Research Article

The q -Chlodowsky and q -Szász-Durrmeyer Hybrid Operators on Weighted Spaces

Harun Çiçek and Aydın İzgi

Department of Mathematics, Harran University, Sanlıurfa 63300, Turkey

Correspondence should be addressed to Harun Çiçek; haruncicek@harran.edu.tr

Received 2 July 2020; Revised 4 November 2020; Accepted 10 December 2020; Published 23 December 2020

Academic Editor: Mehdi Ghatee

Copyright © 2020 Harun Çiçek and Aydın İzgi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The main aim of this article is to introduce a new type of q -Chlodowsky and q -Szász-Durrmeyer hybrid operators on weighted spaces. To this end, we give approximation properties of the modified new q -Hybrid operators. Moreover, in the weighted spaces, we examine the rate of convergence of the modified new q -Hybrid operators by means of moduli of continuity. In addition, we derive Voronovskaja's type asymptotic formula for the related operators.

1. Introduction

Polynomial approach and the classical approximation theory constitute a basic research area in applied mathematics. The development of the approximation theory plays an important role in the numerical solution of partial differential equations, data processing sciences, and many other disciplines. For example, it is widely used in geometric modelling in the aerospace and automotive industries to calculate approximate values with basic functions. Works in this field go back to the 18th century and still continue as a powerful tool in scientific calculations. Furthermore, it is used in civil engineering projects to analyze the energy efficiency and earthquake resistance data of different types of buildings in thermography calculations and earthquake engineering.

Many generalized versions of these polynomials have been studied by many authors. Some of these studies are [1–13]. During the studies of approximation theory, many new operators and their generalizations were introduced with different spaces and variables. For example, on the interval $[0, \infty)$, $[0, 1]$ etc., q -analog type, K -Petree type, King's type, and Weighted space type operators are introduced.

In this study, the q -analog type operator is defined. The studies regarding the q -analog type operator are as follows.

First, the q -Bernstein polynomials were produced by Phillips [14]. When $q = 1$ is used, the results are the same as for classical operators. However, new operators with different properties are obtained for $q \neq 1$. Gupta [15] introduced and analyzed the approach characteristics of q -Durrmeyer operators. Gupta and Heping [16] identified the q -Durrmeyer operators and estimated the rate of convergence for continuous functions with the help of the moduli of continuity. In [2, 17], Mahmudov defined the King-type q -Szász operators. He obtained the rate of convergence on weighted spaces and a Voronovskaya-type theorem for these operators. Some other studies based on classical q -theory are [17–24].

In light of the above studies, the following result has been obtained. For a real-valued function, $f(u)$ defined on the interval $[0, \infty)$, the operators

$$\begin{aligned} & \sum_{k=0}^n P_{n,k,q}(u) \int_0^\infty P_{n,k,q}(t) h(t) d_q t, \\ & \sum_{k=0}^n S_{n,k,q}(u) \int_0^\infty S_{n,k,q}(t) h(t) d_q t, \\ & \sum_{k=0}^n S_{n,k,q}(u) \int_0^\infty P_{n,k,q}(t) h(t) d_q t, \end{aligned} \quad (1)$$

for the general operator kernels

$$P_{n,k,q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k} = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x), \quad (2)$$

$$S_{n,k,q}(y) = \frac{1}{E_q([n]_q y)} q^{k(k-1)/2} \frac{([n]_q y)^k}{[k]_q!}, \quad (3)$$

have been studied by many authors. On the contrary, the operator

$$\sum_{k=0}^n P_{n,k,q}(u) \int_0^\infty P_{n,k,q}(t) h(t) d_q t \quad (4)$$

has not been studied yet.

Therefore, we introduced the following operator:

$$H_{n,q}(h; u) = \frac{[n]_q}{b_n} \sum_{k=0}^n P_{n,k,q}\left(\frac{u}{b_n}\right) q^k \int_0^{\infty/1-q} S_{n,k,q}\left(\frac{t}{b_n}\right) h(q^k t) d_q t, \quad (5)$$

which is q -Chlodowsky and q -Szász-Durrmeyer hybrid operators on weighted spaces. Here, $P_{n,k,q}(x)$ and $S_{n,k,q}(y)$ are defined as in (2)-(3), $u \in [0, b_n]$, $0 < q < 1$, $h \in C[0, b_n]$, and b_n is an increasing and positive sequence with properties $\lim_{n \rightarrow \infty} b_n = \infty$ and $\lim_{n \rightarrow \infty} (b_n/[n]_q) = 0$. In this article, we intend to study the approximation properties of the operator $H_{n,q}(h; u)$. We produced our study by making use of [25].

The important terms of q -analysis which are used in this paper are given below, see [26, 27],

Given value of $q > 0$ and $n \in \mathbb{N}$, we define the q -integer $[n]_q$ by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \neq 1, \\ n, & \text{if } q = 1. \end{cases} \quad (6)$$

The q -factorial $[n]_q!$ is defined by

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \dots [1]_q, & \text{if } n = 1, 2, 3, \dots, \\ 1, & \text{if } n = 0, \end{cases} \quad (7)$$

for $0 \leq k \leq n$; we define the q -binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}. \quad (8)$$

The q -binomial can be written in the following forms:

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q, \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q. \end{aligned} \quad (9)$$

Exponential function e^z has two q -analogs, see [26, 27]:

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!} = \frac{1}{(1-(1-q)z)_q^\infty}, \quad |z| < \frac{1}{1-q}, \quad |q| < 1, \quad (10)$$

$$E_q(z) = \prod_{s=0}^{\infty} (1 + (1-q)q^s z) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{[k]_q!} = (1 + (1-q)z)_q^\infty, \quad |q| < 1,$$

where $(1-u)_q^\infty = \prod_{s=0}^{\infty} (1-q^s u)$. It is easily observed that

$$e_q(z)E_q(-z) = e_q(-z)E_q(z) = 1. \quad (11)$$

The definite q -Jackson integrals and the q -improper integrals of the function h are defined by

$$\int_0^b h(t) d_q t := b(1-q) \sum_{k=0}^{\infty} h(bq^k) q^k, \quad b > 0, \quad (12)$$

$$\int_0^\infty h(t) d_q t := (1-q) \sum_{k=0}^{\infty} h\left(\frac{q^k}{B}\right) \frac{q^k}{B}, \quad B > 0. \quad (13)$$

The series on the right-hand side in (12) is guaranteed to be convergent if the function h has the property $|h(u)| < Cu^\alpha$ in a right neighborhood of $u = 0$ for $C > 0$, $\alpha > -1$.

The q -Gamma functions are defined in two ways. These are

$$\Gamma_q(u) = \int_0^{1/1-q} t^{u-1} E_q(-qt) d_q t, \quad (14)$$

$$\gamma_q^B(u) = \int_0^{\infty/B(1-q)} t^{u-1} e_q(-t) d_q t. \quad (15)$$

For every B and $u > 0$, one has

$$\Gamma_q(u) = K(B; u) \gamma_q^B(u), \quad (16)$$

where $K(B; u) = (1/1+B)B^u (1 + (1/B))_q^u (1+B)_q^{1-u}$. Especially, for any positive integer m ,

$$K(B; m) = q^{m(m-1)/2} \text{ and } \Gamma_q(m) = q^{m(m-1)/2} \gamma_q^B(m), \quad (17)$$

see [28].

The present note deals with the study of the q -Chlodowsky and q -Szász-Durrmeyer hybrid operators on weighted spaces. Firstly, we estimate the moments for the $H_{n,q}(h; u)$ operators.

We also study the rate of convergence for these operators $H_{n,q}(h; u)$. Furthermore, definitions and some properties for weighted spaces are given. We guess the order of approximation by weighted Voronovskaya-type theorem.

2. Estimation Moments

Here, we will prove $H_{n,q}(t^m; u)$ for $m = 0, 1, 2$. By the definition of the q -Gamma function γ_q in (15), we have

$$\begin{aligned}
 \int_0^{\infty/1-q} (q^k t)^m S_{n,k,q}\left(\frac{t}{b_n}\right) d_q t &= \int_0^{\infty/1-q} (q^k t)^m \frac{1}{E_q([n]_q(t/b_n))} q^{k(k-1)/2} \frac{([n]_q(t/b_n))^k}{[k]_q!} d_q t \\
 &= q^{km} \int_0^{\infty/1-q} t^m e_q\left(-[n]_q \frac{t}{b_n}\right) q^{k(k-1)/2} \frac{[n]_q^k t^k}{[k]_q! b_n^k} d_q t \\
 &= q^{km} \left[\frac{b_n^m}{[n]_q^{m+1} [k]_q!} q^{k(k-1)/2} \int_0^{\infty/1-q} \left([n]_q \frac{t}{b_n}\right)^{k+m} e_q\left(-[n]_q \frac{t}{b_n}\right) [n]_q d_q t \right] \\
 &= q^{km} \left[\frac{b_n^{m+1}}{[n]_q^{m+1} [k]_q!} q^{k(k-1)/2} \int_0^{\infty/(1-q)[n]_q^{-1}} (v)^{k+m} e_q(-v) d_q v \right] \\
 &= q^{km} \left[\frac{b_n^{m+1}}{[n]_q^{m+1} [k]_q!} q^{k(k-1)/2} \gamma_q^{[n]_q^{-1}}(k+m+1) \right] \\
 &= q^{km} \left[\frac{b_n^{m+1}}{[n]_q^{m+1} [k]_q!} q^{k(k-1)/2} \frac{\Gamma_q(k+m+1)}{q^{(k+m+1)(k+m)/2}} \right] = q^{km} \left[\frac{b_n^{m+1}}{[n]_q^{m+1} [k]_q!} q^{k(k-1)/2} \frac{[k+m]_q!}{q^{(k+m+1)(k+m)/2}} \right],
 \end{aligned} \tag{18}$$

where, for $m = 0$,

$$\int_0^{\infty/1-q} (q^k t)^0 S_{n,k,q}\left(\frac{t}{b_n}\right) d_q t = \frac{b_n}{[n]_q} q^{-k}, \tag{19}$$

for $m = 1$,

$$\int_0^{\infty/1-q} (q^k t)^1 S_{n,k,q}\left(\frac{t}{b_n}\right) d_q t = \frac{[k+1]_q b_n^2}{[n]_q^2} q^{-k-1}, \tag{20}$$

and for $m = 2$,

$$\int_0^{\infty/1-q} (q^k t)^2 S_{n,k,q}\left(\frac{t}{b_n}\right) d_q t = \frac{[k+1]_q [k+2]_q b_n^3}{[n]_q^3} q^{-k-3}. \tag{21}$$

Lemma 1. For $u \in [0, \infty)$, $0 < q < 1$, and $h \in C[0, \infty)$, we have

$$H_{n,q}(1; u) = 1, \tag{22}$$

$$H_{n,q}(t; u) = u + \frac{b_n}{q[n]_q}, \tag{23}$$

$$H_{n,q}(t^2; u) = u^2 + u \frac{((q^3 + 2q^2 + 1)b_n - q^3 u)}{q^3 [n]_q} + \frac{(q+1)b_n^2}{q^3 [n]_q^2}. \tag{24}$$

Proof. Using (19), we obtain

$$\begin{aligned}
 H_{n,q}(1; u) &= \frac{[n]_q}{b_n} \sum_{k=0}^n P_{n,k,q}\left(\frac{u}{b_n}\right) q^k \int_0^{\infty/1-q} S_{n,k,q}\left(\frac{t}{b_n}\right) d_q t \\
 &= \frac{[n]_q}{b_n} \sum_{k=0}^n P_{n,k,q}\left(\frac{u}{b_n}\right) q^k \left(\frac{b_n}{[n]_q} q^{-k} \right) = \sum_{k=0}^n P_{n,k,q}\left(\frac{u}{b_n}\right) = 1.
 \end{aligned} \tag{25}$$

Using (20) and $[k+1]_q = q[k]_q + 1$, we obtain

$$\begin{aligned}
 H_{n,q}(t, u) &= \frac{[n]_q}{b_n} \sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) q^k \int_0^{\infty/1-q} S_{n,k,q} \left(\frac{t}{b_n} \right) (q^k t) d_q t \\
 &= \frac{[n]_q}{b_n} \sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) q^k \left(\frac{[k+1]_q b_n^2}{[n]_q^2} q^{-k-1} \right) = \sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) \left(\frac{[k+1]_q b_n}{[n]_q q} \right) \\
 &= \frac{b_n}{q} \left[\sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) \frac{q[k]_q}{[n]_q} + \sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) \frac{1}{[n]_q} \right] = \frac{b_n}{q} \left[\sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) \frac{q[k]_q}{[n]_q} + \frac{1}{[n]_q} \right] \\
 &= \frac{b_n}{q} \left[q \frac{u}{b_n} \sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) + \frac{1}{[n]_q} \right] = u + \frac{b_n}{q[n]_q}.
 \end{aligned} \tag{26}$$

Using (20) and $[k+1]_q [k+2]_q = q^3 [k]_q^2 + (2q^2 + 1)[k]_q + q + 1$, we obtain

$$\begin{aligned}
 H_{n,q}(t^2; u) &= \frac{[n]_q}{b_n} \sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) q^k \int_0^{\infty/1-q} S_{n,k,q} \left(\frac{t}{b_n} \right) (q^k t)^2 d_q t \\
 &= \frac{[n]_q}{b_n} \sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) q^k \left(\frac{[k+1]_q [k+2]_q b_n^3}{[n]_q^3} q^{-k-3} \right) = \sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) \left(\frac{[k+1]_q [k+2]_q b_n^2}{[n]_q^2 q^3} \right) \\
 &= \frac{b_n^2}{q^3} \left[\sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) \frac{q^3 [k]_q^2}{[n]_q^2} + \sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) \frac{(2q^2 + 1)[k]_q}{[n]_q^2} + \sum_{k=0}^n P_{n,k,q} \left(\frac{u}{b_n} \right) \frac{q+1}{[n]_q^2} \right] \\
 &= \frac{b_n^2}{q^3} \left[\frac{q^3}{b_n^2} \frac{[n]_q - 1}{[n]_q} u^2 + \frac{(q^3 + 2q^2 + 1)}{[n]_q b_n} u + \frac{q+1}{[n]_q^2} \right] = u^2 - \frac{1}{[n]_q} u^2 + \frac{(q^3 + 2q^2 + 1)b_n}{q^3 [n]_q} u + \frac{(q+1)b_n^2}{q^3 [n]_q^2} \\
 &= u^2 + u \frac{((q^3 + 2q^2 + 1)b_n - q^3 u)}{q^3 [n]_q} + \frac{(q+1)b_n^2}{q^3 [n]_q^2},
 \end{aligned} \tag{27}$$

which completes the proof. $\square$

$$H_{n,q}((t-u); u) = \frac{b_n}{q[n]_q},$$

Lemma 2. Let $u \in [0, \infty)$, $0 < q < 1$, and $h \in C[0, \infty)$, and we have

$$H_{n,q}((t-u)^2; u) = u \frac{((q^3 + 1)b_n - q^3 u)}{[n]_q q^3} + \frac{(q+1)b_n^2}{q^3 [n]_q^2}. \tag{28}$$

Proof. By Lemma 1, we have

$$\begin{aligned}
 H_{n,q}((t-u); u) &= H_{n,q}(t; u) - uH_{n,q}(1; u) = u + \frac{b_n}{q[n]_q} - u = \frac{b_n}{q[n]_q}, \\
 H_{n,q}((t-u)^2; u) &= H_{n,q}(t^2; u) - 2uH_{n,q}(t; u) + u^2H_{n,q}(1; u) \\
 &= u^2 + u \frac{((q^3 + 2q^2 + 1)b_n - q^3u)}{q^3[n]_q} + \frac{(q+1)b_n^2}{q^3[n]_q^2} - 2u\left(u + \frac{b_n}{q[n]_q}\right) + u^2 \\
 &= u \frac{((q^3 + 1)b_n - q^3u)}{[n]_q q^3} + \frac{(q+1)b_n^2}{q^3[n]_q^2},
 \end{aligned} \tag{29}$$

which completes the proof. $\square$

By simple calculations, we obtain

$$H_{n,q}((t-u)^2; u) \leq \frac{b_n^2}{q^6[n]_q} + \frac{2b_n^2}{q^3[n]_q^2} \tag{30}$$

here; since $n \rightarrow \infty$ will be $q_n \rightarrow 1$ and $[n]_q = n$, the following equation is written:

$$H_{n,q}((t-u)^2; u) \leq \frac{3b_n^2}{n}, \tag{31}$$

where $H_{n,q}((t-u)^2; u)$ appears to be constrained in the above inequality, and this result is available in [25]. Likewise, A. İzgi obtained the following result in [25]:

$$H_{n,q}((t-u)^4; u) \leq \frac{144b_n^4}{n^2}. \tag{32}$$

3. Approximation of $H_{n,q}(h; u)$ in Weighted Spaces

In this section, we use Gadjeiev's Korovkin-type theorems on the weighted spaces [4, 29]. Let B_p be the set of all functions h over the real line, $p(u) = 1 + u^2$, where $u \in (-\infty, \infty) = \mathbb{R}$ such that

$$|h(u)| \leq M_{hp(u)}, \tag{33}$$

where M_h is a positive constant depending on the function h . Now, let

$$\begin{aligned}
 C_p(\mathbb{R}) &= \{h \in B_p(\mathbb{R}), h \text{ is continuous}\}, \\
 C_p^k(\mathbb{R}) &= \left\{h \in C_p(\mathbb{R}), \lim_{|x| \rightarrow \infty} \frac{|h(u)|}{p(u)} = K_h < \infty\right\}.
 \end{aligned} \tag{34}$$

It is clear that $C_p^k(\mathbb{R}) \subset C_p(\mathbb{R}) \subset B_p(\mathbb{R})$, where $B_p(\mathbb{R})$ is the linear normed space with the norm

$$\|h\|_p = \sup_{u \in (-\infty, \infty)} \frac{|h(u)|}{p(u)}, \quad h \in B_p. \tag{35}$$

Let b_n be an increasing and positive sequence with features

$$\begin{aligned}
 \lim_{n \rightarrow \infty} q_n &= 1, \\
 \lim_{n \rightarrow \infty} b_n &= \infty \text{ and } \lim_{n \rightarrow \infty} \frac{b_n}{[n]_q} = 0.
 \end{aligned} \tag{36}$$

Lemma 3. $H_{n,q}(h; u)$ defined in (5) is a sequence of positive and linear operators that move $C_p[0, b_n]$ to $\beta_p[0, b_n]$. So, there is

$$\lim_{n \rightarrow \infty} H_{n,q}(\rho(t); u) = \rho(u), \tag{37}$$

for $\rho(u) \in C_p[0, b_n]$ on $[0, b_n]$.

Proof. Taking advantage of equations (22) and (24), we have

$$\begin{aligned}
 H_{n,q}(\rho(t); u) &= \rho(u) + u \frac{((q^3 + 2q^2 + 1)b_n - q^3u)}{q^3[n]_q} \\
 &\quad + \frac{(q+1)b_n^2}{q^3[n]_q^2}.
 \end{aligned} \tag{38}$$

Therefore, $\|H_{n,q}(h; u)\|_{p, [0, b_n]}$ is uniformly bounded on $[0, b_n]$, since

$$\lim_{n \rightarrow \infty} \sup_{u \in [0, b_n]} u \frac{((q^3 + 2q^2 + 1)b_n - q^3u)}{q^3[n]_q} + \frac{(q+1)b_n^2}{q^3[n]_q^2} = 0, \tag{39}$$

under the condition in (36) that completes the proof. $\square$

Theorem 1. Let $h \in C_p^k[0, b_n]$. Then,

$$\lim_{n \rightarrow \infty} \|H_{n,q}h - h\|_{p, [0, b_n]} = 0. \tag{40}$$

Proof. From Lemma 1, we have

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \|H_{n,q}(1; u) - 1\|_{\rho, [0, b_n]} &= 0, \\
 \lim_{n \rightarrow \infty} \|H_{n,q}(t; u) - u\|_{\rho, [0, b_n]} &= \lim_{n \rightarrow \infty} \sup_{u \in [0, b_n]} \left| \frac{b_n}{q[n]_q} \frac{1}{1+u^2} \right| = \lim_{n \rightarrow \infty} \frac{b_n}{q[n]_q} = 0, \\
 \lim_{n \rightarrow \infty} \|H_{n,q}(t^2; u) - u^2\|_{\rho, [0, b_n]} &= \lim_{n \rightarrow \infty} \sup_{u \in [0, b_n]} \left| \left(u \frac{(q^3 + 2q^2 + 1)b_n - q^3 u}{q^3 [n]_q} + \frac{(q+1)b_n^2}{q^3 [n]_q^2} \right) \frac{1}{1+u^2} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{(q^3 + 2q^2 + 1)^2 b_n^2}{4q^6 [n]_q} + \frac{(q+1)b_n^2}{q^3 [n]_q^2} = 0,
 \end{aligned} \tag{41}$$

according to (36), which completes the proof. $\square$

for every $h \in C_\rho^k[0, b_n]$. A similar definition can be found in [5, 25].

4. Main Results

Here, we estimate the rate of approximation of the $H_{n,q}(h; u)$ hybrid operators. The following theorem gives the rate of approximation of the sequence of $H_{n,q}(h; u)$ operators in terms of moduli of continuity of a function $h \in C_\rho^k[0, b_n]$. For $h \in C_\rho^k[0, b_n]$, the moduli of continuity is defined as follows:

$$\omega(h, \delta) = \sup\{|h(t) - h(u)| : t, u \in [a, b], |t - u| \leq \delta\}, \tag{42}$$

where $\delta \rightarrow 0$.

The weighted moduli of continuity is defined as follows:

$$\Lambda_n(h; \delta) = \sup_{\substack{|h| \leq \delta \\ u \in [0, b_n]}} \frac{|h(u+a) - h(u)|}{(1+u^2)(1+a^2)}. \tag{43}$$

It is seen that they provide the characteristics of the continuity module. In what follows,

$$\lim_{\delta \rightarrow 0} \Lambda_n(h, \delta) = 0,$$

$$|h(t) - h(u)| \leq (1 + (t-u)^2)(1+u^2) \left(1 + \frac{|t-u|}{\delta} \right) \Lambda_n(h, \delta), \tag{44}$$

Theorem 2. Let $h \in C_\rho^k[0, b_n]$, and we have

$$\|H_{n,q}(h; u) - h(u)\|_{\rho, [0, b_n]} \leq \mu(b_n, n, q) \Lambda_n \left(h, \sqrt{\frac{b_n^2}{q^3 [n]_q}} \right), \tag{45}$$

where $\mu(b_n, n, q)$ is the positive constant for $\lim_{n \rightarrow \infty} q_n = 1$ and $\lim_{n \rightarrow \infty} b_n/[n]_q = 0$.

Proof. From the well-known features of modulus of continuity, we have

$$|h(t) - h(u)| \leq (1 + (t-u)^2)(1+u^2) \left(1 + \frac{|t-u|}{\delta} \right) \Lambda_n(h, \delta). \tag{46}$$

Then, H_n is a linear-positive operator, and we write

$$\begin{aligned}
 |H_{n,q}(h(t); u) - h(u)| &\leq H_{n,q}(|h(t) - h(u)|; u) \\
 &\leq \Lambda_n(h, \delta) (1+u^2) H_{n,q} \left((1 + (t-u)^2) \left(1 + \frac{|t-u|}{\delta} \right); u \right) \\
 &\leq \Lambda_n(h, \delta) (1+u^2) \left[H_{n,q}((1 + (t-u)^2); u) + H_{n,q} \left(\left(\frac{|t-u|}{\delta} \right); u \right) \right],
 \end{aligned} \tag{47}$$

from Lemmas 1-2, (31), and (32), we obtain

$$\begin{aligned} H_{n,q}\left(\left(1+(t-u)^2\right); u\right) &= H_{n,q}(1; u) + H_{n,q}\left((t-u)^2; u\right) \\ &\leq 1 + \frac{b_n^2}{q^6 [n]_q} + \frac{2b_n^2}{q^3 [n]_q^2} \leq \mu_1(b_n, n, q), \end{aligned} \quad (48)$$

and using Cauchy–Schwartz inequality

$$H_{n,q}\left(\left(1+(t-u)^2\right)\left(1+\frac{|t-u|}{\delta}\right); u\right) \leq H_{n,q}\left(\left(1+(t-u)^2\right)^2; u\right)^{1/2} H_{n,q}\left(\left(\frac{|t-u|}{\delta}\right)^2; u\right)^{1/2}. \quad (49)$$

We can write by (32)-(33) that

$$\begin{aligned} H_{n,q}\left(\left(1+(t-u)^2\right)^2; u\right) &= H_{n,q}(1; u) + 2H_{n,q}\left((t-u)^2; u\right) + H_{n,q}\left((t-u)^4; u\right) \leq 1 + 2\mu_1 + \frac{144b_n^4}{n^2} \leq \mu_2(b_n, n, q), \\ H_{n,q}\left(\left(\frac{|t-u|}{\delta}\right)^2; u\right)^{1/2} &= \frac{1}{\delta} H_{n,q}\left((t-u)^2; u\right)^{1/2} \frac{1}{\delta} \frac{b_n^2}{q^3 [n]_q} \left(\frac{1}{q^3} + \frac{2}{[n]_q}\right) \leq \mu_3(b_n, n, q), \end{aligned} \quad (50)$$

where $\delta = b_n^2/q^3 [n]_q$.

As a result

$$\left\|H_{n,q}(h; u) - h(u)\right\|_{\rho, [0, b_n]} \leq \mu(b_n, n, q) \Lambda_n\left(h, \frac{b_n^2}{q^3 [n]_q}\right), \quad (51)$$

where

$$\mu(b_n, n, q) = \mu_1(b_n, n, q) + \sqrt{\mu_2(b_n, n, q)} \mu_3(b_n, n, q), \quad (52)$$

which completes the proof.

Now, we prove a Voronovskaja-type result for the $H_{n,q}(h(t); u)$ operators. $\square$

Theorem 3. Let $h, D_q h(u), D_q^2 h(u) \in C_\rho^k[0, b_n]$, and $u \in [0, b_n]$ be fixed, and we have

$$\lim_{n \rightarrow \infty} \frac{[n]_q}{b_n} \{H_{n,q}(h; u) - h(u)\} = D_q h(u) + u D_q^2 h(u). \quad (53)$$

Proof. By q -Taylor formula for $h \in C_\rho^k[0, b_n]$ and $0 < q < 1$,

$$\begin{aligned} h(t) &= h(u) + (t-u)_q D_q h(u) \\ &\quad + \frac{1}{[2]_q} (t-u)_q^2 D_q^2 h(u) + \mathcal{O}(t; u) (t-u)_q^2, \end{aligned} \quad (54)$$

where $(t-u)_q^2 = (t-u)(t-qu)$ and

$$\mathcal{O}(t; u) = \begin{cases} \frac{h(t) - h(u) - (t-u)_q D_q h(u) - 1/[2]_q (t-u)_q^2 D_q^2 h(u)}{(t-u)_q^2}, & t \neq u, \\ 0, & t = u. \end{cases} \quad (55)$$

By implementing q -L'Hospital's rule twice,

$$\lim_{t \rightarrow u} \mathcal{O}(t; u) = \lim_{t \rightarrow u} \frac{D_q^2 h(t) - D_q^2 h(u)}{2} = 0. \quad (56)$$

Then, $\mathcal{O}(u; u) = 0$ and $\mathcal{O}(\dots; u) \in C_\rho^k[0, b_n]$. As a result, we can write

$$\begin{aligned} H_{n,q}(h; u) - h(u) &= H_{n,q}\left((t-u)_q; u\right) \\ &= D_q h(u) + \frac{1}{[2]_q} H_{n,q}\left((t-u)_q^2; u\right) D_q^2 h(u) \\ &\quad + H_{n,q}\left(\mathcal{O}(t; u) (t-u)_q^2; u\right). \end{aligned} \quad (57)$$

Now, let us use the Cauchy–Schwarz inequality in the last term of the last equation:

$$H_{n,q}(\bar{\mathcal{O}}(t; u)(t-u)_q^2; u) \leq \{H_{n,q}(t-u)_q^4; u\}^{1/2} \{H_{n,q}(\bar{\mathcal{O}}^2(t; u); u)\}^{1/2}. \quad (58)$$

Since $\bar{\mathcal{O}}(.,.; u) \in C_\rho^k[0, b_n]$ and $\bar{\mathcal{O}}(t; u) \longrightarrow \bar{\mathcal{O}}(u; u) \longrightarrow 0$ as $t \longrightarrow u$, applying Lemma 2 and (31) and (32),

$$\begin{aligned} \lim_{n \rightarrow \infty} H_{n,q}(\bar{\mathcal{O}}^2(t; u); u) &= \bar{\mathcal{O}}^2(u; u) = 0, \\ \lim_{n \rightarrow \infty} H_{n,q}(t-u)_q^4; u &< \infty. \end{aligned} \quad (59)$$

As a result

$$\begin{aligned} H_{n,q}(h; u) - h(u) &= H_{n,q}((t-u)_q; u) D_q h(u) + \frac{1}{[2]_q} H_{n,q}((t-u)_q^2; u) D_q^2 h(u) \\ &= D_q h(u) \frac{b_n}{q[n]_q} + \frac{1}{[2]_q} D_q^2 h(u) \left(u \frac{((q^3+1)b_n - q^3 u)}{[n]_q q^3} + \frac{(q+1)b_n^2}{q^3 [n]_q^2} \right), \end{aligned} \quad (60)$$

from (33), and we have

$$\lim_{n \rightarrow \infty} \frac{[n]_q}{b_n} \{H_{n,q}(h; u) - h(u)\} = D_q h(u) + u D_q^2 h(u), \quad (61)$$

which completes the proof. $\square$

5. Conclusion

In this paper, the approximation properties and rate of convergence of q -Chlodowsky and q -Szász-Durrmeyer hybrid operators in weighted spaces are investigated.

For further research in this topic, it would be interesting to study whether the quality of the approximation pythagorean fuzzy set operators, q -statistical convergence operators, and q -complex operators directly influence the quality of the approximation of the characteristics. Some studies can be used for future research, such as [30–34].

Data Availability

All data generated or analyzed during this study are included in this published article. They are cited at relevant places within the text as references.

Conflicts of Interest

The authors do not have any conflicts of interest.

Authors' Contributions

All authors contributed equally to this paper.

References

- [1] O. Agratini and O. Dogru, “Weighted approximation by sq -szász-king type operators,” *Taiwanese Journal of Mathematics*, vol. 14, no. 4, pp. 1283–1296, 2010.
- [2] M. Becker, “Global approximation theorems for szász-mirakjan and baskakov operators in polynomial weight spaces,” *Indiana University Mathematics Journal*, vol. 27, no. 1, pp. 127–142, 1978.
- [3] T. Coşkun, “Weighted approximation of continuous functions by sequences of linear positive operators,” in *Proceedings of the Indian Academy of Sciences-Mathematical Sciences*, vol. 110, pp. 357–362, Springer, 2000.
- [4] E. Gadjieva and E. Ibikli, “Weighted approximation by bernstein-chlodowsky polynomials,” *Indian Journal of Pure and Applied Mathematics*, vol. 30, pp. 83–88, 1999.
- [5] N. Ispir, “On modified baskakov operators on weighted spaces,” *Turkish Journal of Mathematics*, vol. 25, no. 3, pp. 355–365, 2001.
- [6] A. İzgi, “Approximation by composition of szász-mirakjan and durrmeyer-chlodowsky operators,” *Eurasian Mathematical Journal*, vol. 3, no. 1, pp. 63–71, 2012.
- [7] A.-J. López-Moreno, “Weighted simultaneous approximation with baskakov type operators,” *Acta Mathematica Hungarica*, vol. 104, no. 1–2, pp. 143–151, 2004.
- [8] V. N. Mishra, A. R. Devdhar, and R. Gandhi, “Global approximation theorems for the generalized szász-mirakjan type operators in exponential weight spaces,” *Applied Mathematics and Computation*, vol. 336, pp. 206–214, 2018.
- [9] M. Mursaleen, V. Karakaya, M. Erturk, and F. Gursay, “Weighted statistical convergence and its application to korovkin type approximation theorem,” *Applied Mathematics and Computation*, vol. 218, no. 18, pp. 9132–9137, 2012.
- [10] D. Vishwakarma, A. K. Artee, and A. Kumar, “Multivariate q -bernstein-schurer-kantorovich operators,” *Journal of Mathematics and System Science*, vol. 6, pp. 234–241, 2016.

- [11] Z. Walczak, "On some linear positive operators in exponential weighted spaces," *Mathematical Communications*, vol. 8, no. 1, pp. 77–84, 2003.
- [12] R. Yadav, R. Meher, and V. N. Mishra, *Results on Bivariate Szász-Mirakyan Type Operators in Polynomial Weight Spaces*, <http://arxiv.org/abs/1911.08898>, 2019.
- [13] G. İçöz et al., "Weighted approximation properties of stancu type modification of q -szász-durrmeyer operators," *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, vol. 65, no. 1, pp. 87–104, 2016.
- [14] G. M. Phillips, "Bernstein polynomials based on the q -integers," *Annals of Numerical Mathematics*, vol. 4, pp. 511–518, 1996.
- [15] V. Gupta, "Some approximation properties of q -durrmeyer operators," *Applied Mathematics and Computation*, vol. 197, no. 1, pp. 172–178, 2008.
- [16] V. Gupta and W. Heping, "The rate of convergence of q -Durrmeyer operators for 0," *Mathematical Methods in the Applied Sciences*, vol. 31, no. 16, pp. 1946–1955, 2008.
- [17] N. I. Mahmudov and H. Kaffaoglu, "On q -Szász-Durrmeyer operators," *Central European Journal of Mathematics*, vol. 8, no. 2, pp. 399–409, 2010.
- [18] D. Barbosu, A.-M. Acu, and C. V. Muraru, "On certain GBS-Durrmeyer operators based on q -integers," *Turkish Journal of Mathematics*, vol. 41, no. 2, pp. 368–380, 2017.
- [19] A. Lupas, "A q -analogue of the bernstein operator," in *Seminar On Numerical And Statistical Calculus*, vol. 9, University of Cluj-Napoca, Cluj-Napoca, Romania, 1987.
- [20] H. Oruc and N. Tuncer, "On the convergence and iterates of q -bernstein polynomials," *Journal of Approximation Theory*, vol. 117, no. 2, pp. 301–313, 2002.
- [21] S. Ostrovska, " q -bernstein polynomials and their iterates," *Journal of Approximation Theory*, vol. 123, no. 2, pp. 232–255, 2003.
- [22] R. Ruchi, N. Ispir, and P. N. Agrawal, "Kantorovich variant of a new kind of q -Bernstein-Schurer operators," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 16, pp. 5687–5706, 2017.
- [23] E. Simsek and T. Tunc, "On the construction of q -analogues for some positive linear operators," *Filomat*, vol. 31, no. 13, pp. 4287–4295, 2017.
- [24] V. S. Videnskii, "On some classes of q -parametric positive linear operators," in *Selected Topics in Complex Analysis*, pp. 213–222, Springer, Berlin, Germany, 2005.
- [25] A. Izgi, "Approximation by a composition of chlodowsky operators and szász-durrmeyer operators on weighted spaces," *LMS Journal of Computation and Mathematics*, vol. 16, pp. 388–397, 2013.
- [26] A. Aral, V. Gupta, R. P. Agarwal et al., *Applications of Q-Calculus in Operator Theory*, Springer, Berlin, Germany, 2013.
- [27] V. Kac and P. Cheung, *Quantum Calculus*, Springer Science and Business Media, Berlin, Germany, 2001.
- [28] A. De Sole and V. Kac, "On integral representations of q -gamma and q -beta functions," 2003, <http://arxiv.org/abs/0302032>.
- [29] A. D. O. Gadzhiev, "The convergence problem for a sequence of positive linear operators on unbounded sets, and theorems analogous to that of pp korovkin," in *Doklady Akademii Nauk*, vol. 218, pp. 1001–1004, Russian Academy of Sciences, Moscow, Russia, 1974.
- [30] B. Baxhaku and F. Berisha, "Statistical approximation to Chlodowsky type q -Bernstein-Schurer-Stancu-Kantorovich operators," *Mathematical Sciences and Applications E-Notes*, vol. 5, no. 1, pp. 108–121, 2017.
- [31] S. G. Gal and V. Gupta, "Approximation of vector-valued functions by q -durrmeyer operators with applications to random and fuzzy approximation," *Analele Universitatii Oradea Fasc. Matematica*, vol. 16, pp. 233–242, 2009.
- [32] N. Vijender, A. K. B. Chand, M. A. Navascues, and M. V. Sebastian, "Quantum α -fractal approximation," *International Journal of Computer Mathematics*, pp. 1–14, 2020.
- [33] L. Wang, H. Garg, and N. Li, "Pythagorean fuzzy interactive hamacher power aggregation operators for assessment of express service quality with entropy weight," *Soft Computing*, pp. 1–21, 2020.
- [34] L. Wang and N. Li, "Pythagorean fuzzy interaction power bonferroni mean aggregation operators in multiple attribute decision making," *International Journal of Intelligent Systems*, vol. 35, no. 1, pp. 150–183, 2020.