omega} |u_t|^2 dx + c_2 \int_{\Omega} |\nabla u|^2 dx + c_3 \int_{\Omega} |u|^2 dx + c_4 g^p \circ \nabla u + c_5 \int_{\Omega} |\nabla u_t|^2 dx,
 \end{aligned} \tag{44}$$

where the coefficients c_1, c_2, c_3 , and c_4 are given as follows:

$$\begin{aligned}
 c_1 &= a + b\varepsilon_3 + b\beta\varepsilon_{11} + b(1 - \ell) \left(\frac{1}{4\varepsilon_{12}} - \beta \right), \\
 c_2 &= a \left(\frac{1}{4\varepsilon_1} + \varepsilon_1 \left(1 + \frac{1}{2\varepsilon_2} \right) (1 - \ell)^2 + \frac{C_p^{2\gamma+2}}{2} \left(\frac{(2\gamma + 4)E(0)}{\gamma\ell} \right)^\gamma - \frac{\eta}{2} \right) \\
 &\quad + b \left(\beta\varepsilon_5 + \beta\varepsilon_7(1 - \ell)^2 + \beta\varepsilon_8 C_p^{2\gamma+2} \left(\frac{(2\gamma + 4)E(0)}{\gamma\ell} \right)^\gamma + \alpha\varepsilon_{10} + \alpha\varepsilon_{13}(1 - \ell) \right) - \frac{Mg(t)}{2}, \\
 c_3 &= a \left(\frac{1}{2} - c^2 \right) + b(\beta c^2 \varepsilon_6 + \varepsilon_9 + \varepsilon_{12}(1 - \ell)), \\
 c_4 &= \left(a\varepsilon_1(1 + 2\varepsilon_2) + b \left(\frac{C_p}{4\varepsilon_3} + \frac{\beta\eta - \alpha}{4\varepsilon_4} + \frac{c^2\beta}{4\varepsilon_6} + \beta \left(\varepsilon_7 + \frac{1}{4\varepsilon_7} \right) + \frac{\beta}{4\varepsilon_8} \right) \right) \left(\int_0^t g^{2-p}(\zeta) d\zeta \right) \\
 &\quad + b\xi \left(\frac{g(0)C_p}{4\varepsilon_9} + \frac{g(0)\alpha}{4\varepsilon_{10}} + \frac{\beta g(0)C_p}{4\varepsilon_{11}} \right) - \frac{M\xi}{2}, \\
 c_5 &= a \left(1 - \frac{\eta}{2} \right) + b \left(\varepsilon_4(\beta\eta - \alpha) + \frac{\alpha}{\varepsilon_{13}} \right).
 \end{aligned} \tag{45}$$

Next, it is time to carry out a discussion on the different values of p . □

Case 1. $p = 1$

In such a case, choosing $\tilde{c}_1 = c_1, \tilde{c}_2 = c_2 + (h/\gamma + 2)(C_p^{2\gamma+2}/2)((2\gamma + 4)E(0)/\gamma\ell)^\gamma, \tilde{c}_3 =$

$c_3 + (h/2)(\gamma + 2), \tilde{c}_4 = c_4, \tilde{c}_5 = c_5 = 0$ and taking the positive constant h as

$$h \leq \min \left\{ -2\tilde{c}_1, \frac{-2\tilde{c}_2}{1 - \int_0^t g(\zeta) d\zeta}, \frac{-2\tilde{c}_3}{c^2}, -2\tilde{c}_4 \right\}, \tag{46}$$

we find

$$\begin{aligned}
 L'(t) & - \frac{h}{\gamma + 2} \int_{\Omega} |u|^{\gamma+2} dx \\
 & \leq L'(t) + \frac{h}{\gamma + 2} \int_{\Omega} |u|^{\gamma+2} dx \\
 & \leq L'(t) + \frac{h}{\gamma + 2} \frac{C_p^{2\gamma+2}}{2} \left(\frac{(2\gamma + 4)E(0)}{\gamma\ell} \right)^{\gamma} \int_{\Omega} |\nabla u|^2 dx + \frac{h}{2(\gamma + 2)} \int_{\Omega} |u|^2 dx \\
 & \leq \tilde{c}_1 \int_{\Omega} |u_t|^2 dx + \tilde{c}_2 \int_{\Omega} |\nabla u|^2 dx + \tilde{c}_3 \int_{\Omega} |u|^2 dx + \tilde{c}_4 g^p \circ \nabla u + \tilde{c}_5 \int_{\Omega} |\nabla u_t|^2 dx \\
 & \leq -h \left(\frac{1}{2} \int_{\Omega} |u_t|^2 dx + \frac{1}{2} \left(1 - \int_0^t g(\zeta) d\zeta \right) \int_{\Omega} |\nabla u|^2 dx + \frac{c^2}{2} \int_{\Omega} |u|^2 dx + \frac{1}{2} g \circ \nabla u \right).
 \end{aligned} \tag{47}$$

Therefore, (45) yields

$$L'(t) \leq -h\varepsilon(t). \tag{48}$$

Remark 2. Using the computational technique proposed in [10] and selecting some appropriate numerical values by means of the calculation software Mathematica, it is obvious that $\tilde{c}_i < 0 (1 \leq i \leq 4)$.

Considering the interval of the fuzzy number η , taking

$$\begin{aligned}
 M = 10, \eta = 100, \varepsilon_i = 0.01, (1 \leq i \leq 13), \\
 \alpha = 1, \beta = 2, a = 1, b = -1, \gamma = 2,
 \end{aligned} \tag{49}$$

for example, with the aid of Mathematica, we easily verify that

$$\begin{aligned}
 \tilde{c}_1 & = -0.580816 - 23(1 - \ell), \\
 \tilde{c}_2 & = 13.4295 + 5g(t) + 0.611833\ell \\
 & - 0.300916\ell^2 - \frac{(4.72653 - 2h)E(0)^2 C_p^6}{\ell^2},
 \end{aligned} \tag{50}$$

$$\begin{aligned}
 \tilde{c}_3 & = -0.295408 + 0.530816c^2 + \frac{h}{8} + 0.01\ell, \\
 \tilde{c}_4 & = -(5100.03 + 50c^2)(1 - \ell) - 5\xi - 100\xi g(0).
 \end{aligned}$$

Taking another example into consideration,

$$\begin{aligned}
 M = 10, \eta = 10, \varepsilon_i = \frac{i}{1000}, (1 \leq i \leq 13), \\
 \alpha = 1, \beta = 2, a = 1, b = -1, \gamma = 2,
 \end{aligned} \tag{51}$$

we find that

$$\begin{aligned}
 \tilde{c}_1 & = -4.85169 - 18.8333(1 - \ell), \\
 \tilde{c}_2 & = -1183.8 + 5g(0) + 2.464\ell - 1.2255\ell^2 \\
 & - \frac{(38.8695 - 2h)E(0)^2 C_p^6}{\ell^2}, \\
 \tilde{c}_3 & = -2.43435 + 4.81469c^2 + \frac{h}{8} + 0.012\ell, \\
 \tilde{c}_4 & = -(1321.45 + 83.3333(c^2 + C_p)) (1 - \ell) - 5\xi \\
 & - 25\xi g(0) - 73.2323C_p g(0)\xi.
 \end{aligned} \tag{52}$$

Additionally, we may retrieve more effective data of \tilde{c}_i , and the data provide an intuitive understanding in the discussion of the auxiliary functionals.

Considering the equivalence between $L(t)$ and $\varepsilon(t)$, it is easy for us to see that

$$M_1\varepsilon(t) \leq L(t) \leq M_2\varepsilon(t), \tag{53}$$

where the coefficients M_1 and M_2 in (53) have many possibilities to be chosen.

The following expression can be presented through the combination of (48) and (53):

$$L'(t) \leq -\frac{h}{M_1}\varepsilon(t). \tag{54}$$

After a simple integral on the interval (t_0, t) , (54) leads to

$$L(t) \leq e^{-(h(t-t_0)/M_1)} L(t_0). \tag{55}$$

As a consequence, taking $k = -(h/M_1)$, (55) leads to

$$\varepsilon(t) \leq \frac{L(t_0)}{M_1 e^{(h/M_1)(t-t_0)}} \leq \frac{K_1\varepsilon(0)}{e^{kt}}, \tag{56}$$

where K_1 is some appropriate positive constant.

Case 2. $p > 1$.

Using H_1, H_2 , and (9), we easily verify that

$$\int_0^\infty g^{1-\theta}(\zeta)d\zeta \leq \int_0^{+\infty} \left(\frac{1}{d_1t + d_2}\right)^{(1-\theta/p-1)} d\zeta < \infty, \quad 0 \leq \theta < 2 - p. \tag{57}$$

Moreover, considering the inequality in [6], for some constant C ,

$$g \circ \nabla u \leq C \left(E(0) \int_0^\infty g^{1-\theta}(\zeta)d\zeta\right)^{(p-1/(p-1+\theta))} (g^p \circ \nabla u)^{(\theta/p-1+\theta)}. \tag{58}$$

Taking $\theta = (1/2), \gamma = 2p - 1$, we find $(\gamma\theta/p - 1 + \theta) = 1$. Therefore, for some $m > 1$, we get

$$\varepsilon^m(t) \leq C \left(\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega |u_t|^2 dx + g^p \circ \nabla u \right), \tag{59}$$

that is,

$$-\left(\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega |u_t|^2 dx + g^p \circ \nabla u \right) \leq -\frac{1}{C} \varepsilon^m(t). \tag{60}$$

Taking (48) into account, we have

$$-\varepsilon^m(t) \leq -\frac{1}{M_2^m} L^m(t). \tag{61}$$

For each $t \geq t_0$, it is straight for us to get

$$\begin{aligned} L'(t) &\leq \frac{h}{\gamma + 2} \int_\Omega |u|^{\gamma+2} dx + L'(t) \\ &\leq -\frac{h}{2} \left(\int_\Omega |u_t|^2 dx + \ell \int_\Omega |\nabla u|^2 dx + c^2 \int_\Omega |u|^2 dx + g \circ \nabla u \right) \\ &\leq -\frac{h}{C} \varepsilon^m(t) \\ &\leq -\frac{h}{CM_2^m} L^m(t) \\ &\leq -CL^m(t). \end{aligned} \tag{62}$$

As we mentioned before, the symbol C denotes different constants in different places. Executing a simple integral on the interval (t_0, t) , (62) yields

$$L(t) \leq (C_a t + C_b)^{-(1/m-1)} = (C_a t + C_b)^{-(1/2(p-1))}. \tag{63}$$

Considering the equivalence between $L(t)$ and $\varepsilon(t)$, it is apparently for us to get

$$\varepsilon(t) \leq \frac{K_2}{(t + 1)^{(1/2(p-1))}}. \tag{64}$$

This completes the proof.

4. Conclusion

Under the assumptions on the relaxation function and the interval of the fuzzy number η , applying the computational technique, a lot of auxiliary functionals can be constructed numerically. Two decay results, the exponential one and the polynomial one, are derived for the model (1) eventually. The result shows a new way for the decay rates, which is quite different from other literatures.

Data Availability

All datasets generated for this study are included in the manuscript.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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