Research Article

Trapezium-Type Inequalities for k -Fractional Integral via New Exponential-Type Convexity and Their Applications

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In this paper, the authors investigated the concept of (s, m) -exponential-type convex functions and their algebraic properties. New generalizations of Hermite–Hadamard-type inequality for the (s, m) -exponential-type convex function ψ and for the products of two (s, m) -exponential-type convex functions ψ and ϕ are proved. Many refinements of the (H–H) inequality via (s, m) -exponential-type convex are obtained. Finally, several new bounds for special means and new error estimates for the trapezoidal and midpoint formula are provided as well. The ideas and techniques of this paper may stimulate further research in different areas of pure and applied sciences.

1. Introduction

Theory of convexity also played a significant role in the development of theory of inequalities. Many famously known results in inequalities theory can be obtained using the convexity property of the functions. Hermite–Hadamard's double inequality is one of the most intensively studied results involving convex functions. This result provides us necessary and sufficient condition for a function to be convex. It is also known as classical equation of H–H inequality.

The Hermite–Hadamard inequality assert that if a function $\psi: J \subset \mathbb{R} \rightarrow \mathbb{R}$ is convex in J for $a_1, a_2 \in J$ and $a_1 < a_2$, then

$$\psi\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \psi(\chi) d\chi \leq \frac{\psi(a_1) + \psi(a_2)}{2}. \quad (1)$$

Interested readers can refer to [1–29].

Definition 1 (see [30]). A function $\psi: [0, +\infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense for a real number $s \in (0, 1]$ or ψ belongs to the class of K_s^2 , if

$$\psi(\chi a_1 + (1 - \chi)a_2) \leq \chi^s \psi(a_1) + (1 - \chi)^s \psi(a_2), \quad (2)$$

holds for all $a_1, a_2 \in [0, +\infty)$ and $\chi \in [0, 1]$.

A s -convex function was introduced in Breckner's article in [30], and a number of properties and connections with s -convexity in the first sense are discussed in [10]. Usually, convexity means for s -convexity when $s = 1$. Dragomir et al. proved a variant of Hadamard's inequality in [5], which holds for s -convex functions in the second sense. In the last decade, many mathematicians added the rich literature in the field of mathematical inequalities involving fractional calculus (see [8, 9, 13, 15, 22, 25, 28, 31]).

Toader introduced the class of m -convex functions in [26].

Definition 2 (see [26]). A function $\psi: [0, a_2] \longrightarrow \mathfrak{R}$, $a_2 > 0$, is said to be m -convex, where $m \in (0, 1]$, if

$$\psi(\chi\theta_1 + m(1-\chi)\theta_2) \leq \chi\psi(\theta_1) + m(1-\chi)\psi(\theta_2) \quad (3)$$

holds $\forall \theta_1, \theta_2 \in [0, a_2]$ and $\chi \in [0, 1]$. Otherwise, ψ is m -concave if $(-\psi)$ is m -convex.

In a recent paper, Eftekhari [6] defined the class of (s, m) -convex functions in the second sense as follows.

Definition 3. A function $\psi: [0, +\infty) \longrightarrow \mathfrak{R}$ is said to be (s, m) -convex for some fixed real numbers $s, m \in (0, 1]$ if

$$\psi(\chi a_1 + m(1-\chi)a_2) \leq \chi^s \psi(a_1) + m(1-\chi)^s \psi(a_2) \quad (4)$$

holds $\forall a_1, a_2 \in [0, +\infty)$ and $\chi \in [0, 1]$.

Fractional integral inequalities are useful to find the uniqueness of solutions for certain fractional partial differential equations (see [19, 32]).

Let $\psi \in L[a_1, a_2]$. Then, Riemann–Liouville fractional integrals of order $\alpha > 0$ with $a_1 \geq 0$ are defined as follows:

$$J_{a_1^+}^\alpha \psi(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^x (x-\chi)^{\alpha-1} \psi(\chi) d\chi, \quad x > a_1, \quad (5)$$

$$J_{a_2^-}^\alpha \psi(x) = \frac{1}{\Gamma(\alpha)} \int_x^{a_2} (\chi-x)^{\alpha-1} \psi(\chi) d\chi, \quad x < a_2.$$

For further details, one may see [14, 16, 18].

In [7, 17], there is given definition of k -fractional Riemann–Liouville integrals.

Let $\psi \in L[a_1, a_2]$. Then, k -fractional integrals of order $\alpha, k > 0$ with $a_1 \geq 0$ are defined as follows:

$${}^k J_{a_1^+}^\alpha \psi(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^x (x-\chi)^{\alpha/k-1} \psi(\chi) d\chi, \quad x > a_1,$$

$${}^k J_{a_2^-}^\alpha \psi(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{a_2} (\chi-x)^{\alpha/k-1} \psi(\chi) d\chi, \quad x < a_2, \quad (6)$$

where $\Gamma_k(\alpha)$ is the k -gamma function defined as

$$\Gamma_k(\alpha) = \int_0^{+\infty} \chi^{\alpha-1} e^{-\chi^k/k} d\chi. \quad (7)$$

We can notice that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha),$$

$${}^1 J_{a_1^+}^0 \psi(x) = {}^1 J_{a_2^-}^0 \psi(x) = \psi(x). \quad (8)$$

By choosing $k = 1$, the above k -fractional integrals yield Riemann–Liouville integrals.

Also that, the incomplete gamma function $\gamma(\vartheta, \theta)$ is defined for $\vartheta > 0$ and $\theta \geq 0$ by integral

$$\gamma(\vartheta, \theta) = \int_0^\theta e^{-\mu} \mu^{\vartheta-1} d\mu. \quad (9)$$

The gamma function $\Gamma(\vartheta)$ is defined for $\vartheta > 0$ by integral

$$\Gamma(\vartheta) = \int_0^{+\infty} e^{-\mu} \mu^{\vartheta-1} d\mu. \quad (10)$$

Motivated by above results and literatures, we will give first in Section 2 the concept of (s, m) -exponential-type convex function, and we will study some of their algebraic properties. In Section 3, we will prove new generalizations of Hermite–Hadamard-type inequality for the (s, m) -exponential-type convex function ψ and for the products of two (s, m) -exponential-type convex functions ψ and ϕ . In Section 4, we will obtain some refinements of the (H–H) inequality for functions whose first derivative in absolute value at certain power is (s, m) -exponential-type convex. In Section 5, some new bounds for special means and error estimates for the trapezoidal and midpoint formula will be provided. In Section 6, a briefly conclusion will be given as well.

2. Some Algebraic Properties of (s, m) -Exponential-Type Convex Functions

In this section, we will introduce a new definition, called (s, m) -exponential-type convex function, and we will study some basic algebraic properties of it.

Definition 4. A nonnegative function $\psi: J \longrightarrow \mathfrak{R}$ is said (s, m) -exponential-type convex for some fixed $s, m \in (0, 1]$ if

$$\psi(\chi\theta_1 + m(1-\chi)\theta_2) \leq (e^{s\chi} - 1)\psi(\theta_1) + m(e^{(1-\chi)s} - 1)\psi(\theta_2) \quad (11)$$

holds for all $\theta_1, \theta_2 \in J$ and $\chi \in [0, 1]$.

Remark 1. For $m = s = 1$, we get exponential-type convexity given by İşcan in [11].

Remark 2. (s, m) -exponential-type convex functions for some fixed $m \in (0, 1]$ and $s \in [\ln 2.5, 1]$ have the range $[0, +\infty)$.

Proof. Let $\theta \in J$ be arbitrary for some fixed $m \in (0, 1]$ and $s \in [\ln 2.5, 1]$. Using the Definition 4 for $\chi = 1$, we have

$$\psi(\theta) \leq (e^s - 1)\psi(\theta) \implies (e^s - 2)\psi(\theta) \geq 0 \implies \psi(\theta) \geq 0. \quad (12) \quad \square$$

Lemma 1. For all $\chi \in [0, 1]$ and for some fixed $m \in (0, 1]$ and $s \in [\ln 2.5, 1]$, the following inequalities $(e^{s\chi} - 1) \geq \chi^s$ and $(e^{(1-\chi)s} - 1) \geq (1-\chi)^s$ hold.

Proof. The proof is evident. $\square$

Proposition 1. Every nonnegative (s, m) -convex function is (s, m) -exponential-type convex function for some fixed $m \in (0, 1]$ and $s \in [\ln 2.5, 1]$.

Proof. By using Lemma 1, for some fixed $m \in (0, 1]$ and $s \in [\ln 2.5, 1]$, we have

$$\begin{aligned} \psi(\chi\theta_1 + m(1-\chi)\theta_2) &\leq \chi^s \psi(\theta_1) + m(1-\chi)^s \psi(\theta_2) \\ &\leq (e^{s\chi} - 1)\psi(\theta_1) + m(e^{(1-\chi)s} - 1)\psi(\theta_2). \end{aligned} \quad (13)$$

Theorem 1. Let $\psi, \phi: [a_1, a_2] \rightarrow \mathfrak{R}$. If ψ and ϕ are (s, m) -exponential-type convex functions for some fixed $s, m \in (0, 1]$, then the following holds:

- (1) $\psi + \phi$ is (s, m) -exponential-type convex function.
- (2) For nonnegative real number c , $c\psi$ is (s, m) -exponential-type convex function.

Proof. By Definition 4, for some fixed $s, m \in (0, 1]$, the proof is obvious. $\square$

Theorem 2. Let $\psi: [0, a_2] \rightarrow J$ be m -convex function for $a_2 > 0$ and some fixed $m \in (0, 1]$ and $\phi: J \rightarrow \mathfrak{R}$ is nondecreasing and (s, m) -exponential-type convex function for some fixed $s, m \in (0, 1]$. Then, for the same fixed numbers $s, m \in (0, 1]$, the function $\phi^\circ \psi: [0, a_2] \rightarrow \mathfrak{R}$ is (s, m) -exponential-type convex.

Proof. For all $\theta_1, \theta_2 \in [0, a_2]$ and $\chi \in [0, 1]$ and for some fixed numbers $s, m \in (0, 1]$, we have

$$\begin{aligned} (\phi^\circ \psi)(\chi\theta_1 + m(1-\chi)\theta_2) &= \phi(\psi(\chi\theta_1 + m(1-\chi)\theta_2)) \\ &\leq \phi(\chi\psi(\theta_1) + m(1-\chi)\psi(\theta_2)) \\ &\leq (e^{s\chi} - 1)(\phi^\circ \psi)(\theta_1) + m(e^{(1-\chi)s} - 1)(\phi^\circ \psi)(\theta_2). \end{aligned} \quad (14)$$

Theorem 3. Let $\psi_i: [a_1, a_2] \rightarrow \mathfrak{R}$ be an arbitrary family of (s, m) -exponential-type convex functions for the same fixed $s, m \in (0, 1]$ and let $\psi(\theta) = \sup_i \psi_i(\theta)$. If $A = \{\theta \in [a_1, a_2]: \psi(\theta) < +\infty\} \neq \emptyset$, then A is an interval and ψ is (s, m) -exponential-type convex function on A .

Proof. For all $\theta_1, \theta_2 \in A$ and $\chi \in [0, 1]$ and for the same fixed numbers $s, m \in (0, 1]$, we have

$$\begin{aligned} \psi(\chi\theta_1 + m(1-\chi)\theta_2) &= \sup_i \psi_i(\chi\theta_1 + m(1-\chi)\theta_2) \\ &\leq \sup_i [(e^{s\chi} - 1)\psi_i(\theta_1) + m(e^{(1-\chi)s} - 1)\psi_i(\theta_2)] \\ &\leq (e^{s\chi} - 1)\sup_i \psi_i(\theta_1) + m(e^{(1-\chi)s} - 1)\sup_i \psi_i(\theta_2) \\ &= (e^{s\chi} - 1)\psi(\theta_1) + m(e^{(1-\chi)s} - 1)\psi(\theta_2) < +\infty. \end{aligned} \quad (15)$$

This means simultaneously that A is an interval, and ψ is (s, m) -exponential-type convex function on A . $\square$

Theorem 4. If the function $\psi: [a_1, a_2] \rightarrow \mathfrak{R}$ is (s, m) -exponential-type convex for some fixed $s, m \in (0, 1]$, then ψ is bounded on $[a_1, ma_2]$.

Proof. Let $L = \max\{\psi(a_1), \psi(a_2/m)\}$ and $x \in [a_1, a_2]$ be an arbitrary point for some fixed $m \in (0, 1]$. Then, there exists $\chi \in [0, 1]$ such that $x = \chi a_1 + (1-\chi)a_2$. Thus, since $e^{s\chi} \leq e^s$ and $e^{(1-\chi)s} \leq e^s$ for some fixed $s \in (0, 1]$, we have

$$\begin{aligned} \psi(x) &= \psi(\chi a_1 + (1-\chi)a_2) \leq (e^{s\chi} - 1)\psi(a_1) \\ &\quad + m(e^{(1-\chi)s} - 1)\psi\left(\frac{a_2}{m}\right) \\ &\leq (e^s - 1)L + m(e^s - 1)L = (m+1)L(e^s - 1) = M. \end{aligned} \quad (16)$$

We have shown that ψ is bounded above from real number M . Interested reader can also prove the fact that ψ is bounded below using the same idea as in Theorem 4 in [11]. $\square$

3. New Generalizations of (H-H)-Type Inequality

In this section, we will establish some new generalizations of Hermite–Hadamard-type inequality for the (s, m) -exponential-type convex function ψ and for the products of two (s, m) -exponential-type convex functions ψ and ϕ .

Theorem 5. Let $\psi: [a_1, ma_2] \rightarrow \mathfrak{R}$ be (s, m) -exponential-type convex function for some fixed $s, m \in (0, 1]$ and $a_1 < ma_2$. If $\psi \in L_1([a_1, ma_2])$, then

$$\begin{aligned} \frac{1}{(e^{s/2} - 1)} \psi\left(\frac{a_1 + ma_2}{2}\right) &\leq \frac{1}{(ma_2 - a_1)} \\ &\quad \left\{ \int_{a_1}^{ma_2} \psi(x) dx + m \int_{a_1/m}^{a_2} \psi(x) dx \right\} \\ &\leq \left(\frac{e^s - s - 1}{s} \right) \left[\psi(a_1) + \psi(a_2) + m \left(\psi\left(\frac{a_1}{m^2}\right) + \psi(a_2) \right) \right]. \end{aligned} \quad (17)$$

Proof. Let denote, respectively,

$$\theta_1 = \chi a_1 + m(1-\chi)a_2, \theta_2 = (1-\chi)\frac{a_1}{m} + \chi a_2, \quad \forall \chi \in [0, 1]. \quad (18)$$

Using (s, m) -exponential-type convexity of ψ , we have

$$\begin{aligned} \psi\left(\frac{a_1 + ma_2}{2}\right) &= \psi\left(\frac{\theta_1 + m\theta_2}{2}\right) \\ &= \psi\left(\frac{[\chi a_1 + m(1-\chi)a_2] + [(1-\chi)a_1 + m\chi a_2]}{2}\right) \\ &\leq (e^{s/2} - 1) [\psi(\chi a_1 + m(1-\chi)a_2) + \psi((1-\chi)a_1 + m\chi a_2)]. \end{aligned} \quad (19)$$

Now, integrating on both sides in the last inequality with respect to χ over $[0, 1]$, we get

$$\begin{aligned}
& \psi\left(\frac{a_1 + ma_2}{2}\right) \leq (e^{s/2} - 1) \times \left[\int_0^1 \psi(\chi a_1 + m(1-\chi)a_2) d\chi \right. \\
& \quad \left. + \int_0^1 \psi\left((1-\chi)\frac{a_1}{m} + \chi a_2\right) d\chi \right] \\
& = \frac{(e^{s/2} - 1)}{(ma_2 - a_1)} \left\{ \int_{a_1}^{ma_2} \psi(x) dx + m \int_{a_1/m}^{a_2} \psi(x) dx \right\}, \quad (20)
\end{aligned}$$

which completes the left-side inequality. For the right-side inequality, using (s, m) -exponential-type convexity of ψ , we obtain

$$\begin{aligned}
& \frac{1}{(ma_2 - a_1)} \left\{ \int_{a_1}^{ma_2} \psi(x) dx + m \int_{\frac{a_1}{m}}^{a_2} \psi(x) dx \right\} \\
& = \int_0^1 \psi(\chi a_1 + m(1-\chi)a_2) d\chi + \int_0^1 \psi\left((1-\chi)\frac{a_1}{m} + \chi a_2\right) d\chi \\
& \leq \int_0^1 [(e^{s\chi} - 1)\psi(a_1) + m(e^{(1-\chi)s} - 1)\psi(a_2)] d\chi \\
& \quad + \int_0^1 [(e^{s\chi} - 1)\psi(a_2) + m(e^{(1-\chi)s} - 1)\psi\left(\frac{a_1}{m^2}\right)] d\chi \\
& = \left(\frac{e^s - s - 1}{s}\right) \left[\psi(a_1) + \psi(a_2) + m\left(\psi\left(\frac{a_1}{m^2}\right) + \psi(a_2)\right) \right], \quad (21)
\end{aligned}$$

which gives the right-side inequality. $\square$

Corollary 1. By taking $m = s = 1$ in Theorem 5, we get (Theorem 5, [11]).

Theorem 6. Assume that $\psi, \phi: [a_1, ma_2] \rightarrow \mathfrak{R}$ are, respectively, (s_1, m) and (s_2, m) -exponential-type convex functions for the same fixed $m \in (0, 1]$ and for some fixed $s_1, s_2 \in (0, 1]$, where $s_1 < s_2$ and $a_1 < ma_2$. If ψ, ϕ are synchronous functions and $\psi, \phi, \psi\phi \in L_1([a_1, ma_2])$, then

$$\begin{aligned}
& \frac{1}{(ma_2 - a_1)} \int_{a_1}^{ma_2} \psi(\theta) d\theta \cdot \frac{1}{(ma_2 - a_1)} \int_{a_1}^{ma_2} \phi(\theta) d\theta \\
& \leq \frac{1}{(ma_2 - a_1)} \int_{a_1}^{ma_2} \psi(\theta) \phi(\theta) d\theta \quad (22) \\
& \leq A(s_1, s_2) M_m(a_1, a_2) + B(s_1, s_2) N(a_1, a_2),
\end{aligned}$$

where

$$\begin{aligned}
M_m(a_1, a_2) & = \psi(a_1)\phi(a_1) + m^2\psi(a_2)\phi(a_2), N(a_1, a_2) \\
& = \psi(a_1)\phi(a_2) + \psi(a_2)\phi(a_1), \\
A(s_1, s_2) & = \frac{e^{s_1+s_2} + s_1 + s_2 - 1}{s_1 + s_2} - \frac{s_1(e^{s_2} - 1) + s_2(e^{s_1} - 1)}{s_1 s_2}, \\
B(s_1, s_2) & = \frac{e^{s_2} - e^{s_1}}{s_2 - s_1} - \frac{s_1(e^{s_2} - 1) + s_2(e^{s_1} - 1)}{s_1 s_2} + 1. \quad (23)
\end{aligned}$$

Proof. Let us denote $\theta = \chi a_1 + m(1-\chi)a_2$ for all $\chi \in [0, 1]$. Using the property of the (s_1, m) and (s_2, m) -exponential-type convex functions ψ and ϕ , respectively, we have

$$\begin{aligned}
\psi(\chi a_1 + m(1-\chi)a_2) & \leq (e^{s_1\chi} - 1)\psi(a_1) + m(e^{(1-\chi)s_1} - 1)\psi(a_2), \\
\phi(\chi a_1 + m(1-\chi)a_2) & \leq (e^{s_2\chi} - 1)\phi(a_1) + m(e^{(1-\chi)s_2} - 1)\phi(a_2). \quad (24)
\end{aligned}$$

Multiplying above inequalities on both sides, we get

$$\begin{aligned}
& \psi(\chi a_1 + m(1-\chi)a_2)\phi(\chi a_1 + m(1-\chi)a_2) \\
& \leq [(e^{s_1\chi} - 1)\psi(a_1) + m(e^{(1-\chi)s_1} - 1)\psi(a_2)] \\
& \quad \times [(e^{s_2\chi} - 1)\phi(a_1) + m(e^{(1-\chi)s_2} - 1)\phi(a_2)] \\
& = (e^{s_1\chi} - 1)(e^{s_2\chi} - 1)\psi(a_1)\phi(a_1) \quad (25) \\
& \quad + m[(e^{s_1\chi} - 1)(e^{(1-\chi)s_2} - 1)\psi(a_1)\phi(a_2) \\
& \quad + (e^{s_2\chi} - 1)(e^{(1-\chi)s_1} - 1)\psi(a_2)\phi(a_1)] \\
& \quad + m^2(e^{(1-\chi)s_1} - 1)(e^{(1-\chi)s_2} - 1)\psi(a_2)\phi(a_2).
\end{aligned}$$

Applying Chebyshev integral inequality (see [23]), we obtain

$$\begin{aligned}
& \int_0^1 \psi(\chi a_1 + m(1-\chi)a_2)\phi(\chi a_1 + m(1-\chi)a_2) d\chi \\
& \geq \int_0^1 \psi(\chi a_1 + m(1-\chi)a_2) d\chi \cdot \int_0^1 \phi(\chi a_1 + m(1-\chi)a_2) d\chi. \quad (26)
\end{aligned}$$

Changing the variable of integration, we get

$$\begin{aligned}
& \frac{1}{(ma_2 - a_1)} \int_{a_1}^{ma_2} \psi(\theta)\phi(\theta) d\theta \\
& \geq \frac{1}{(ma_2 - a_1)} \int_{a_1}^{ma_2} \psi(\theta) d\theta \cdot \frac{1}{(ma_2 - a_1)} \int_{a_1}^{ma_2} \phi(\theta) d\theta, \quad (27)
\end{aligned}$$

which completes the left-side inequality. For the right-side inequality, integrating on both sides of the inequality (25) with respect to χ over $[0, 1]$, we have

$$\begin{aligned}
& \frac{1}{(ma_2 - a_1)} \int_{a_1}^{ma_2} \psi(\theta) d\theta \cdot \frac{1}{(ma_2 - a_1)} \int_{a_1}^{ma_2} \phi(\theta) d\theta \\
& \leq \left[\int_0^1 (e^{s_1 \chi} - 1)(e^{s_2 \chi} - 1) d\chi \right] \psi(a_1) \phi(a_1) \\
& \quad + m \left[\left(\int_0^1 (e^{s_1 \chi} - 1)(e^{(1-\chi)s_2} - 1) d\chi \right) \psi(a_1) \phi(a_2) \right. \\
& \quad \left. + \left(\int_0^1 (e^{s_2 \chi} - 1)(e^{(1-\chi)s_1} - 1) d\chi \right) \psi(a_2) \phi(a_1) \right] \\
& \quad + m^2 \left[\int_0^1 (e^{(1-\chi)s_1} - 1)(e^{(1-\chi)s_2} - 1) d\chi \right] \psi(a_2) \phi(a_2) \\
& = A(s_1, s_2) M_m(a_1, a_2) + B(s_1, s_2) N(a_1, a_2),
\end{aligned} \tag{28}$$

which give the right-side inequality. $\square$

4. Refinements of (H–H)-Type Inequality for k – Fractional Integral

In this section, we will obtain some refinements of the (H–H) inequality via (s, m) -exponential-type convex functions.

Lemma 2. Suppose $0 < w \leq 1$ and a mapping $\psi: [wa_1, a_2] \rightarrow \mathfrak{R}$ is differentiable on (wa_1, a_2) with $0 < a_1 < a_2$. If $\psi' \in L_1[wa_1, a_2]$, then the following equality for k -fractional integral holds:

$$\begin{aligned}
& \frac{\psi(wa_1) + \alpha/k \psi(a_2)}{\alpha/k + 1} - \frac{\Gamma_k(\alpha + k)}{(a_2 - wa_1)^{\alpha/k}} J_{a_2^-}^\alpha \psi(wa_1) \\
& = \left(\frac{a_2 - wa_1}{\alpha/k + 1} \right) \int_0^1 \left[\left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} - 1 \right] \psi'(w(1 - \chi)a_1 + \chi a_2) d\chi,
\end{aligned} \tag{29}$$

where $\alpha, k > 0$ and $\Gamma(\cdot)$ is the Euler Gamma function.

Proof. Using the integrating by parts, we have

$$\begin{aligned}
& \left(\frac{a_2 - wa_1}{\alpha/k + 1} \right) \int_0^1 \left[\left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} - 1 \right] \psi'(w(1 - \chi)a_1 + \chi a_2) d\chi \\
& = \left(\frac{a_2 - wa_1}{\alpha/k + 1} \right) \left\{ \int_0^1 \left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} \psi'(w(1 - \chi)a_1 + \chi a_2) d\chi \right. \\
& \quad \left. - \int_0^1 \psi'(w(1 - \chi)a_1 + \chi a_2) d\chi \right\} \\
& = \left(\frac{a_2 - wa_1}{\alpha/k + 1} \right) \left[\left(\frac{\alpha}{k} + 1 \right) \left\{ \frac{\chi^{\alpha/k} \psi(w(1 - \chi)a_1 + \chi a_2)}{a_2 - wa_1} \Big|_0^1 \right. \right. \\
& \quad \left. \left. - \int_0^1 \frac{\psi(w(1 - \chi)a_1 + \chi a_2)}{a_2 - wa_1} \frac{\alpha}{k} \chi^{\alpha/k-1} d\chi \right\} \right. \\
& \quad \left. - \frac{\psi(w(1 - \chi)a_1 + \chi a_2)}{a_2 - wa_1} \Big|_0^1 \right] \\
& = \left(\frac{a_2 - wa_1}{\alpha/k + 1} \right) \left[\left(\frac{\alpha}{k} + 1 \right) \times \left\{ \frac{\psi(a_2)}{a_2 - wa_1} - \frac{\alpha}{k(a_2 - wa_1)} \right. \right. \\
& \quad \left. \left. \int_0^1 \chi^{\alpha/k-1} \psi(w(1 - \chi)a_1 + \chi a_2) d\chi \right\} \right] \\
& = \frac{\psi(wa_1) + \alpha/k \psi(a_2)}{\alpha/k + 1} - \frac{\Gamma_k(\alpha + k)}{(a_2 - wa_1)^{\alpha/k}} J_{a_2^-}^\alpha \psi(wa_1),
\end{aligned} \tag{30}$$

which completes the proof.

Lemma 3. Suppose $0 < w \leq 1$ and a mapping $\psi: [wa_1, a_2] \rightarrow \mathfrak{R}$ is differentiable on (wa_1, a_2) with $0 < a_1 < a_2$. If $\psi' \in L_1[wa_1, a_2]$, then the following equality for k -fractional integral holds:

$$\begin{aligned}
& \frac{\psi(wa_1) + \psi(a_2)}{w + 1} - \frac{\Gamma_k(\alpha + k)}{(w + 1)(a_2 - wa_1)^{\alpha/k}} \\
& \quad \left\{ {}^k J_{a_1^+}^\alpha \psi(a_2) + {}^k J_{a_2^-}^\alpha \psi(wa_1) \right\} \\
& = \left(\frac{a_2 - wa_1}{w + 1} \right) \int_0^1 \left[\chi^{\alpha/k} - (1 - \chi)^{\alpha/k} \right] \psi'(w(1 - \chi)a_1 + \chi a_2) d\chi.
\end{aligned} \tag{31}$$

Proof. Using the integrating by parts, we have

$$\begin{aligned}
& \left(\frac{a_2 - wa_1}{w+1} \right) \int_0^1 [\chi^{\alpha/k} - (1-\chi)^{\alpha/k}] \psi'(w(1-\chi)a_1 + \chi a_2) d\chi \\
&= \left(\frac{a_2 - wa_1}{w+1} \right) \left[\int_0^1 \chi^{\alpha/k} \psi'(w(1-\chi)a_1 + \chi a_2) d\chi \right. \\
&\quad \left. - \int_0^1 (1-\chi)^{\alpha/k} \psi'(w(1-\chi)a_1 + \chi a_2) d\chi \right] \\
&= \left(\frac{a_2 - wa_1}{w+1} \right) [I_1 - I_2],
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
I_1 &= \int_0^1 \chi^{\alpha/k} \psi'(w(1-\chi)a_1 + \chi a_2) d\chi \\
&= \frac{\chi^{\alpha/k} \psi(w(1-\chi)a_1 + \chi a_2)}{a_2 - wa_1} \Big|_0^1 \\
&\quad - \int_0^1 \frac{\psi(w(1-\chi)a_1 + \chi a_2)}{a_2 - wa_1} \frac{\alpha}{k} \chi^{\alpha/k-1} d\chi \\
&= \frac{\psi(a_2)}{a_2 - wa_1} - \frac{\alpha}{k(a_2 - wa_1)} \int_0^1 \chi^{\alpha/k-1} \psi(w(1-\chi)a_1 + \chi a_2) d\chi \\
&= \frac{\psi(a_2)}{a_2 - wa_1} - \frac{\Gamma_k(\alpha+k)}{(a_2 - wa_1)^{\alpha/k+1}} J_{a_2^-}^{\alpha} \psi(wa_1),
\end{aligned} \tag{33}$$

$$\begin{aligned}
I_2 &= \int_0^1 (1-\chi)^{\alpha/k} \psi'(w(1-\chi)a_1 + \chi a_2) d\chi \\
&= \frac{(1-\chi)^{\alpha/k} \psi(w(1-\chi)a_1 + \chi a_2)}{a_2 - wa_1} \Big|_0^1 \\
&\quad - \int_0^1 \frac{\psi(w(1-\chi)a_1 + \chi a_2)}{a_2 - wa_1} \frac{\alpha}{k} (1-\chi)^{\alpha/k-1} (-1) d\chi \\
&= -\frac{\psi(wa_1)}{a_2 - wa_1} + \frac{\alpha}{k(a_2 - wa_1)} \\
&\quad \int_0^1 (1-\chi)^{\alpha/k-1} \psi(w(1-\chi)a_1 + \chi a_2) d\chi \\
&= -\frac{\psi(wa_1)}{a_2 - wa_1} + \frac{\Gamma_k(\alpha+k)}{(a_2 - wa_1)^{\alpha/k+1}} J_{a_1^+}^{\alpha} \psi(a_2).
\end{aligned} \tag{34}$$

Combining equations (33) and (34) in (32) and multiplying by $a_2 - wa_1/w + 1$, we get (31), which completes the proof. $\square$

Theorem 7. Suppose $0 < w \leq 1$ and a mapping $\psi: (0, a_2/mw] \rightarrow \mathfrak{R}$ is differentiable on $(0, a_2/mw)$ with $0 < a_1 < a_2$. If $|\psi'|^q$ is (s, m) -exponential-type convex on $(0, a_2/mw]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed

$s, m \in (0, 1]$, then the following inequality for k -fractional integral holds:

$$\begin{aligned}
& \left| \frac{\psi(wa_1) + \alpha/k \psi(a_2)}{\alpha/k + 1} - \frac{\Gamma_k(\alpha+k)}{(a_2 - wa_1)^{\alpha/k}} J_{a_2^-}^{\alpha} \psi(wa_1) \right| \\
&\leq \left(\frac{a_2 - wa_1}{\alpha/k + 1} \right) [U_1(\alpha, k, p) + U_2(\alpha, k, p)]^{1/p} \\
&\quad \times \left[\left(\frac{e^s - s - 1}{s} \right) \left(|\psi'(wa_1)|^q + m \left| \psi'\left(\frac{a_2}{m}\right) \right|^q \right) \right]^{1/q},
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
U_1(\alpha, k, p) &= \int_0^{1/\sqrt{[\alpha]}(\alpha/k+1)^k} \left(1 - \left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} \right)^p d\chi, \\
U_2(\alpha, k, p) &= \int_{1/\sqrt{[\alpha]}(\alpha/k+1)^k}^1 \left(\left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} - 1 \right)^p d\chi.
\end{aligned} \tag{36}$$

Proof. From Lemma 2, Hölder's inequality and (s, m) -exponential-type convexity of $|\psi'|^q$, we have

$$\begin{aligned}
& \left| \frac{\psi(wa_1) + \alpha/k \psi(a_2)}{\alpha/k + 1} - \frac{\Gamma_k(\alpha+k)}{(a_2 - wa_1)^{\alpha/k}} J_{a_2^-}^{\alpha} \psi(wa_1) \right| \\
&\leq \left(\frac{a_2 - wa_1}{\alpha/k + 1} \right) \int_0^1 \left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} - 1 \left| \psi'(w(1-\chi)a_1 + \chi a_2) \right| d\chi \\
&\leq \left(\frac{a_2 - wa_1}{\alpha/k + 1} \right) \left(\int_0^1 \left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} - 1 \right)^p d\chi \\
&\quad \left(\int_0^1 |\psi'(w(1-\chi)a_1 + \chi a_2)|^q d\chi \right)^{1/q} \\
&\leq \left(\frac{a_2 - wa_1}{\alpha/k + 1} \right) \left(\int_0^1 \left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} - 1 \right)^p d\chi \\
&\quad \times \left(\int_0^1 \left(e^{(1-\chi)s} - 1 \right) |\psi'(wa_1)|^q + m(e^{s\chi} - 1) \left| \psi'\left(\frac{a_2}{m}\right) \right|^q d\chi \right)^{1/q} \\
&= \left(\frac{a_2 - wa_1}{\alpha/k + 1} \right) [U_1(\alpha, k, p) + U_2(\alpha, k, p)]^{1/p} \\
&\quad \times \left[\left(\frac{e^s - s - 1}{s} \right) \left(|\psi'(wa_1)|^q + m \left| \psi'\left(\frac{a_2}{m}\right) \right|^q \right) \right]^{1/q},
\end{aligned} \tag{37}$$

which completes the proof. $\square$

Theorem 8. Suppose $0 < w \leq 1$ and a mapping $\psi: (0, a_2/mw] \rightarrow \mathfrak{R}$ is differentiable on $(0, a_2/mw)$ with $0 < a_1 < a_2$. If $|\psi'|^q$ is (s, m) -exponential-type convex on $(0, a_2/mw]$ for $q \geq 1$, then for some fixed $s, m \in (0, 1]$, then the following inequality for k -fractional integral holds:

$$\begin{aligned}
& \left| \frac{\psi(wa_1) + \alpha/k\psi(a_2)}{\alpha/k+1} - \frac{\Gamma_k(\alpha+k)}{(a_2-wa_1)^{\alpha/k}} J_{a_2^-}^\alpha \psi(wa_1) \right| \\
& \leq \left(\frac{a_2-wa_1}{\alpha/k+1} \right) \left(\frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}} \right)^{1-1/q} \\
& \quad \times \left[|\psi'(wa_1)|^q \left\{ \frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}} - \frac{2}{s} e^{s(1-1/\sqrt{[\alpha]}(\alpha/k+1)^k)} - \left(\frac{\alpha}{k} + 1 \right) \frac{e^s}{s^{\alpha/k}} \gamma \left(\frac{\alpha}{k} + 1, \frac{1}{\sqrt{[\alpha]}(\alpha/k+1)^k} \right) \right. \right. \\
& \quad \left. \left. + \left(\frac{\alpha}{k} + 1 \right) e^s (s^{-\alpha/k}) \gamma_{(1-1/\sqrt{[\alpha]}(\alpha/k+1)^k)} \left(\frac{\alpha}{k} + 1, \frac{1}{\sqrt{[\alpha]}(\alpha/k+1)^k} \right) + \frac{1}{s} \right\} \right. \\
& \quad \left. + m \left| \psi' \left(\frac{a_2}{m} \right) \right| \left\{ \frac{2e^{s/\sqrt{[\alpha]}(\alpha/k+1)^k}}{s} - \frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}} - \left(\frac{\alpha}{k} + 1 \right) \left(\frac{1}{(-s)^{\alpha/k}} \right) \gamma \left(\frac{\alpha}{k} + 1, \frac{1}{\sqrt{[\alpha]}(\alpha/k+1)^k} \right) \right. \right. \\
& \quad \left. \left. + \left(\frac{\alpha}{k} + 1 \right) \left(\frac{1}{(-s)^{\alpha/k}} \right) \gamma_{1-1/\sqrt{[\alpha]}(\alpha/k+1)^k} \left(\frac{\alpha}{k} + 1, \frac{1}{\sqrt{[\alpha]}(\alpha/k+1)^k} \right) - \frac{e^s}{s} \right\} \right]^{1/q}.
\end{aligned} \tag{38}$$

Proof. From Lemma 2, power mean inequality and (s, m) -exponential-type convexity of $|\psi'|^q$, we have

$$\begin{aligned}
& \left| \frac{\psi(wa_1) + \alpha/k\psi(a_2)}{\alpha/k+1} - \frac{\Gamma_k(\alpha+k)}{(a_2-wa_1)^{\alpha/k}} J_{a_2^-}^\alpha \psi(wa_1) \right| \\
& \leq \left(\frac{a_2-wa_1}{\alpha/k+1} \right) \int_0^1 \left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} - 1 \left| \psi'(w(1-\chi)a_1 + \chi a_2) \right| d\chi \\
& \leq \left(\frac{a_2-wa_1}{\alpha/k+1} \right) \left(\int_0^1 \left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} - 1 d\chi \right)^{1-1/q} \times \left(\int_0^1 \left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} - 1 \left| \psi'(w(1-\chi)a_1 + \chi a_2) \right|^q d\chi \right)^{1/q} \\
& \leq \left(\frac{a_2-wa_1}{\alpha/k+1} \right) \left(\int_0^1 \left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} - 1 d\chi \right)^{1-1/q} \times \left(\int_0^1 \left(\frac{\alpha}{k} + 1 \right) \chi^{\alpha/k} - 1 \left[\left(e^{(1-\chi)s} - 1 \right) |\psi'(wa_1)|^q + m(e^{s\chi} - 1) \left| \psi' \left(\frac{a_2}{m} \right) \right|^q \right] d\chi \right)^{1/q} \\
& = \left(\frac{a_2-wa_1}{\alpha/k+1} \right) \left(\frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}} \right)^{1-1/q} \\
& \quad \times \left[|\psi'(wa_1)|^q \left\{ \frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}} - \frac{2}{s} e^{s(1-1/\sqrt{[\alpha]}(\alpha/k+1)^k)} - \left(\frac{\alpha}{k} + 1 \right) \frac{e^s}{s^{\alpha/k}} \gamma \left(\frac{\alpha}{k} + 1, \frac{1}{\sqrt{[\alpha]}(\alpha/k+1)^k} \right) \right. \right. \\
& \quad \left. \left. + \left(\frac{\alpha}{k} + 1 \right) \frac{e^s}{s^{\alpha/k}} \gamma_{1-1/\sqrt{[\alpha]}(\alpha/k+1)^k} \left(\frac{\alpha}{k} + 1, \frac{1}{\sqrt{[\alpha]}(\alpha/k+1)^k} \right) + \frac{1}{s} \right\} \right. \\
& \quad \left. + m \left| \psi' \left(\frac{a_2}{m} \right) \right| \left\{ \frac{2e^{s/\sqrt{[\alpha]}(\alpha/k+1)^k}}{s} - \frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}} - \left(\frac{\alpha}{k} + 1 \right) \left(\frac{1}{(-s)^{\alpha/k}} \right) \gamma \left(\frac{\alpha}{k} + 1, \frac{1}{\sqrt{[\alpha]}(\alpha/k+1)^k} \right) \right. \right. \\
& \quad \left. \left. \cdot \gamma \left(\frac{\alpha}{k} + 1, \frac{1}{\sqrt{[\alpha]}(\alpha/k+1)^k} \right) + \left(\frac{\alpha}{k} + 1 \right) \left(\frac{1}{(-s)^{\alpha/k}} \right) \gamma_{1-1/\sqrt{[\alpha]}(\alpha/k+1)^k} \left(\frac{\alpha}{k} + 1, \frac{1}{\sqrt{[\alpha]}(\alpha/k+1)^k} \right) - \frac{e^s}{s} \right\} \right]^{1/q},
\end{aligned} \tag{39}$$

which completes the proof. $\square$

Theorem 9. Suppose $0 < w \leq 1$ and a mapping $\psi: (0, a_2/m] \rightarrow \mathfrak{R}$ is differentiable on $(0, a_2/m)$ with

$0 < a_1 < a_2$. If $|\psi'|^q$ is (s, m) -exponential-type convex on $(0, a_2/m]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $s, m \in (0, 1]$, then the following inequality for k -fractional integral holds:

$$\begin{aligned} & \left| \frac{\psi(wa_1) + \psi(a_2)}{w+1} - \frac{\Gamma_k(\alpha+k)}{(w+1)(a_2-wa_1)^{\alpha/k}} \left\{ {}^k J_{a_1^+}^\alpha \psi(a_2) + {}^k J_{a_2^-}^\alpha \psi(wa_1) \right\} \right| \\ & \leq \frac{2(a_2-wa_1)}{w+1} \left(\frac{k}{\alpha p + k} \right)^{1/p} \left[\left(\frac{e^s - s - 1}{s} \right) \left(|\psi'(wa_1)|^q + m \left| \psi' \left(\frac{a_2}{m} \right) \right|^q \right) \right]^{1/q}. \end{aligned} \quad (40)$$

Proof. From Lemma 3, Hölder's inequality and (s, m) -exponential-type convexity of $|\psi'|^q$, we have

$$\begin{aligned} & \left| \frac{\psi(wa_1) + \psi(a_2)}{w+1} - \frac{\Gamma_k(\alpha+k)}{(w+1)(a_2-wa_1)^{\alpha/k}} \left\{ {}^k J_{a_1^+}^\alpha \psi(a_2) + {}^k J_{a_2^-}^\alpha \psi(wa_1) \right\} \right| \\ & \leq \left(\frac{a_2-wa_1}{w+1} \right) \int_0^1 |\chi^{\alpha/k} - (1-\chi)^{\alpha/k}| |\psi'(w(1-\chi)a_1 + \chi a_2)| d\chi \\ & \leq \left(\frac{a_2-wa_1}{w+1} \right) \left[\int_0^1 \chi^{\alpha/k} |\psi'(w(1-\chi)a_1 + \chi a_2)| d\chi + \int_0^1 (1-\chi)^{\alpha/k} |\psi'(w(1-\chi)a_1 + \chi a_2)| d\chi \right] \\ & \leq \left(\frac{a_2-wa_1}{w+1} \right) \left[\left(\int_0^1 \chi^{\alpha/kp} d\chi \right)^{1/p} \left(\int_0^1 |\psi'(w(1-\chi)a_1 + \chi a_2)|^q d\chi \right)^{1/q} + \left(\int_0^1 (1-\chi)^{\alpha/kp} d\chi \right)^{1/p} \left(\int_0^1 |\psi'(w(1-\chi)a_1 + \chi a_2)|^q d\chi \right)^{1/q} \right] \\ & \leq \left(\frac{a_2-wa_1}{w+1} \right) \left[\left(\int_0^1 \chi^{\alpha/kp} d\chi \right)^{1/p} \times \left(\int_0^1 \left[(e^{(1-\chi)s} - 1) |\psi'(wa_1)|^q + m(e^{s\chi} - 1) \left| \psi' \left(\frac{a_2}{m} \right) \right|^q \right] d\chi \right)^{1/q} \right. \\ & \quad \left. + \left(\int_0^1 (1-\chi)^{\alpha/kp} d\chi \right)^{1/p} \times \left(\int_0^1 \left[(e^{(1-\chi)s} - 1) |\psi'(wa_1)|^q + m(e^{s\chi} - 1) \left| \psi' \left(\frac{a_2}{m} \right) \right|^q \right] d\chi \right)^{1/q} \right] \\ & = \frac{2(a_2-wa_1)}{w+1} \left(\frac{k}{\alpha p + k} \right)^{1/p} \left[\left(\frac{e^s - s - 1}{s} \right) \left(|\psi'(wa_1)|^q + m \left| \psi' \left(\frac{a_2}{m} \right) \right|^q \right) \right]^{1/q}, \end{aligned} \quad (41)$$

which completes the proof. $\square$

Theorem 10. Suppose $0 < w \leq 1$ and a mapping $\psi: (0, a_2/m] \rightarrow \mathfrak{R}$ is differentiable on $(0, a_2/m)$ with

$0 < a_1 < a_2$. If $|\psi'|^q$ is (s, m) -exponential-type convex on $(0, a_2/m]$ for $q \geq 1$, then for some fixed $s, m \in (0, 1]$, then the following inequality for k -fractional integral holds:

$$\begin{aligned}
& \left| \frac{\psi(wa_1) + \psi(a_2)}{w+1} - \frac{\Gamma_k(\alpha+k)}{(w+1)(a_2-wa_1)^{\alpha/k}} \left\{ {}^k J_{a_1^+}^\alpha \psi(a_2) + {}^k J_{a_2^-}^\alpha \psi(wa_1) \right\} \right| \\
& \leq \left(\frac{a_2-wa_1}{w+1} \right) \left(\frac{k}{\alpha+k} \right)^{1-1/q} \\
& \quad \times \left[\left\{ |\psi'(wa_1)|^q \left(\Gamma\left(\frac{\alpha}{k}+1\right) - \Gamma\left(\frac{\alpha}{k}+1, s\right) e^s s^{-\alpha/k-1} - \frac{1}{\alpha/k+1} \right) + m \left| \psi'\left(\frac{a_2}{m}\right) \right|^q \left(\frac{\Gamma(\alpha/k+1, -s) - \Gamma(\alpha/k+1)}{(-s)^{\alpha/k}s} - \frac{1}{\alpha/k+1} \right) \right\}^{1/q} \right. \\
& \quad + \left\{ |\psi'(wa_1)|^q \left((-1)^{\alpha/k-1} s^{-\alpha/k-1} \left(\Gamma\left(\frac{\alpha}{k}+1\right) - \Gamma\left(\frac{\alpha}{k}+1, -s\right) \right) - \frac{1}{\alpha/k+1} \right) \right. \\
& \quad \left. \left. + m \left| \psi'\left(\frac{a_2}{m}\right) \right|^q \left(\frac{(\alpha/k+1)e^s k s^{-\alpha/k-1} (\Gamma(\alpha/k+1, s) - \Gamma(\alpha/k+1))}{k+\alpha} - \frac{1}{\alpha/k+1} \right) \right\}^{1/q} \right].
\end{aligned} \tag{42}$$

Proof. From Lemma 3, power mean inequality and (s, m) -exponential-type convexity of $|\psi'|^q$, we have

$$\begin{aligned}
& \left| \frac{\psi(wa_1) + \psi(a_2)}{w+1} - \frac{\Gamma_k(\alpha+k)}{(w+1)(a_2-wa_1)^{\alpha/k}} \left\{ {}^k J_{a_1^+}^\alpha \psi(a_2) + {}^k J_{a_2^-}^\alpha \psi(wa_1) \right\} \right| \\
& \leq \left(\frac{a_2-wa_1}{w+1} \right) \int_0^1 \left| \chi^{\alpha/k} - (1-\chi)^{\alpha/k} \right| |\psi'(w(1-\chi)a_1 + \chi a_2)| d\chi \\
& \leq \left(\frac{a_2-wa_1}{w+1} \right) \times \left[\int_0^1 \chi^{\alpha/k} |\psi'(w(1-\chi)a_1 + \chi a_2)| d\chi + \int_0^1 (1-\chi)^{\alpha/k} |\psi'(w(1-\chi)a_1 + \chi a_2)| d\chi \right] \\
& \leq \left(\frac{a_2-wa_1}{w+1} \right) \left[\left(\int_0^1 \chi^{\alpha/k} d\chi \right)^{1-1/q} \left(\int_0^1 \chi^{\alpha/k} |\psi'(w(1-\chi)a_1 + \chi a_2)|^q d\chi \right)^{1/q} + \left(\int_0^1 (1-\chi)^{\alpha/k} d\chi \right)^{1-1/q} \right. \\
& \quad \left. \left(\int_0^1 (1-\chi)^{\alpha/k} |\psi'(w(1-\chi)a_1 + \chi a_2)|^q d\chi \right)^{1/q} \right] \\
& \leq \left(\frac{a_2-wa_1}{w+1} \right) \left(\int_0^1 \chi^{\alpha/k} d\chi \right)^{1-1/q} \times \left[\left(\int_0^1 \chi^{\alpha/k} \left[(e^{(1-\chi)s} - 1) |\psi'(wa_1)|^q + m(e^{s\chi} - 1) \left| \psi'\left(\frac{a_2}{m}\right) \right|^q \right] d\chi \right)^{1/q} \right. \\
& \quad \left. + \left(\int_0^1 (1-\chi)^{\alpha/k} \left[(e^{(1-\chi)s} - 1) |\psi'(wa_1)|^q + m(e^{s\chi} - 1) \left| \psi'\left(\frac{a_2}{m}\right) \right|^q \right] d\chi \right)^{1/q} \right] \\
& = \left(\frac{a_2-wa_1}{w+1} \right) \left(\frac{k}{\alpha+k} \right)^{1-1/q} \\
& \quad \times \left[\left\{ |\psi'(wa_1)|^q \left(\Gamma\left(\frac{\alpha}{k}+1\right) - \Gamma\left(\frac{\alpha}{k}+1, s\right) e^s s^{-\alpha/k-1} - \frac{1}{\alpha/k+1} \right) + m \left| \psi'\left(\frac{a_2}{m}\right) \right|^q \left(\frac{\Gamma(\alpha/k+1, -s) - \Gamma(\alpha/k+1)}{(-s)^{\alpha/k}s} - \frac{1}{\alpha/k+1} \right) \right\}^{1/q} \right. \\
& \quad + \left\{ |\psi'(wa_1)|^q \left((-1)^{\alpha/k-1} s^{-\alpha/k-1} \left(\Gamma\left(\frac{\alpha}{k}+1\right) - \Gamma\left(\frac{\alpha}{k}+1, -s\right) \right) - \frac{1}{\alpha/k+1} \right) \right. \\
& \quad \left. \left. + m \left| \psi'\left(\frac{a_2}{m}\right) \right|^q \left(\frac{(\alpha/k+1)e^s k s^{-\alpha/k-1} (\Gamma(\alpha/k+1, s) - \Gamma(\alpha/k+1))}{k+\alpha} - \frac{1}{\alpha/k+1} \right) \right\}^{1/q} \right],
\end{aligned} \tag{43}$$

which completes the proof. $\square$

5. Applications

Let consider the following two special means for different positive-real numbers $a_1 < a_2$.

(1) The arithmetic mean:

$$\mathcal{A}(a_1, a_2) = \frac{a_1 + a_2}{2}. \quad (44)$$

(2) The generalized log-mean:

$$\mathcal{L}_l(a_1, a_2) = \left[\frac{a_2^{l+1} - a_1^{l+1}}{(l+1)(a_2 - a_1)} \right]^{1/l}; \quad l \in \mathfrak{R} \setminus \{-1, 0\}. \quad (45)$$

Dragomir et al. [5] have proved that, for $s \in (0, 1)$, where $1 \leq l \leq 1/s$, the function $f(x) = x^{ls}$, $x > 0$ is s -convex function. Then, from Proposition 1, it is also s -exponential convex function for some fixed $s \in [\ln 2.5, 1)$.

Using Section 4, we have the following interesting results:

Proposition 2. Let $0 < a_1 < a_2$, $0 < w \leq 1$, and $q > 1$ such that $p^{-1} + q^{-1} = 1$. Then, for some fixed $s \in [\ln 2.5, 1)$, where $1 \leq l \leq 1/s$, we have

$$\begin{aligned} \left| \mathcal{A}((wa_1)^{ls}, a_2^{ls}) - \mathcal{L}_s^{ls}(wa_1, a_2) \right| &\leq \frac{ls(a_2 - wa_1)}{\sqrt{[p]2}} \\ &\times \left(\frac{1}{p+1} \right)^{1/p} \left(\frac{e^s - s - 1}{s} \right)^{1/q} \mathcal{A}^{1/q}((wa_1)^{(ls-1)q}, a_2^{(ls-1)q}). \end{aligned} \quad (46)$$

Proof. Consider the s -exponential convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 7 for $\alpha = 1 = k$, we obtain the required result. $\square$

Proposition 3. Let $0 < a_1 < a_2$, $0 < w \leq 1$ and $q \geq 1$. Then, for some fixed $s \in [\ln 2.5, 1)$, where $1 \leq l \leq 1/s$, we get

$$\begin{aligned} \left| \mathcal{A}((wa_1)^{ls}, a_2^{ls}) - \mathcal{L}_s^{ls}(wa_1, a_2) \right| &\leq \frac{ls(a_2 - wa_1)}{4^{(1-1/q)}} \\ &\times \left(\frac{2(s-2)e^s + 8e^{s/2} - s^2 - 2s - 4}{2s^2} \right)^{1/q} \\ &\mathcal{A}^{1/q}((wa_1)^{(ls-1)q}, a_2^{(ls-1)q}). \end{aligned} \quad (47)$$

Proof. Consider the s -exponential convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 8 for $\alpha = 1 = k$, the result is obvious.

Proposition 4. Let $0 < a_1 < a_2$, $0 < w \leq 1$ and $q > 1$, such that $p^{-1} + q^{-1} = 1$. Then, for some fixed $s \in [\ln 2.5, 1)$, where $1 \leq l \leq 1/s$, we obtain

$$\begin{aligned} &\left| \frac{2}{w+1} \left(\mathcal{A}((wa_1)^{ls}, a_2^{ls}) - \mathcal{L}_s^{ls}(wa_1, a_2) \right) \right| \\ &\leq \sqrt{[q]2} \frac{ls(a_2 - wa_1)}{w+1} \times \left(\frac{1}{p+1} \right)^{1/p} \left(\frac{e^s - s - 1}{s} \right)^{1/q} \\ &\mathcal{A}^{1/q}((wa_1)^{(ls-1)q}, a_2^{(ls-1)q}). \end{aligned} \quad (48)$$

Proof. Consider the s -exponential convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 9 for $\alpha = 1 = k$, the result is evident. $\square$

Proposition 5. Let $0 < a_1 < a_2$, $0 < w \leq 1$ and $q \geq 1$. Then, for some fixed $s \in [\ln 2.5, 1)$, where $1 \leq l \leq 1/s$, we have

$$\begin{aligned} &\left| \frac{2}{w+1} \left(\mathcal{A}((wa_1)^{ls}, a_2^{ls}) - \mathcal{L}_s^{ls}(wa_1, a_2) \right) \right| \leq \frac{ls(a_2 - wa_1)}{2(w+1)} \\ &\times \left(\frac{2(s-2)e^s + 8e^{s/2} - s^2 - 2s - 4}{2s^2} \right)^{1/q} \\ &\mathcal{A}^{1/q}((wa_1)^{(ls-1)q}, a_2^{(ls-1)q}). \end{aligned} \quad (49)$$

Proof. Consider the s -exponential convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 10 for $\alpha = 1 = k$, we obtain the required result.

At the end, let consider some applications of the integral inequalities obtained above, to find new bounds for the trapezoidal and midpoint formula.

For $a_2 > 0$, let $\mathcal{U}: 0 = \chi_0 < \chi_1 < \dots < \chi_{n-1} < \chi_n = a_2$ is a partition of $[0, a_2]$.

We denote

$$\mathcal{T}(\mathcal{U}, \psi) = \sum_{i=0}^{n-1} \left(\frac{\psi(\chi_i) + \psi(\chi_{i+1})}{2} \right) h_i, \quad (50)$$

$$\int_0^{a_2} \psi(x) dx = \mathcal{T}(\mathcal{U}, \psi) + \mathcal{R}(\mathcal{U}, \psi),$$

where $\mathcal{R}(\mathcal{U}, \psi)$ is the remainder term and $h_i = \chi_{i+1} - \chi_i$ for $i = 0, 1, 2, \dots, n-1$.

Using above notations, we are in position to prove the following error estimations. $\square$

Proposition 6. Suppose a mapping $\psi: (0, a_2] \rightarrow \mathfrak{R}$ is differentiable on $(0, a_2]$ with $a_2 > 0$. If $|\psi'|^q$ is s -exponential-type convex on $(0, a_2]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $s \in (0, 1]$, the remainder term satisfies the following error estimation:

$$|\mathcal{R}(\mathcal{U}, \psi)| \leq \frac{1}{2} \left(\frac{1}{p+1} \right)^{1/p} \left(\frac{e^s - s - 1}{s} \right)^{1/q} \times \sum_{i=0}^{n-1} h_i^2 \left[(|\psi'(\chi_i)|^q + |\psi'(\chi_{i+1})|^q) \right]^{1/q}. \quad (51)$$

Proof. Using Theorem 7 on subinterval $[\chi_i, \chi_{i+1}]$ of closed interval $[0, a_2]$, for all $i = 0, 1, 2, \dots, n-1$ and $w = \alpha = k = m = 1$, we get

$$\left| \left(\frac{\psi(\chi_i) + \psi(\chi_{i+1})}{2} \right) h_i - \int_{\chi_i}^{\chi_{i+1}} \psi(x) dx \right| \leq \frac{1}{2} \left(\frac{1}{p+1} \right)^{1/p} \left(\frac{e^s - s - 1}{s} \right)^{1/q} \times h_i^2 \left[(|\psi'(\chi_i)|^q + |\psi'(\chi_{i+1})|^q) \right]^{1/q}. \quad (52)$$

Summing inequality (52) over i from 0 to $n-1$ and using the property of modulus, we obtain the desired inequality (51). $\square$

Proposition 7. Suppose a mapping $\psi: (0, a_2] \rightarrow \mathfrak{R}$ is differentiable on $(0, a_2]$ with $a_2 > 0$. If $|\psi'|^q$ is s -exponential-type convex on $(0, a_2]$ for $q \geq 1$, then for some fixed $s \in (0, 1]$, the remainder term satisfies the following error estimation:

$$|\mathcal{R}(\mathcal{U}, \psi)| \leq \left(\frac{e^s - s - 1}{2s} \right)^{1/q} \sum_{i=0}^{n-1} h_i^2 \left[(|\psi'(\chi_i)|^q + |\psi'(\chi_{i+1})|^q) \right]^{1/q}. \quad (53)$$

Proof. Apply the same technique as in Proposition 6 but using Theorem 8.

6. Conclusion

In this article, the authors showed new generalizations of trapezium-type inequality for the new class of functions, the so-called (s, m) -exponential-type convex function ψ and for the products of two (s, m) -exponential-type convex functions ψ and ϕ . We have obtained refinements of the (H–H) inequality for functions using (s, m) -exponential-type convex and founded new bounds for special means and for the error estimates for the trapezoidal and midpoint formula. We hope that current work will attract the attention of researchers working in mathematical analysis, fractional calculus, quantum calculus, postquantum calculus, and other related fields.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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