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*Research Article*

# On Marginal Automorphisms of Free Nilpotent Lie Algebras

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Let $L$ be the free nilpotent Lie algebra of finite rank over a field of characteristic zero. We define the concepts of marginal ideals and marginal automorphisms of $L$, and we give some results on marginal automorphisms.

## 1. Introduction

The concept of marginal subgroups and marginal automorphisms of a group $G$ has been already studied by many authors (for details, see [1–3]). The group of marginal automorphisms of $G$ is very important in studying its automorphism group $\mathrm{Aut}(G)$. For a suitable nonempty subset $W$ of a free group, one-way construct marginal automorphism group of $G$ carries the required properties. Lie algebra analogues to the marginal subgroup of a group were considered by Stewart [4]. Stewart has given nonassociative algebra versions of the concepts of verbal and marginal subgroups of a group and specialized in Lie algebras to obtain stronger results.

In this paper, we give some properties of marginal ideal and marginal automorphism groups of free nilpotent Lie algebra of finite rank.

## 2. Preliminaries

Let $F$ be the free Lie algebra freely generated by a finite set $\{x_1, x_2, \ldots, x_m\}$, $m \geq 2$, over a field $K$ of characteristic zero, and let $L = F/\gamma_n(F)$, where $\gamma_n(F)$ is the $n$th lower central term of $F$. It is clear that $L$ is the free nilpotent of class $n-1$ Lie algebra. For the Lie multiplication, we use the commutator notation.

For any element $w \in F$, we denote by $\overline{w}$ the natural image of $w$ in $L$ and we say that $x_i$ appears in $w$ if the word $w$ contains $x_i$.

*Definition 1.* Let $G$ be any Lie algebra and let $W$ be a nonempty subset of $F$. For $w = w(x_1, \ldots, x_m) \in W$, we introduce the set of values of $w$ in $G$ as

$$w(G) := \{w(u_1, \ldots, u_m) \mid u_1, \ldots, u_m \in G\} \tag{1}$$

and the set of all $w$ values in $G$ as

$$W(G) := \{w(u_1, \ldots, u_m) \mid w \in W\}. \tag{2}$$

(1) The verbal subalgebra $A_W(G)$ of $G$ with respect to $W$ is the subalgebra generated by the set $W(G)$

(2) The verbal ideal $I_W(G)$ of $G$ with respect to $W$ is ideal which is generated by the set $W(G)$

*Definition 2.* Let $G$ be any Lie algebra, and let $W$ be a nonempty subset of $F$. The marginal subspace, $S_W^M(G)$ of $G$ with respect to $W$, is defined to be the following set:

$$\begin{aligned} S_W^M(G) = \{\alpha \in L \mid & w(u_1, \ldots, u_i + \beta\alpha, \ldots u_m) \\ & = w(u_1, \ldots, u_i, \ldots u_m), \\ & \text{for all } w \in W, \beta \in K \text{ and } u_i \in G,\ i = 1, \ldots, m\}. \end{aligned} \tag{3}$$

(1) The largest subalgebra of $G$ contained in $S_W^M(G)$ is called the marginal subalgebra with respect to $W$ and it is denoted by $A_W^M(G)$

(2) The largest ideal of $G$ contained in $S_W^M(G)$ is called the marginal ideal with respect to $W$ and it is denoted by $I_W^M(G)$

If the set $W$ has a single element $w$, then we will write $A_w(G), I_w(G), A_w^M(G)$, and $I_w^M(G)$, instead of $A_W(G)$, $I_W(G)$, $A_W^M(G)$, and $I_W^M(G)$, respectively.

Throughout this paper, we take the subset $W$ of $F$ as $W = \{w_1, w_2, \ldots, w_t\}, x$ where $w_r$'s $(r = 1, \ldots, t, t \geq 1)$ are monomials of $F$.

## 3. Main Results

In this section, we prove our main results which are analogue to the results of [1, 2].

**Lemma 1.** *Let $w = w(x_1, x_2, \ldots, x_m)$ be a monomial of $F$. Assume that $x_1, x_2, \ldots, x_m$ appear in $w$. Then, $\overline{\alpha} \in S_w^M(L)$ if and only if $w(\overline{u_1}, \ldots, \overline{u_{\imath-1}}, \overline{\alpha}, \overline{u_{\imath+1}}, \ldots, \overline{u_m}) = 0$ for all $\overline{u_\imath} \in L$ and $\imath = 1, \ldots, m$.*

*Proof.* Let $\overline{\alpha}$ be any element of $S_w^M(L)$. If $\overline{\alpha} = \overline{0}$, then it is clear that

$$w(\overline{u_1}, \ldots, \overline{u_{\imath-1}}, \overline{0}, \overline{u_{\imath+1}}, \ldots, \overline{u_m}) = 0. \tag{4}$$

Let $\overline{\alpha} \neq \overline{0}$. Then,

$$\begin{aligned} &w(\overline{u_1}, \ldots, \overline{u_{\imath-1}}, \overline{u_\imath} + \beta\overline{\alpha}, \overline{u_{\imath+1}}, \ldots, \overline{u_m}) \\ &\quad = w(\overline{u_1}, \ldots, \overline{u_{\imath-1}}, \overline{u_\imath}, \overline{u_{\imath+1}}, \ldots, \overline{u_m}), \end{aligned} \tag{5}$$

where $\beta \in K$. This gives $w(\overline{u_1}, \ldots, \overline{u_{\imath-1}}, \overline{\alpha}, \overline{u_{\imath+1}}, \ldots, \overline{u_m}) = 0$.

The "only if" part of the statement is clear.

**Lemma 2.** *Let $H$ and $S$ be subalgebras of $L$ such that $S \subset H$. Then, a marginal subalgebra of $H$ may not be the marginal subalgebra of $S$.*

*Proof.* Let $H = \gamma_k(L)$ and $S = \gamma_{k+1}(L)$, $2 \leq k < n$, and $w = [x_1, x_2, \ldots, x_{n-1}]$ be a monomial in $F$. Then, it is clear that $A_w^M(H) = H$ and $A_w^M(S) = S$. Therefore, $A_w^M(H) \neq A_w^M(S)$.

**Proposition 1.** *Let $W$ be a nonempty subset of $F$. For each $w_r \in W$ and $\overline{b_1}, \ldots, \overline{b_m} \in S_W^M(L)$, we have $w_r(\overline{b_1}, \overline{b_2}, \ldots, \overline{b_m}) = 0$.*

*Proof.* Let $w_r = w_r(x_1, x_2, \ldots, x_m) \in W$ and let $\overline{b_1}, \overline{b_2}, \ldots, \overline{b_m} \in S_W^M(L)$:

$$\begin{aligned} w_r(\overline{b_1}, \overline{b_2}, \ldots, \overline{b_m}) &= w_r(0 + \overline{b_1}, \overline{b_2}, \ldots, \overline{b_m}) \\ &= w_r(0, \overline{b_2}, \ldots, \overline{b_m}) \\ &= w_r(0, 0 + \overline{b_2}, \ldots, \overline{b_m}) \\ &\vdots \\ &= w_r(0, 0, \ldots, 0) \\ &= 0. \end{aligned} \tag{6}$$

□

*Definition 3.* We call an endomorphism $\varphi$ of $L$ a marginal endomorphism if $\varphi(\overline{u}) - \overline{u} \in I_W^M(L)$ for each $\overline{u} \in L$. The set of all marginal endomorphisms of $L$ will be denoted by $\mathrm{End}_W^M(L)$. If a marginal endomorphism is an automorphism, then it is called a marginal automorphism. The set of all marginal automorphisms of $L$ is a subgroup of $\mathrm{Aut}(L)$, and it is denoted by $\mathrm{Aut}_W^M(L)$.

*Example 1.* Let $F$ be the free Lie algebra freely generated by a finite set $\{x_1, x_2\}$ and let $L = F/\gamma_6(F)$.

If we choose the set $W$ as $W = \{w_1 = [x_1, x_2]\}$, then $I_W^M(L) = I_{w_1}^M(L) = \gamma_5(L)$ and the automorphisms

$$x_i \longrightarrow x_i + \sum_k a_k [x_1, x_2, x_{i_1}, x_{i_2}, x_{i_3}] \tag{7}$$

are the marginal (with respect to $w_1$) automorphisms of $L$ where $i, i_1, i_2, i_3 \in \{1, 2\}$ and $\alpha \in K$.

Let $w_2 = [x_1, x_2, x_1]$; then, one can easily see that the marginal (with respect to $w_2$) ideal $I_{w_2}^M(L)$ of $L$ is $\gamma_6(L) = \gamma_4(F)/\gamma_6(F)$ and the automorphisms defined as

$$x_i \longrightarrow x_i + \sum_k \alpha_k [x_1, x_2, x_{i_1}, x_{i_2}] + \sum_k \beta_k [x_1, x_2, x_{i_1}, x_{i_2}, x_{i_3}] \tag{8}$$

are the marginal (with respect to $w_2$) automorphisms of $L$, where $i, i_1, i_2, i_3 \in \{1, 2\}$ and $\alpha, \beta \in K$.

**Lemma 3.** *Every marginal endomorphism $\varphi$ of $L$ fixes all elements of the verbal subalgebra $A_W(L)$.*

*Proof.* Let $\overline{u_1}, \overline{u_2}, \ldots, \overline{u_m} \in L$ and $w = w(x_1, x_2, \ldots, x_m) \in W$. By definition of a marginal endomorphism for $\varphi \in \mathrm{End}_W^M(L)$, there is an element $\overline{\alpha_i} \in I_w^M(L)$ such that $\varphi(\overline{u_i}) = \overline{u_i} + \overline{\alpha_i}$ for $i = 1, \ldots, m$. Now, we have

$$\begin{aligned} \varphi(w(\overline{u_1}, \overline{u_2}, \ldots, \overline{u_m})) &= w(\varphi((\overline{u_1}), \varphi(\overline{u_2}) \ldots, \varphi(\overline{u_m}))) \\ &= w(\overline{u_1} + \overline{\alpha_1}, \overline{u_2} + \overline{\alpha_2}, \ldots, \overline{u_m} + \overline{\alpha_m}) \\ &= w(\overline{u_1}, \overline{u_2} + \overline{\alpha_2}, \ldots, \overline{u_m} + \overline{\alpha_m}) \\ &\vdots \\ &= w(\overline{u_1}, \overline{u_2}, \ldots, \overline{u_m}). \end{aligned} \tag{9}$$

Let $G$ and $H$ be two Lie algebras; we denote by $\mathrm{Hom}(G, H)$ the set of all Lie algebra homomorphisms from $G$ to $H$.

**Proposition 2.** *Let $\varnothing \neq W \subset F$. For each $\psi \in \mathrm{Hom}(L, A_W^M(L))$ and $w \in W$, we have $\psi(w) = 0$.*

*Proof.* Let $\psi \in \mathrm{Hom}(L, A_W^M(L)), w \in W$ and $\overline{u_1}, \overline{u_2}, \ldots, \overline{u_m} \in L$. Since $\psi(\overline{u_i}) \in A_W^M(L)$ from Proposition 1, we have

$$\psi(w(\overline{u_1}, \overline{u_2}, \ldots, \overline{u_m})) = w(\psi((\overline{u_1}), \psi(\overline{u_2}), \ldots, \psi(\overline{u_m}))) = 0. \tag{10}$$

**Theorem 1.** *If $I_W^M(L) \subseteq I_W(L)$, then*

$$\mathrm{Hom}\left(\frac{L}{I_W(L)}, I_W^M(L)\right) \cong \mathrm{Hom}\left(L, I_W^M(L)\right). \tag{11}$$

*Proof.* The map $f\colon \psi \longrightarrow f_{\psi}$ defines a homomorphism from $\mathrm{Hom}(L/I_W(L), I_W^M(L))$ to $\mathrm{Hom}(L, I_W^M(L))$, where $f_{\psi}(\overline{u}) = \psi(\overline{u} + I_W(L))$ for all $\overline{u} \in L$. It is clear that $f$ is well-defined and it is easy to see that $\psi \longrightarrow f_{\psi}$ is an isomorphism from $\mathrm{Hom}(L/I_W(L), I_W^M(L))$ to $\mathrm{Hom}(L, I_W^M(L))$.

If $I_W^M(L)$ is contained in the center of $L$ and $\theta \in \mathrm{Aut}_W^M(L)$, then for all $\overline{u} \in L$, one can easily see that the map $f_{\theta}\colon \overline{u} \longrightarrow \theta(\overline{u}) - \overline{u}$ is a homomorphism from $L$ into $I_W^M(L)$. On the other hand, for every $f \in \mathrm{Hom}(L, I_W^M(L))$, the map $\theta_f\colon \overline{u} \longrightarrow \overline{u} + f(\overline{u})$ is a marginal endomorphism of $L$. Note that the endomorphism $\theta_f$ is an automorphism if and only if $f(\overline{u}) \neq -\overline{u}$, for all $0 \neq \overline{u} \in L$.

**Theorem 2.** *If $I_W^M(L) \subseteq I_W(L)$, then $\mathrm{Aut}_W^M(L)$ acts trivially on $I_W^M(L)$.*

*Proof.* If $I_W^M(L) \subseteq I_W(L)$, then $W \subseteq L^k$ such that $I_W^M(L) \subseteq L^{n-k+1}$ and $k \leq n-k+1$. Now, let $\overline{u} = \overline{u}(x_1, x_2, \ldots, x_m) \in I_W^M(L)$ and $\theta \in \mathrm{Aut}_W^M(L)$:

$$\begin{aligned}\theta(\overline{u}(x_1, x_2, \ldots, x_m)) &= \overline{u}(\theta(x_1), \theta(x_2), \ldots, \theta(x_m)) \\ &= \overline{u}(x_1 + \overline{\alpha_1}, x_2 + \overline{\alpha_2}, \ldots, x_m + \overline{\alpha_m}) \\ &= \overline{u}(x_1, x_2, \ldots, x_m) \\ &\quad + \overline{u}(\overline{\alpha_1}, x_2 + \overline{\alpha_2}, \ldots, x_m + \overline{\alpha_m}) + \ldots + \\ &\quad + \overline{u}(x_1 + \overline{\alpha_1}, x_2 + \overline{\alpha_2}, \ldots, \overline{\alpha_m}),\end{aligned} \tag{12}$$

where $\overline{\alpha_i} \in I_w^M(L)$, $i = 1, \ldots, m$. Then, $\overline{u}(x_1 + \overline{\alpha_1}, \ldots, \overline{\alpha_i}, \ldots, x_m + \overline{\alpha_m}) = 0$, $i = 1, \ldots, m$, by Lemma 1. Therefore, we get $\theta(\overline{u}(x_1, x_2, \ldots, x_m)) = \overline{u}(x_1, x_2, \ldots, x_m)$.

The next theorem determines the structure of $\mathrm{Aut}_W^M(L)$.

**Theorem 3.** *Let $\varnothing \neq W \subseteq L$ such that $I_W^M(L) \subseteq I_W(L)$. Then,*

$$\mathrm{Aut}_W^M(L) \cong \mathrm{Hom}\left(\frac{L}{I_W(L)}, I_W^M(L)\right). \tag{13}$$

*Proof.* The map

$$\psi\colon \theta \longrightarrow \psi(\theta), \tag{14}$$

defines an isomorphism from $\mathrm{Aut}_W^M(L)$ to $\mathrm{Hom}(L/I_W(L), I_W^M(L))$, where $\psi(\theta)(\overline{u} + I_W(L)) = \theta(\overline{u}) - \overline{u}$ for all $\overline{u} \in L$.

(i) $\psi(\theta)$ is well-defined: Let $\overline{u_1}, \overline{u_2} \in L$ and $\overline{u_1} + I_W(L) = \overline{u_2} + I_W(L)$; then, $\overline{u_1} - \overline{u_2} \in I_W(L)$. By Lemma 3, we have $\theta(\overline{u_1} - \overline{u_2}) = \overline{u_1} - \overline{u_2}$ and $\theta(\overline{u_1}) - \overline{u_1} = \theta(\overline{u_2}) - \overline{u_2}$. Therefore, $\psi(\theta)(\overline{u_1} + I_W(L)) = \psi(\theta)(\overline{u_2} + I_W(L))$, as required.

(ii) $\psi(\theta) \in \mathrm{Hom}(L/I_W(L), I_W^M(L))$: Let $\overline{u_1}, \overline{u_2} \in L$; then,

$$\begin{aligned}\psi(\theta)((\overline{u_1} + I_W(L)) + (\overline{u_2} + I_W(L))) &= \psi(\theta)(\overline{u_1} + \overline{u_2} + I_W(L)) \\ &= \theta(\overline{u_1} + \overline{u_2}) - (\overline{u_1} + \overline{u_2}) \\ &= (\theta(\overline{u_1}) - \overline{u_1}) + (\theta(\overline{u_2}) - \overline{u_2}) \\ &= \psi(\theta)(\overline{u_1} + I_W(L)) + \psi(\theta)(\overline{u_2} + I_W(L)),\end{aligned} \tag{15}$$

and similarly, we get

$$\begin{aligned}&\psi(\theta)[c\overline{u_1} + I_W(L), \overline{u_2} + I_W(L)] \\ &\quad = c[\psi(\theta)(\overline{u_1} + I_W(L)), \psi(\theta)(\overline{u_2} + I_W(L))].\end{aligned} \tag{16}$$

(iii) $\psi$ is a homomorphism: Let $\theta_1, \theta_2 \in \mathrm{Aut}_W^M(L)$ and $\overline{u} \in L$. Since $\theta_2(\overline{u}) - \overline{u} \in I_W^M(L)$, Theorem 2 implies that

$$\begin{aligned}\psi(\theta_1\theta_2)(\overline{u} + I_W(L)) &= (\theta_1\theta_2)(\overline{u}) - \overline{u} \\ &= \theta_1(\theta_2(\overline{u}) - \overline{u} + \overline{u}) - \overline{u} \\ &= \theta_1(\theta_2(\overline{u}) - \overline{u}) + \theta_1(\overline{u}) - \overline{u} \\ &= (\theta_2(\overline{u}) - \overline{u}) + (\theta_1(\overline{u}) - \overline{u}) \\ &= \psi(\theta_1)(\overline{u} + I_W(L)) \\ &\quad + \psi(\theta_2)(\overline{u} + I_W(L)).\end{aligned} \tag{17}$$

(iv) $\psi$ is injective: Let $\theta_1, \theta_2 \in \mathrm{Aut}_W^M(L)$ such that $\psi(\theta_1) = \psi(\theta_2)$. Then, $\theta_1(\overline{u}) - \overline{u} = \theta_2(\overline{u}) - \overline{u}$ for all $\overline{u} \in L$, and so $\theta_1 = \theta_2$.

(v) $\psi$ is onto: Let $\beta \in \mathrm{Hom}(L/I_W(L), I_W^M(L))$. Define $\theta\colon L \longrightarrow L$ by $\theta(\overline{u}) = \overline{u} + \beta(\overline{u} + I_W(L))$ for every $\overline{u} \in L$. It is clear that $\theta(\overline{u}) - \overline{u} \in I_W^M(L)$ and $\theta \in \mathrm{End}_W^M(L)$:

$$\begin{aligned}\beta(\overline{u} + I_W(L)) &= \theta(\overline{u}) - \overline{u} = \psi(\theta)(\overline{u} + I_W(L)) \\ &= \psi(\theta)(\overline{u} + I_W(L)).\end{aligned} \tag{18}$$

From this, we have $\beta = \psi(\theta)$. So $\psi$ is onto.

Therefore, $\mathrm{Aut}_W^M(L) \cong \mathrm{Hom}(L/I_W(L), I_W^M(L))$. □

**Theorem 4.** *Let $\varnothing \neq W \subseteq L$ such that $I_W^M(L) \subseteq I_W(L)$. Then,*

$$\mathrm{Aut}_W^M(L) \cong \mathrm{Hom}(L, I_W^M(L)). \tag{19}$$

*Proof.* By Theorems 1 and 3, the result is obtained clearly. □

## Data Availability

All the data in the manuscript are available upon request to the corresponding author.

## Conflicts of Interest

The author declares that they have no conflicts of interest.

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## References

[1] M. S. Attar, "On the marginal automorphisms of a group," *Communications in Algebra*, vol. 37, no. 7, pp. 2300–2308, 2009.

[2] M. Khademi and A. Gholami, "On marginal automorphism of infinite groups," *Italian Journal of Pure and Applied Mathematics*, vol. 36, pp. 965–974, 2016.

[3] M. R. R. Moghaddam and H. Safa, "Some properties of marginal automorphisms of groups," *Bulletin of the Iranian Mathematical Society*, vol. 39, no. 6, pp. 1181–1188, 2013.

[4] I. Stewart, "Verbal and marginal properties of non-associative algebras," *Proceedings of the London Mathematical Society*, vol. s3-28, no. 1, pp. 129–140, 1974.