Review Article

Nontrivial Solutions for the $2n^{\text{th}}$ Lidstone Boundary Value Problem

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In this paper, we study the existence of nontrivial solutions for the $2n^{\text{th}}$ Lidstone boundary value problem with a sign-changing nonlinearity. Under some conditions involving the eigenvalues of a linear operator, we use the topological degree theory to obtain our main results.

1. Introduction

In this paper, we investigate the existence of nontrivial solutions for the following $2n^{\text{th}}$ Lidstone boundary value problem with a sign-changing nonlinearity:

$$\begin{cases} (-1)^n u^{(2n)}(t) = f(t, u(t), -u''(t), \dots, (-1)^{n-1} u^{(2n-2)}(t)), & 0 < t < 1, \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, & i = 0, 1, \dots, n-1, \end{cases} \quad (1)$$

where the nonlinearity f satisfies the following condition:

- (i) (C0) $f \in C([0, 1] \times \mathbb{R}^n, \mathbb{R})$ and there exist three nonnegative functions $a(t)$, $b(t)$, and $K(x_1, x_2, \dots, x_n) (\equiv 0)$ such that

$$\begin{aligned} f(t, x_1, x_2, \dots, x_n) &\geq -a(t) - b(t)K(x_1, x_2, \dots, x_n), \\ \text{for } t &\in [0, 1], x_i \in \mathbb{R}, i = 1, 2, \dots, n. \end{aligned} \quad (2)$$

The Lidstone boundary value problem arises in many different areas of applied mathematics and physics. When $n = 2$, problem (1) describes the deformation of an elastic beam in which both ends are simply supported. Recently, this problem has been extensively studied, and the authors refer the reader to [1–11] and references cited therein. For example, in [1], the authors used a cone-theoretic fixed point theorem to study the existence of nontrivial solutions for the nonlinear Lidstone boundary value problem:

$$\begin{cases} y^{(2m)}(t) = \lambda a(t)f(y(t), \dots, y^{(2j)}(t), \dots, y^{(2(m-1))}(t)), & 0 < t < 1, \\ y^{(2i)}(0) = 0 = y^{(2i)}(1), & i = 0, \dots, m-1, \end{cases} \quad (3)$$

where $(-1)^m f > 0$ is continuous and a is nonnegative. In [2], the authors investigated the existence and uniqueness of positive solutions for the following generalized Lidstone boundary value problem:

$$\begin{cases} (-1)^n u^{(2n)} = f(t, u, -u'', \dots, (-1)^{n-1} u^{(2n-2)}), \\ \alpha_0 u^{(2i)}(0) - \beta_0 u^{(2i+1)}(0) = 0 \quad (i = 0, 1, 2, \dots, n-1), \\ \alpha_1 u^{(2i)}(1) - \beta_1 u^{(2i+1)}(1) = 0 \quad (i = 0, 1, 2, \dots, n-1), \end{cases} \quad (4)$$

where $\alpha_j \geq 0, \beta_j \geq 0$ ($j = 0, 1$) and $\alpha_0 \alpha_1 + \alpha_0 \beta_1 + \alpha_1 \beta_0 > 0$. In view of symmetry, these results demonstrate that problem (4) is essentially identical with Dirichlet boundary condition (1).

Meanwhile, we also note that there are a large number of papers in the literature devoted to sign-changing nonlinearities, and some results can be found in a series of papers [12–32] and the references cited therein. For example, in [12], the authors studied the following higher-order nonlinear fractional boundary value problem involving Riemann–Liouville fractional derivatives:

$$\begin{cases} D_{0+}^\alpha u(t) = -f(t, u(t), D_{0+}^{\beta_1} u(t), D_{0+}^{\beta_2} u(t), \dots, D_{0+}^{\beta_{n-1}} u(t)), & 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = D_{0+}^\beta u(1) = 0, \end{cases} \quad (5)$$

where f is a sign-changing nonlinearity. Under some appropriate conditions involving the eigenvalues of the relevant linear operators, they utilized the topological degree to obtain a nontrivial solution for (5). In [13], the authors adopted the similar method in [12] to study the existence of nontrivial solutions for the following system of fractional q -difference equations with q -integral boundary conditions:

$$\begin{cases} D_q^\alpha x(t) + f_1(t, y(t)) = 0, & t \in (0, 1), \\ D_q^\alpha y(t) + f_2(t, x(t)) = 0, & t \in (0, 1), \\ x(0) = 0, D_q x(0) = 0, D_q^\nu x(1) = \int_0^1 h(t) D_q^\nu x(t) d_q t, \\ y(0) = 0, D_q y(0) = 0, D_q^\nu y(1) = \int_0^1 h(t) D_q^\nu y(t) d_q t, \end{cases} \quad (6)$$

where $\alpha \in (2, 3), \nu \in (1, 2)$, and D_q^α is the α -order Riemann–Liouville's fractional q -derivative.

Inspired by the aforementioned works, in this paper, we study the existence of nontrivial solutions for (1) where the nonlinearity f is sign-changing. Under some conditions involving the eigenvalues of the relevant linear operators, we use the topological degree to obtain our results.

2. Preliminaries

Let $E = C([0, 1], \mathbb{R})$, $\|v\| = \max_{t \in [0, 1]} |v(t)|$, $P = \{v \in E : v(t) \geq 0, \forall t \in [0, 1]\}$, $B_r = \{v \in E : \|v\| < r\}$ for $r > 0$. Clearly, $(E, \|\cdot\|)$ is a real Banach space and P is a solid cone in E . In (1), let $(-1)^{n-1} u^{(2n-2)}(t) = v(t)$ and from [2, P224], we can obtain that (1) is equivalent to the following integral equation:

$$v(t) = \int_0^1 G_1(t, s) f\left(s, \int_0^1 G_{n-1}(s, \tau) v(\tau) d\tau, \dots, \int_0^1 G_1(s, \tau) v(\tau) d\tau, v(s)\right) ds, \quad (7)$$

where

$$\begin{aligned} G_1(t, s) &= \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases} \\ G_i(t, s) &= \int_0^1 G_1(t, \tau) G_{i-1}(\tau, s) d\tau, \quad i = 2, 3, \dots \end{aligned} \quad (8)$$

Next, we provide a lemma, which expresses some vital properties of the functions G_i ($i = 1, 2, \dots$).

Lemma 1 (i) G_i are nonnegative continuous functions on $[0, 1]^2$, and $G_i(t, s) > 0, (t, s) \in (0, 1)^2$

(ii) G_1 has the inequalities $t(1-t)G_1(s, s) \leq G_1(t, s) \leq G_1(s, s), \forall t, s \in [0, 1]$

(iii) $G_i(t, s) = G_i(s, t), \forall t, s \in [0, 1]$

(iv) $\int_0^1 G_i(t, s) \sin \pi s ds = (1/\pi^2)^i \sin \pi t, t \in [0, 1]$, and $\int_0^1 G_i(t, s) \sin \pi t dt = (1/\pi^2)^i \sin \pi s, s \in [0, 1]$

Proof. We only prove (iii) and (iv). For (iii), $i = 1$ holds obviously. From the definition of G_i , we have

$$\begin{aligned}
G_i(s, t) &= \int_0^1 G_1(s, \tau_1) G_{i-1}(\tau_1, t) d\tau_1 \\
&= \int_0^1 \int_0^1 G_1(s, \tau_1) G_1(\tau_1, \tau_2) G_{i-2}(\tau_2, t) d\tau_2 d\tau_1 \\
&= \dots \\
&= \underbrace{\int_0^1 \dots \int_0^1}_{i-1} G_1(s, \tau_1) G_1(\tau_1, \tau_2) G_1(\tau_2, \tau_3) \dots G_1(\tau_{i-1}, t) d\tau_{i-1} \dots d\tau_2 d\tau_1 \\
&= \underbrace{\int_0^1 \dots \int_0^1}_{i-1} G_1(t, \tau_{i-1}) \dots G_1(\tau_3, \tau_2) G_1(\tau_2, \tau_1) G_1(\tau_1, s) d\tau_{i-1} \dots d\tau_2 d\tau_1 \\
&= G_i(t, s), \quad \text{for } t, s \in [0, 1].
\end{aligned} \tag{9}$$

For (iv), when $i = 1$, we have

$$\begin{aligned}
\int_0^1 G_1(t, s) \sin \pi s ds &= \int_0^t s(1-t) \sin \pi s ds + \int_t^1 t(1-s) \sin \pi s ds \\
&= \frac{1}{\pi^2} \sin \pi t.
\end{aligned} \tag{10}$$

Noting that $G_1(s, t) = G_1(t, s)$, we have

$$\int_0^1 G_1(t, s) \sin \pi t dt = \int_0^1 G_1(s, t) \sin \pi t dt = \frac{1}{\pi^2} \sin \pi s. \tag{11}$$

When $i \geq 1$, we have

$$\begin{aligned}
\int_0^1 G_i(t, s) \sin \pi s ds &= \underbrace{\int_0^1 \dots \int_0^1}_{i-1} G_1(t, \tau_{i-1}) \dots G_1(\tau_3, \tau_2) G_1(\tau_2, \tau_1) G_1(\tau_1, s) \sin \pi s ds d\tau_{i-1} \dots d\tau_2 d\tau_1 \\
&= \frac{1}{\pi^2} \underbrace{\int_0^1 \dots \int_0^1}_{i-1} G_1(t, \tau_{i-1}) \dots G_1(\tau_3, \tau_2) G_1(\tau_2, \tau_1) \sin \pi \tau_1 d\tau_{i-1} \dots d\tau_2 d\tau_1 \\
&= \dots \\
&= \left(\frac{1}{\pi^2} \right)^i \sin \pi t.
\end{aligned} \tag{12}$$

Using the symmetry of G_i , we easily have

$$\int_0^1 G_i(t, s) \sin \pi t dt = \left(\frac{1}{\pi^2} \right)^i \sin \pi s. \tag{13}$$

This completes the proof.

Let $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i^2 \neq 0$. Then, we have the following equations:

$$\begin{aligned}
\int_0^1 G_{\alpha_1, \dots, \alpha_n}(t, s) \sin \pi s ds &= \left[\alpha_1 \left(\frac{1}{\pi^2} \right)^n + \dots + \alpha_n \left(\frac{1}{\pi^2} \right)^1 \right] \sin \pi t, \\
t &\in [0, 1],
\end{aligned} \tag{14}$$

$$\begin{aligned}
\int_0^1 G_{\alpha_1, \dots, \alpha_n}(t, s) \sin \pi t dt &= \left[\alpha_1 \left(\frac{1}{\pi^2} \right)^n + \dots + \alpha_n \left(\frac{1}{\pi^2} \right)^1 \right] \sin \pi s, \\
s &\in [0, 1],
\end{aligned} \tag{15}$$

where

$$G_{\alpha_1, \dots, \alpha_n}(t, s) = \alpha_1 G_n(t, s) + \dots + \alpha_n G_1(t, s), \quad \text{for } t, s \in [0, 1]. \tag{16}$$

Lemma 2. Let $(Lv)(t) = \int_0^1 G_1(t, s) v(s) ds$, for $t \in [0, 1]$. Then, if $v \in P$, we have $Lv \in P_0$, where

$$P_0 = \{v \in P: v(t) \geq t(1-t)\|v\|, \quad \forall t \in [0, 1]\}. \tag{17}$$

This is a direct result from Lemma 1 (ii), so we omit its proof.

Remark 1. $\sin \pi t \in P_0$, for $t \in [0, 1]$.

Lemma 3 (see [33], Theorem 1 [3]). Let Ω be a bounded open set in a Banach space E and $T: \Omega \longrightarrow E$ be a continuous compact operator. If there exists $x_0 \in E \setminus \{0\}$ such that

$$x - Tx \neq \mu x_0, \quad \forall x \in \partial\Omega, \mu \geq 0, \quad (18)$$

then the topological degree $\deg(I - T, \Omega, 0) = 0$.

Lemma 4 (see [33], Lemma 4 [1]). *Let Ω be a bounded open set in a Banach space E with $0 \in \Omega$ and $T: \Omega \rightarrow E$ be a continuous compact operator. If*

$$Tx \neq \mu x, \quad \forall x \in \partial\Omega, \mu \geq 1, \quad (19)$$

then the topological degree $\deg(I - T, \Omega, 0) = 1$.

3. Main Results

Define the operator $A: E \rightarrow E$ by

$$(Av)(t) = \int_0^1 G_1(t, s) f\left(s, \int_0^1 G_{n-1}(s, \tau) v(\tau) d\tau, \dots, \int_0^1 G_1(s, \tau) v(\tau) d\tau, v(s)\right) ds. \quad (20)$$

Moreover, the continuity of f implies that A is completely continuous and the existence of solutions for (1) is equivalent to that of fixed points of A .

Now, we list some assumptions for the functions f and K :

(C1) There exist $\beta_i \geq 0$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n \beta_i^2 \neq 0$ such that

$$\lim_{\beta_1|x_1|+\beta_2|x_2|+\dots+\beta_n|x_n| \rightarrow +\infty} \frac{K(x_1, x_2, \dots, x_n)}{\beta_1|x_1| + \beta_2|x_2| + \dots + \beta_n|x_n|} = 0. \quad (21)$$

(C2) $\liminf_{\beta_1|x_1|+\beta_2|x_2|+\dots+\beta_n|x_n| \rightarrow +\infty} \frac{|f(t, x_1, x_2, \dots, x_n)|}{(\beta_1|x_1| + \beta_2|x_2| + \dots + \beta_n|x_n|)} > \lambda_{\beta_1, \beta_2, \dots, \beta_n}$, uniformly for $t \in [0, 1]$.

(C3) There exist $\gamma_i \geq 0$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n \gamma_i^2 \neq 0$ such that

$$\limsup_{\gamma_1|x_1|+\gamma_2|x_2|+\dots+\gamma_n|x_n| \rightarrow 0} \frac{|f(t, x_1, x_2, \dots, x_n)|}{\gamma_1|x_1| + \gamma_2|x_2| + \dots + \gamma_n|x_n|} < \lambda_{\gamma_1, \gamma_2, \dots, \gamma_n}, \quad (22)$$

where

$$\lambda_{\beta_1, \beta_2, \dots, \beta_n}^{-1} = \beta_1 \left(\frac{1}{\pi^2}\right)^n + \dots + \beta_n \left(\frac{1}{\pi^2}\right)^1, \quad (23)$$

$$\lambda_{\gamma_1, \gamma_2, \dots, \gamma_n}^{-1} = \gamma_1 \left(\frac{1}{\pi^2}\right)^n + \dots + \gamma_n \left(\frac{1}{\pi^2}\right)^1.$$

Theorem 1. *Suppose that (C0)–(C3) hold. Then, (1) has at least one nontrivial solution.*

Proof. We divide the following two steps:

(i) *Step 1.* By (C3), there exist $\varepsilon_1 \in (0, \lambda_{\gamma_1, \gamma_2, \dots, \gamma_n})$ and $r > 0$ such that

$$|f(t, x_1, x_2, \dots, x_n)| \leq (\lambda_{\gamma_1, \gamma_2, \dots, \gamma_n} - \varepsilon_1)(\gamma_1|x_1| + \gamma_2|x_2| + \dots + \gamma_n|x_n|), \quad (24)$$

for all $t \in [0, 1]$, $x_i \in \mathbb{R}$ ($i = 1, 2, \dots, n$) with $0 \leq \gamma_1|x_1| + \gamma_2|x_2| + \dots + \gamma_n|x_n| \leq r$. Substituting this inequality into (20), we have

$$\begin{aligned} |(Av)(t)| &\leq \int_0^1 G_1(t, s) \left| f\left(s, \int_0^1 G_{n-1}(s, \tau) v(\tau) d\tau, \dots, \int_0^1 G_1(s, \tau) v(\tau) d\tau, v(s)\right) \right| ds \\ &\leq (\lambda_{\gamma_1, \gamma_2, \dots, \gamma_n} - \varepsilon_1) \int_0^1 G_1(t, s) \left(\gamma_1 \left| \int_0^1 G_{n-1}(s, \tau) v(\tau) d\tau \right| + \dots + \gamma_{n-1} \left| \int_0^1 G_1(s, \tau) v(\tau) d\tau \right| + \gamma_n |v(s)| \right) ds \\ &\leq (\lambda_{\gamma_1, \gamma_2, \dots, \gamma_n} - \varepsilon_1) \int_0^1 (\gamma_1 G_n(t, s) + \dots + \gamma_{n-1} G_2(t, s) + \gamma_n G_1(t, s)) |v(s)| ds \\ &= (\lambda_{\gamma_1, \gamma_2, \dots, \gamma_n} - \varepsilon_1) \int_0^1 G_{\gamma_1, \dots, \gamma_n}(t, s) |v(s)| ds. \end{aligned} \quad (25)$$

Now for this r , we claim

$$Av \neq \lambda v, \quad \forall v \in \partial B_r, \lambda \geq 1. \quad (26)$$

Suppose the contrary. Then, there exist $v_0 \in \partial B_r$ and $\lambda_0 \geq 1$ such that $Av_0 = \lambda_0 v_0$. Therefore, we obtain

$$\begin{aligned} |v_0(t)| &\leq \lambda_0 |v_0(t)| = |(Av_0)(t)| \leq (\lambda_{\gamma_1, \gamma_2, \dots, \gamma_n} - \varepsilon_1) \\ &\int_0^1 G_{\gamma_1, \dots, \gamma_n}(t, s) |v_0(s)| ds. \end{aligned} \quad (27)$$

Multiply by $\sin \pi t$ on both sides and integrate over $[0, 1]$ and use (15) to obtain

$$\begin{aligned} \int_0^1 |v_0(t)| \sin \pi t dt &\leq (\lambda_{\gamma_1, \gamma_2, \dots, \gamma_n} - \varepsilon_1) \int_0^1 \int_0^1 G_{\gamma_1, \dots, \gamma_n}(t, s) |v_0(s)| \sin \pi t ds dt \\ &= (\lambda_{\gamma_1, \gamma_2, \dots, \gamma_n} - \varepsilon_1) \lambda_{\gamma_1, \gamma_2, \dots, \gamma_n}^{-1} \int_0^1 |v_0(s)| \sin \pi s ds. \end{aligned} \quad (28)$$

This indicates that $\int_0^1 |v_0(t)| \sin \pi t dt = 0$, and thus, $|v_0(t)| \equiv 0$, $t \in [0, 1]$. This contradicts to $v_0 \in \partial B_r$. Consequently, (26) holds and Lemma 4 yields that

$$\deg(I - A, B_r, 0) = 1. \quad (29)$$

Step 2. By virtue of (C2), there exist $\varepsilon_2 > 0$ and $X_0 > 0$ such that

$$\begin{aligned} f(t, x_1, x_2, \dots, x_n) &\geq (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2) \\ &\quad \cdot (\beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n|), \end{aligned} \quad (30)$$

for all $t \in [0, 1]$ and $\beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n| > X_0$. For any given ε with $\varepsilon_2 - \|b\|\varepsilon > 0$, and using (C1), there exists $X_1 > X_0$ such that

$$\begin{aligned} K(x_1, x_2, \dots, x_n) &\leq \varepsilon (\beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n|), \\ &\text{for all } \beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n| > X_1. \end{aligned} \quad (31)$$

Consequently, when $t \in [0, 1]$, $\beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n| > X_1$, we have

$$\begin{aligned} f(t, x_1, x_2, \dots, x_n) &\geq (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2) (\beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n|) - a(t) - b(t)K(x_1, x_2, \dots, x_n) \\ &\geq (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2) (\beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n|) - a(t) - b(t)\varepsilon (\beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n|) \\ &\geq (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) (\beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n|) - a(t). \end{aligned} \quad (32)$$

Noting that when $t \in [0, 1]$, $\beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n| \leq X_1$, $f(t, x_1, \dots, x_n)$, and $K(x_1, \dots, x_n)$ are bounded, we can let

$$\begin{aligned} C_{X_1} &= \max_{0 \leq t \leq 1, \beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n| \leq X_1} |f(t, x_1, \dots, x_n)| \\ &\quad + (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) X_1, \end{aligned} \quad (33)$$

$$K^* = \max_{\beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n| \leq X_1} K(x_1, \dots, x_n).$$

Then, we can obtain

$$\begin{aligned} f(t, x_1, x_2, \dots, x_n) &\geq (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) \\ &\quad \cdot (\beta_1 |x_1| + \beta_2 |x_2| + \dots + \beta_n |x_n|) \\ &\quad - a(t) - C_{X_1}, \end{aligned} \quad (34)$$

for all $t \in [0, 1]$, $x_i \in \mathbb{R}$, $i = 1, 2, \dots, n$. Note that ε can be chosen arbitrarily small, and let

$$R > \max \left\{ \frac{\int_0^1 G_1(s, s) (a(s) + K^* b(s) + C_{X_1}) ds}{1 - \|b\|\varepsilon \max_{t \in [0, 1]} \int_0^1 G_{\beta_1, \dots, \beta_n}(t, \tau) d\tau}, S_1 \right\}, \quad (35)$$

where

$$S_1 = \frac{\int_0^1 [(\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) + (\varepsilon_2 - \|b\|\varepsilon) G_1(y, y)] (a(y) + K^* b(y) + C_{X_1}) dy}{(\varepsilon_2 - \|b\|\varepsilon) \left(1 - \|b\|\varepsilon \max_{t \in [0, 1]} \int_0^1 G_{\beta_1, \dots, \beta_n}(t, \tau) d\tau \right) - \varepsilon (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) \int_0^1 b(y) \sum_{i=1}^n \int_0^1 \beta_i G_{n-i}(y, \tau) d\tau dy}, \quad (36)$$

and $G_0(y, \tau) \equiv 1$. Now, we claim

$$v - Av \neq \mu \varphi, \quad \forall v \in \partial B_R, \mu \geq 0, \quad (37)$$

where $\varphi(t) = \sin \pi t$, $t \in [0, 1]$. Suppose that (37) is not satisfied. Then, there exist $v_1 \in \partial B_R$ and $\mu_1 > 0$ such that

$$v_1 - Av_1 = \mu_1 \varphi. \quad (38)$$

Let

$$\tilde{v}(t) = \int_0^1 G_1(t, s) \left(a(s) + b(s) K \left(\int_0^1 G_{n-1}(s, \tau) v_1(\tau) d\tau, \dots, \int_0^1 G_1(s, \tau) v_1(\tau) d\tau, v_1(s) \right) + C_{X_1} \right) ds. \quad (39)$$

Then, from Lemma 2, we have $\tilde{v} \in P_0$. Now, we estimate the norm of $\tilde{v}$. Noting that $v_1 \in \partial B_R$ ($\|v_1\| = R$), we obtain

$$\begin{aligned} \tilde{v}(t) &\leq \int_0^1 G_1(t, s) \left(a(s) + b(s) \left[\varepsilon \left(\beta_1 \left| \int_0^1 G_{n-1}(s, \tau) v_1(\tau) d\tau \right| + \dots + \beta_{n-1} \left| \int_0^1 G_1(s, \tau) v_1(\tau) d\tau \right| + \beta_n |v_1(s)| \right) + K^* \right] + C_{X_1} \right) ds \\ &\leq \int_0^1 G_1(s, s) (a(s) + K^* b(s) + C_{X_1}) ds + \|b\| \varepsilon \int_0^1 G_{\beta_1, \dots, \beta_n}(t, \tau) |v_1(\tau)| d\tau \\ &\leq \int_0^1 G_1(s, s) (a(s) + K^* b(s) + C_{X_1}) ds + \|b\| \|v_1\| \varepsilon \max_{t \in [0, 1]} \int_0^1 G_{\beta_1, \dots, \beta_n}(t, \tau) d\tau \\ &< R. \end{aligned} \quad (40)$$

We calculate $v_1 + \tilde{v}$. By (38) we have

$$\begin{aligned} v_1(t) + \tilde{v}(t) &= (Av_1)(t) + \mu_1 \varphi(t) + \tilde{v}(t) \\ &= \int_0^1 G_1(t, s) \left[f \left(s, \int_0^1 G_{n-1}(s, \tau) v_1(\tau) d\tau, \dots, \int_0^1 G_1(s, \tau) v_1(\tau) d\tau, v_1(s) \right) \right. \\ &\quad \left. + a(s) + b(s) K \left(\int_0^1 G_{n-1}(s, \tau) v_1(\tau) d\tau, \dots, \int_0^1 G_1(s, \tau) v_1(\tau) d\tau, v_1(s) \right) + C_{X_1} \right] ds + \mu_1 \varphi(t). \end{aligned} \quad (41)$$

Using (C0), Lemma 2 and Remark 1, we find

$$v_1 + \tilde{v} \in P_0. \quad (42)$$

Therefore, (34) enables us to calculate

$$\begin{aligned} (Av_1)(t) + \tilde{v}(t) &\geq (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\| \varepsilon) \int_0^1 G_1(t, s) \left(\beta_1 \left| \int_0^1 G_{n-1}(s, \tau) v_1(\tau) d\tau \right| + \dots + \beta_{n-1} \left| \int_0^1 G_1(s, \tau) v_1(\tau) d\tau \right| + \beta_n |v_1(s)| \right) ds \\ &\quad - \int_0^1 G_1(t, s) (a(s) + C_{X_1}) ds \\ &\quad + \int_0^1 G_1(t, s) \left[a(s) + b(s) K \left(\int_0^1 G_{n-1}(s, \tau) v_1(\tau) d\tau, \dots, \int_0^1 G_1(s, \tau) v_1(\tau) d\tau, v_1(s) \right) + C_{X_1} \right] ds \\ &\geq (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\| \varepsilon) \int_0^1 G_1(t, s) \left(\beta_1 \left| \int_0^1 G_{n-1}(s, \tau) v_1(\tau) d\tau \right| + \dots + \beta_{n-1} \left| \int_0^1 G_1(s, \tau) v_1(\tau) d\tau \right| + \beta_n |v_1(s)| \right) ds \\ &= (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\| \varepsilon) \left| \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) v_1(s) ds \right| \\ &\geq (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\| \varepsilon) \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) v_1(s) ds. \end{aligned} \quad (43)$$

Consequently, we have

$$\begin{aligned}
 & (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) (v_1(s) + \tilde{v}(s)) ds \\
 & \quad - (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) \tilde{v}(s) ds \\
 & = \lambda_{\beta_1, \beta_2, \dots, \beta_n} \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) (v_1(s) + \tilde{v}(s)) ds + (\varepsilon_2 - \|b\|\varepsilon) \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) (v_1(s) + \tilde{v}(s)) ds \\
 & \quad - (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) \tilde{v}(s) ds.
 \end{aligned} \tag{44}$$

Now, we estimate $(\varepsilon_2 - \|b\|\varepsilon) \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) (v_1(s) + \tilde{v}(s)) ds - (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) \tilde{v}(s) ds$. From (42), we have $v_1(t) + \tilde{v}(t) \geq t(1-t)\|v_1\| +$

$\tilde{v} \geq t(1-t)(\|v_1\| - \|\tilde{v}\|)$, $t \in [0, 1]$, and using $G_1(t, s) \leq t(1-t)$, $t, s \in [0, 1]$, we have

$$\begin{aligned}
 & (\varepsilon_2 - \|b\|\varepsilon) \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) (v_1(s) + \tilde{v}(s)) ds - (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) \tilde{v}(s) ds \\
 & \geq (\varepsilon_2 - \|b\|\varepsilon) \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) s(1-s) (\|v_1\| - \|\tilde{v}\|) ds \\
 & \quad - (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) \\
 & \quad \cdot \int_0^1 s(1-s) \left(a(y) + b(y) \left[\varepsilon \left(\beta_1 \left| \int_0^1 G_{n-1}(y, \tau) v_1(\tau) d\tau \right| + \dots + \beta_{n-1} \left| \int_0^1 G_1(y, \tau) v_1(\tau) d\tau \right| + \beta_n |v_1(y)| \right) + K^* \right] + C_{X_1} \right) dy ds \\
 & \geq \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) s(1-s) ds \\
 & \quad \cdot \left[(\varepsilon_2 - \|b\|\varepsilon) (\|v_1\| - \|\tilde{v}\|) - (\lambda_{\beta_1, \beta_2, \dots, \beta_n} + \varepsilon_2 - \|b\|\varepsilon) \int_0^1 \left(a(y) + b(y) \left(\varepsilon \|v_1\| \sum_{i=1}^n \beta_i \int_0^1 G_{n-i}(y, \tau) d\tau + K^* \right) + C_{X_1} \right) dy \right] \\
 & \geq 0.
 \end{aligned} \tag{45}$$

As a result, by means of (43) and (44), we obtain

$$(Av_1)(t) + \tilde{v}(t) \geq \lambda_{\beta_1, \beta_2, \dots, \beta_n} \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) (v_1(s) + \tilde{v}(s)) ds. \tag{46}$$

Combining this with (38), we have

$$\begin{aligned}
 v_1(t) + \tilde{v}(t) & = (Av_1)(t) + \mu_1 \varphi(t) + \tilde{v}(t) \\
 & \geq \lambda_{\beta_1, \beta_2, \dots, \beta_n} \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) (v_1(s) + \tilde{v}(s)) ds + \mu_1 \varphi(t) \\
 & \geq \mu_1 \varphi(t).
 \end{aligned} \tag{47}$$

Define $\mu^* = \sup S = \sup \{\mu > 0: v_1 + \tilde{v} \geq \mu \varphi\}$. Then, $S \neq \emptyset$ ($\mu_1 \in S$) and $\mu^* \geq \mu_1$. Hence, from (14), we have

$$\begin{aligned}
 v_1(t) + \tilde{v}(t) & \geq \lambda_{\beta_1, \beta_2, \dots, \beta_n} \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) (v_1(s) + \tilde{v}(s)) ds + \mu_1 \varphi(t) \\
 & \geq \lambda_{\beta_1, \beta_2, \dots, \beta_n} \int_0^1 G_{\beta_1, \dots, \beta_n}(t, s) \mu^* \varphi(s) ds + \mu_1 \varphi(t) \\
 & = \lambda_{\beta_1, \beta_2, \dots, \beta_n} \lambda_{\beta_1, \beta_2, \dots, \beta_n}^{-1} \mu^* \varphi(t) + \mu_1 \varphi(t) \\
 & = (\mu^* + \mu_1) \varphi(t),
 \end{aligned} \tag{48}$$

which contradicts the definition of μ^* . Therefore, (37) holds, and from Lemma 3, we obtain

$$\deg(I - A, B_R, 0) = 0. \tag{49}$$

Now, (29) and (49) together imply that

$$\deg\left(I - A, \frac{B_R}{B_r}, 0\right) = \deg(I - A, B_R, 0) - \deg(I - A, B_r, 0) = -1. \tag{50}$$

Therefore, the operator A has at least one fixed point in $B_R/\overline{B}_r$. Equivalently, (1) has at least one nontrivial solution. This completes the proof.

4. Conclusion

In this paper, we use the topological degree to study the nontrivial solutions for the $2n^{\text{th}}$ Lidstone boundary value problem (1). To the best of our knowledge, there are few works that deal with the problem where the nonlinear terms may be unbounded and sign-changing. Moreover, it is remarked that the main result is discussed under some conditions concerning the first eigenvalues corresponding to the relevant linear operators. These mean that our main result is an improvement in some related works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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