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Research Article

An Intertwining of Curvelet and Linear Canonical Transforms

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In this article, we introduce a novel curvelet transform by combining the merits of the well-known curvelet and linear canonical transforms. The motivation towards the endeavour spurts from the fundamental question of whether it is possible to increase the flexibility of the curvelet transform to optimize the concentration of the curvelet spectrum. By invoking the fundamental relationship between the Fourier and linear canonical transforms, we formulate a novel family of curvelets, which is comparatively flexible and enjoys certain extra degrees of freedom. The preliminary analysis encompasses the study of fundamental properties including the formulation of reconstruction formula and Rayleigh's energy theorem. Subsequently, we develop the Heisenberg-type uncertainty principle for the novel curvelet transform. Nevertheless, to extend the scope of the present study, we introduce the semidiscrete and discrete analogues of the novel curvelet transform. Finally, we present an example demonstrating the construction of novel curvelet waveforms in a lucid manner.

1. Introduction

The wavelet transform is a multiscale integral transform, which serves as one of the corner stones of nonstationary signal processing. It can be used in time-frequency analysis, wherein the scale and frequency are inverse to each other. The wavelet transform decomposes a signal into components determined by the translations and dilations of a single function known as the mother wavelet. By applying these local decomposition filters, the wavelet transform has proved to be of substantial importance in capturing the local characteristics of nonstationary signals and has paved its way to a number of fields including signal and image processing, sampling theory, geophysics, astrophysics, and quantum mechanics [1–4]. However, the efficiency of the wavelet transform fades away in the realm of higher-dimensional signal processing due to the fact that the wavelet transform employs isotropic scalings in dimensions $n \geq 2$. Such isotropic scalings are incompetent to capture the edges and corners in higher-dimensional signals appearing due to the spatial occlusion between different objects; for instance, in medical imaging curves separate bones and different kinds of soft tissue. Therefore, the key problem in multidimensional signal analysis is to extract and characterize the relevant and directional information regarding the occurrence of curves and boundaries in signals. As a result, some off-shoots of the wavelet transform, such as the Stockwell transform [5, 6], ridgelet transform [7], curvelet transform [8, 9], contourlet transform [10], and the shearlet transform [11], have been introduced to address these shortcomings of the wavelet transform.

The curvelet transform aims to deal with certain interesting phenomena occurring along curved edges in higher-dimensional signals. Unlike the wavelet transform, the curvelet transform provides time-frequency localization with a reasonable directionality and anisotropy by using angled polar wedges or angled trapezoid windows in frequency domain. The intrinsic multiscale and anisotropic nature of curvelet waveforms leads to optimally sparse representations of objects which display curve-punctuated smoothness, that is, smoothness except for discontinuity along a general curve with bounded curvature. Another remarkable property of curvelets is that they elegantly model the geometry of wave propagation; curvelets may be viewed as coherent waveforms with enough frequency localization to behave like waves but, at the same time, with sufficient spatial localization to behave like particles [12]. For more about curvelets and their applications, we refer to the monographs in [12–19]. Keeping in view the merits of the curvelet transform, in the present study, we aim to answer the fundamental question of whether it is possible to increase the flexibility of the curvelet transform to optimize the concentration of the curvelet spectrum. The answer to this question is affirmative and lies in intertwining the curvelet transform with the well-known linear canonical transform, an integral transform known for its flexibility and higher degrees of freedom in modelling physical phenomenon [20].

The highlights of the article are given as follows:

(i) We introduce the notion of novel curvelet transform by combining the merits of the curvelet and linear canonical transforms

(ii) We study the fundamental properties of the proposed transform including the reconstruction and Rayleigh's energy formulae

(iii) We formulate a Heisenberg-type uncertainty principle associated with the novel curvelet transform

(iv) To extend the scope of the study, we introduce both the semidiscrete and discrete analogues of the novel curvelet transform

(v) Finally, we present an example regarding the construction of novel curvelets

The rest of the article is structured as follows: In Section 2, we recapitulate the linear canonical transform and the ordinary curvelet transform. In Section 3, we present the formal aspects of the study, which are continued to Section 4, and Section 5 is devoted to illustrating the construction of novel curvelets. Finally, in Section 6, we extract a conclusion and provide an impetus to the future research work in the realm of novel curvelet transform.

2. Linear Canonical and Curvelet Transforms

In this section, we shall present a gentle overview of the linear canonical and curvelet transforms, which facilitates the formulation of the proposed novel curvelet transform.

2.1. Two-Dimensional Linear Canonical Transform. The origin of the theory of linear canonical transforms dates back to early 1970s with the independent seminal works of Collins [21] in paraxial optics and Moshinsky and Quesne [22] in quantum mechanics to study the conservation of information and uncertainty under linear maps of phase space. It was only in 1990s that both these independent works began to be referred to jointly in the open literature. The linear canonical transform (LCT) encompasses several well-known signal processing transforms as special cases including the Fourier transform, the fractional Fourier transform, the Fresnel transform, and even simple multiplication by quadratic phase factors [20]. As of now, the theory of linear canonical transforms has expanded into an independent and broad field of research with numerous applications to optics, mathematical physics, and signal and image processing. For more about LCT and its applications, the reader is referred to the monographs in [20–27].

Below, we shall present the formal definition of the two-dimensional LCT [25]. For notational convenience, we shall write a 2×2 matrix $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as $M=(A,B\colon C,D)$.

Definition 1. For any $f\in L^2(\mathbb{R}^2)$, the two-dimensional LCT with respect to a real, unimodular matrix $M=(A,B\colon C,D)$ is denoted by $\mathscr{L}_M[f]$ and is defined as

$$\mathscr{L}_M[f](\xi)=\begin{cases}\displaystyle\int_{\mathbb{R}^2} f(\mathbf{t})\mathscr{K}_M(\mathbf{t},\xi)d\mathbf{t}, & B\neq 0,\\[2ex] \sqrt{D}\,\exp\left\{\dfrac{iCD|\xi|^2}{2}\right\} f(D\xi), & B=0,\end{cases}\tag{1}$$

where $\mathscr{K}_M(\mathbf{t},\xi)$, with $\mathbf{t}=(t_1,t_2)^T$ and $\xi=(\xi_1,\xi_2)^T$, denotes the kernel of the two-dimensional LCT and is given by

$$\mathscr{K}_M(\mathbf{t},\xi)=\frac{1}{2\pi B}\exp\left\{\frac{i\left(A|\mathbf{t}|^2-2\mathbf{t}^T\xi+D|\xi|^2\right)}{2B}\right\},\quad B\neq 0.\tag{2}$$

It is pertinent to mention that, for the case $B=0$, the two-dimensional LCT (1) corresponds to a chirp multiplication operation. Moreover, the case $B<0$ is also of no particular interest to us. As such, in the rest of the article, we shall focus our attention on the case $B>0$. We also note that the phase-space transform (1) is lossless if and only if the matrix M is unimodular; that is, $AD-BC=1$. The inversion formula corresponding to the two-dimensional LCT (1) is given by

$$f(\mathbf{t})=\mathscr{L}_M^{-1}\left(\mathscr{L}_M[f](\xi)\right)(\mathbf{t})=\int_{\mathbb{R}^2}\mathscr{L}_M[f](\xi)\overline{\mathscr{K}_M(\mathbf{t},\xi)}d\xi.\tag{3}$$

Also, Parseval's formula associated with (1) reads

$$\langle f,g\rangle_2=\langle\mathscr{L}_M[f],\mathscr{L}_M[g]\rangle_2,\quad\forall f,g\in L^2\left(\mathbb{R}^2\right).\tag{4}$$

In the remaining part of this subsection, we shall present an analogue of the two-dimensional LCT using the polar coordinates. We emphasize that the polar LCT plays a key role in the development of the novel curvelet transform. For $\xi_1=r\cos\omega$, $\xi_2=r\sin\omega$ and $t_1=\rho\cos\eta$, $t_2=\rho\sin\eta$, where $r,\rho\geq 0$ and $\omega,\eta\in[0,2\pi)$, the polar LCT is given by

$$\mathscr{L}_M[f](r,\omega)=\frac{1}{2\pi B}\int_0^{2\pi}\int_0^{\infty} f(\rho,\eta)\exp\left\{\frac{i\left(A\rho^2+Dr^2-2\rho r\cos(\eta-\omega)\right)}{2B}\right\}\rho\, d\rho\, d\eta. \tag{5}$$

Also, the inversion formula corresponding to (5) is given by

$$f(\rho,\eta)=\frac{1}{2\pi B}\int_0^{2\pi}\int_0^{\infty}\mathscr{L}_M[f](r,\omega)\exp\left\{-\frac{i\left(A\rho^2+Dr^2-2\rho r\cos(\eta-\omega)\right)}{2B}\right\} r\mathrm{d}r\mathrm{d}\omega. \tag{6}$$

Remarks 1. The aforementioned definitions (1) and (5) embody several well-known integral transforms, some of which are listed below:

(i) As a special case when $M=(0,1\colon -1,0)$, the LCT definitions (1) and (5) reduce to their respective counterparts of the Fourier transform

(ii) Plugging the matrix $M=(\cos\alpha,\sin\alpha\colon -\sin\alpha,\cos\alpha)$, $\alpha\neq n\pi, n\in\mathbb{Z}$ (1) and (5), yields the respective counterparts of the fractional Fourier transform

(iii) For the matrix $M=(1,B\colon 0,1)$, $B\neq 0$, the LCT definitions (1) and (5) boil down to their analogues for the Fresnel transform

2.2. Ordinary Curvelet Transform. In this subsection, we shall recapitulate the mathematical frameworks of the classical curvelet transform, which serve as preliminaries for the development of the novel curvelet transform.

Consider the frequency plane $\mathbb{R}^2$ and let (r,ω), $r\geq 0, \omega\in[0,2\pi)$, denote the polar coordinates of an arbitrary point $\xi\in\mathbb{R}^2$. We choose a pair of window functions $W\colon (0,\infty)\longrightarrow(0,\infty)$, called "radial window," and $V\colon(-\infty,\infty)\longrightarrow(0,\infty)$, called "angular window," satisfying the following admissibility conditions:

$$\int_0^{\infty}|W(r)|^2\frac{\mathrm{d}r}{r}=1, \operatorname{supp}(W)\subseteq\left(\frac{1}{2},2\right), \tag{7}$$

$$(2\pi)^2\int_{-1}^{1}|V(\omega)|^2\mathrm{d}\omega=1, \operatorname{supp}(V)\subseteq[-1,1]. \tag{8}$$

The window functions (7) and (8) are used to construct a family of complex-valued waveforms adopted to scale $a>0$ location $\mathbf{b}\in\mathbb{R}^2$ and orientation $\theta\in[0,2\pi)$ or $(-\pi,\pi)$ according to convenience. For a fixed scale $a\in(0,a_0)$ where $a_0<\pi^2$ the basic curvelet $\Psi_a\colon\mathbb{R}^2\longrightarrow\mathbb{C}$ is defined via the polar Fourier transform as

$$\mathscr{F}[\Psi_a](r,\omega)=a^{3/4}W(ar)V\left(\frac{\omega}{\sqrt{a}}\right), \tag{9}$$

where $\mathscr{F}$ denotes the well-known Fourier transform defined by

$$\mathscr{F}[f](\xi)=\frac{1}{2\pi}\int_{\mathbb{R}^2}f(\mathbf{t})e^{-i t^T\xi}\mathrm{d}\mathbf{t}, \tag{10}$$

which can be expressed via the polar coordinates as

$$\mathscr{F}[f](r,\omega)=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}f(\rho,\eta)\exp\{-i\rho r\cos(\eta-\omega)\}\rho\mathrm{d}\rho\mathrm{d}\eta. \tag{11}$$

Consequently, the family of analyzing waveforms $\Psi_{a,\mathbf{b},\theta}(\mathbf{t})$ called curvelets is generated by translation and rotation of the basic element $\Psi_a(\mathbf{t})$; that is,

$$\Psi_{a,\mathbf{b},\theta}(\mathbf{t})=\Psi_a\left(R_\theta(\mathbf{t}-\mathbf{b})\right),\quad \mathbf{t}\in\mathbb{R}^2, \tag{12}$$

where $R_\theta=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ denotes the 2×2 rotation matrix affecting the planar rotation by θ radians. From (9), we note that the support of the basic element Ψ_a in the frequency domain is a polar wedge governed by the respective supports of the radial and angular windows. The scaling in the radial and angular windows is parabolic in nature with ω being the "thin" variable. The coarsest scale a_0 is fixed once for all and must obey $a_0<\pi^2$. These elements become increasingly needle-like at fine scales. Formally, we have the following definition of the ordinary curvelet transform [8, 9].

Definition 2. Given a function $f\in L^2(\mathbb{R}^2)$, the ordinary curvelet transform is defined as

$$\left[\Gamma_\Psi f\right](a,\mathbf{b},\theta)=\int_{\mathbb{R}^2}f(\mathbf{t})\overline{\Psi_{a,b,\theta}}(\mathbf{t})\mathrm{d}\mathbf{t}, \tag{13}$$

where $a<a_0, \mathbf{b}\in\mathbb{R}^2$, $\theta\in[0,2\pi)$, and $\Psi_{a,\mathbf{b},\theta}(\mathbf{t})$ is given by (12).

3. Novel Curvelet Transform

In this section, our aim is to introduce the notion of the novel curvelet transform and formulate the associated reconstruction formula and Rayleigh's energy theorem. Subsequently, we shall also study the support and oscillation properties of the proposed novel curvelet transform.

For a fixed scale $a\in(0,a_0)$ where $a_0<\pi^2$, consider a basic waveform $\Psi_a\colon\mathbb{R}^2\longrightarrow\mathbb{C}$ defined via the polar LCT (5) as

$$\mathscr{L}_M[\Psi_a](r,\omega)=a^{3/4}W(ar)V\left(\frac{\omega}{\sqrt{a}}\right), \tag{14}$$

where the radial and angular windows $W(r)$ and $V(\omega)$ satisfy the slightly modified set of admissibility conditions given by

$$B^2 \int_0^{\infty} |W(r)|^2 \frac{dr}{r} = 1, \operatorname{supp}(W) \subseteq \left(\frac{1}{2}, 2\right), \tag{15}$$

$$(2\pi)^2 \int_{-1}^{1} |V(\omega)|^2 d\omega = 1, \ \operatorname{supp}(V) \subseteq [-1, 1]. \tag{16}$$

Applying the inverse LCT (6) on both sides of the expression (14), we have

$$\begin{aligned} \Psi_a(\rho, \eta) &= \frac{a^{3/4}}{2\pi B} \int_0^{2\pi} \int_0^{\infty} W(ar) V\left(\frac{\omega}{\sqrt{a}}\right) \exp\left\{-\frac{i\left(A\rho^2 + Dr^2 - 2\rho r \cos(\eta - \omega)\right)}{2B}\right\} r dr d\omega, \\ &= \frac{a^{3/4}}{2\pi B} \exp\left\{-\frac{iA\rho^2}{2B}\right\} \int_0^{2\pi} \int_0^{\infty} \exp\left\{-\frac{i Dr^2}{2B}\right\} W(ar) V\left(\frac{\omega}{\sqrt{a}}\right) \times \exp\left\{\frac{i\rho r \cos(\eta - \omega)}{B}\right\} r dr d\omega. \end{aligned} \tag{17}$$

and upon simplifying (17), we obtain a novel basic waveform $\Psi_a^M(\mathbf{t})$ via the following expression:

$$\mathscr{F}\left[\Psi_a^M\right](r, \omega) = a^{3/4} B \exp\left\{-\frac{i DB r^2}{2}\right\} W(aBr) V\left(\frac{\omega}{\sqrt{a}}\right), \tag{18}$$

where $\Psi_a^M(\mathbf{t}) = \exp\{iA|\mathbf{t}|^2/2B\}\Psi_a(\mathbf{t})$.

Hence, the family of novel curvelets $\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})$ (or linear canonical curvelets) is obtained by translating the basic waveform $\Psi_a^M(\mathbf{t})$ by $\mathbf{b} \in \mathbb{R}^2$ and then inducing a rotation of $\theta \in [0, 2\pi)$ radians; that is,

$$\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t}) = \Psi_a^M\left(R_\theta(\mathbf{t} - \mathbf{b})\right), \quad \mathbf{t} \in \mathbb{R}^2. \tag{19}$$

Having formulated a new family of curvelets $\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})$ by invoking the two-dimensional linear canonical transform (1), we are ready to introduce the formal definition of the novel curvelet transform.

Definition 3. Given a real, unimodular matrix $M = (A, B: C, D)$ with $B > 0$, for any square-integrable function f on $\mathbb{R}^2$, the novel curvelet transform is defined as

$$\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta) = \langle f, \Psi_{a,\mathbf{b},\theta}^M \rangle_2 = \int_{\mathbb{R}^2} f(\mathbf{t}) \overline{\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})} d\mathbf{t}, \tag{20}$$

where $a < a_0, \mathbf{b} \in \mathbb{R}^2, \theta \in [0, 2\pi)$, and $\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})$ is given by (19).

Definition 3 embodies many new integral transforms that are yet to be reported in the open literature. Below we point out some important deductions.

(i) Choosing the matrix $M = (\cos\alpha, \sin\alpha: -\sin\alpha, \cos\alpha), \alpha \neq n\pi, n \in \mathbb{Z}$, Definition 3 yields a new curvelet transform combining the merits of the ordinary curvelet transform and the well-known fractional Fourier transform

(ii) For $M = (1, B: 0, 1)$, $B \neq 0$, Definition 3 intertwines the advantages of the ordinary curvelet and the well-known Fresnel transforms into a new curvelet transform

(iii) Nevertheless, when $M = (0, 1: -1, 0)$, Definition 3 boils down to the ordinary curvelet transform (13)

Next, we shall present a proposition that interlinks the Fourier transform of the novel curvelet transform $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ as a function of the translation variable $\mathbf{b}$, with the respective Fourier transforms of the given function f and the basic waveform Ψ_a^M.

Proposition 1. *Given any $f \in L^2(\mathbb{R}^2)$, the novel curvelet transform $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ defined in (20) can be expressed as*

$$\mathscr{F}\left(\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right)(\xi) = 2\pi \mathscr{F}[f](\xi) \mathscr{F}\left[\Psi_a^M\right](R_\theta \xi). \tag{21}$$

Proof. To accomplish the motive, we shall firstly compute the Fourier transform of the novel curvelet family $\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})$ defined in (19). We proceed as

$$\begin{aligned} \mathscr{F}\left[\Psi_{a,\mathbf{b},\theta}^M\right](\xi) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Psi_{a,\mathbf{b},\theta}^M(\mathbf{t}) e^{-i t^T \xi} d\mathbf{t} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Psi_a^M\left(R_\theta(\mathbf{t} - \mathbf{b})\right) e^{-i t^T \xi} d\mathbf{t} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Psi_a^M(\mathbf{z}) e^{-i\left(b + R_{-\theta} z\right)^T \xi} d\mathbf{z} \\ &= \frac{e^{-i b^T \xi}}{2\pi} \int_{\mathbb{R}^2} \Psi_a^M(z) e^{-i z^T (R_\theta \xi)} d\mathbf{z} \\ &= e^{-i b^T \xi} \mathscr{F}\left[\Psi_a^M\right](R_\theta \xi). \end{aligned} \tag{22}$$

Let (σ, μ), (ρ, η), and (r, ω) denote the polar coordinates of the variables $\mathbf{b}$, $\mathbf{t}$, and ξ, respectively. Then, we can rewrite (22) as follows:

$$\mathscr{F}\left[\Psi^M_{a,b,\theta}\right](r,\omega) = e^{-ir\sigma\cos(\mu-\omega)}\mathscr{F}\left[\Psi^M_a\right](r,\omega-\theta) = e^{-ir\sigma\cos(\mu-\omega)}a^{3/4}B\exp\left\{-\frac{iDBr^2}{2}\right\} W(aBr)V\left(\frac{\omega-\theta}{\sqrt{a}}\right). \tag{23}$$

Finally, using Definition 3 and invoking the well-known Parseval's formula in polar coordinates, we have

$$\left[\Gamma^M_\Psi f\right](a,\mathbf{b},\theta) = \langle f, \Psi^M_{a,b,\theta}\rangle_2 = a^{3/4}B\int_0^{2\pi}\int_0^\infty e^{ir\sigma\cos(\mu-\omega)}\mathscr{F}[f](r,\omega)\exp\left\{\frac{iDBr^2}{2}\right\}W(aBr)V\left(\frac{\omega-\theta}{\sqrt{a}}\right)r\mathrm{d}r\mathrm{d}\omega, \tag{24}$$

This completes the proof of Proposition 1.

Next, translating the expression (24) into cartesian coordinates yields the following:

$$\left[\Gamma^M_\Psi f\right](a,\mathbf{b},\theta) = \int_{\mathbb{R}^2}\mathscr{F}[f](\xi)\overline{\mathscr{F}\left[\Psi^M_a\right]}(R_\theta\xi)e^{ib^T\xi}\mathrm{d}\xi = 2\pi\mathscr{F}^{-1}\left(\mathscr{F}[f](\xi)\overline{\mathscr{F}\left[\Psi^M_a\right]}(R_\theta\xi)\right)(\mathbf{b}), \tag{25}$$

Applying the Fourier transform on both sides of (25), we obtain the desired result

$$\mathscr{F}\left(\left[\Gamma^M_\Psi f\right](a,\mathbf{b},\theta)\right)(\xi) = 2\pi\mathscr{F}[f](\xi)\mathscr{F}\left[\Psi^M_a\right](R_\theta\xi). \tag{26}$$

Next, we shall analyze the support and oscillatory behaviour of the novel curvelet transform by invoking Proposition 1. We shall demonstrate that the proposed transform enjoys a certain degree of freedom as the radial window is comparatively more flexible with the degree of flexibility governed by the matrix parameter B. As such, the proposed transform is capable of optimizing the concentration of the curvelet spectrum.

Let $(\sigma,\mu), \sigma\geq 0, \mu\in[0,2\pi)$ be the polar coordinates of the translation variable $\mathbf{b}$. Then, as a consequence of Proposition 1, we can express the novel curvelet transform (20) as

$$\left[\Gamma^M_\Psi f\right](a,\mathbf{b},\theta) = \left[\Gamma^M_\Psi f\right](a,(\sigma,\mu),\theta) = a^{3/4}B\int_0^{2\pi}\int_0^\infty e^{ir\sigma\cos(\mu-\omega)}\mathscr{F}[f](r,\omega)\exp\left\{\frac{iDBr^2}{2}\right\}W(aBr)V\left(\frac{\omega-\theta}{\sqrt{a}}\right)r\mathrm{d}r\mathrm{d}\omega. \tag{27}$$

From (23), we observe that the support of the analyzing elements $\Psi^M_{a,\mathbf{b},\theta}$ in the frequency domain is completely determined by the support of the radial window $W(aBr)$ and the angular window $V(\omega-\theta/\sqrt{a})$. Moreover, we observe that

$$\operatorname{supp}(W(aBr))\subseteq\left(\frac{1}{2aB},\frac{2}{aB}\right)\text{and}\operatorname{supp}\left(V\left(\frac{\omega-\theta}{\sqrt{a}}\right)\right)\subseteq[-\sqrt{a}+\theta,\sqrt{a}+\theta]. \tag{28}$$

Hence, we conclude that the support of the analyzing elements $\Psi^M_{a,\mathbf{b},\theta}$ in the frequency domain depends upon the choice of the matrix parameter B and is completely independent of the translation parameter $\mathbf{b}$. Therefore, an appropriate matrix parameter B can be chosen to optimize the concentration of novel curvelet spectrum.

On the other hand, since the curvelet functions $\Psi^M_{a,\mathbf{b},\theta}$ have compact support in the frequency domain, the well-known Heisenberg's uncertainty principle implies that the novel curvelet functions cannot have compact support in the time domain. We note that, for large $|\mathbf{t}|$, the decay of the novel curvelet functions $\Psi^M_{a,\mathbf{b},\theta}(\mathbf{t})$ depends upon the smoothness of the corresponding Fourier transform; the smoother $\mathscr{F}[\Psi^M_{a,\mathbf{b},\theta}](\xi)$ is, the faster the decay is. Moreover, by definition, $\mathscr{F}[\Psi^M_a](\xi)$ is supported away from the vertical axis $\xi_1 = 0$ but near the horizontal axis $\xi_2 = 0$. Hence, for smaller values of $a < a_0$, the basic waveform $\Psi^M_a(\mathbf{t})$ is less oscillatory in t_2 direction and more oscillatory in t_1 direction.

Below, we shall present the formal reconstruction formula associated with the novel curvelet transform. We note that the said reconstruction formula is valid for high-frequency signals. The analogue for low-frequency signals will be dealt with afterwards. To facilitate the narrative, we need the following definition. □

Definition 4. Given any two functions $f, g\in L^2(\mathbb{R}^2)$, the convolution operation is denoted by $\circledast$ and is defined as

$$(f \circledast g)(\mathbf{z}) = \int_{\mathbb{R}^2} f(\mathbf{t}) g(\mathbf{z}-\mathbf{t}) \mathrm{d}t, \tag{29}$$

Moreover, the convolution theorem corresponding to (29) reads

$$\mathscr{F}[f \circledast g](\xi) = 2\pi \mathscr{F}[f](\xi) \mathscr{F}[g](\xi). \tag{30}$$

Theorem 1 (Reconstruction Formula). *For any $f \in L^2(\mathbb{R}^2)$ satisfying $\mathscr{F}[f](\xi) = 0, \forall |\xi| < 2/a_0 B, a_0 < \pi^2$, the reconstruction formula for the novel curvelet transform $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ defined in (20) is given by*

$$f(\mathbf{t}) = \int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} [\Gamma_\Psi^M f](a, b, \theta) \Psi_{a,b,\theta}^M(t) \frac{\mathrm{d}a \mathrm{d}b \mathrm{d}\theta}{a^3}, \tag{31}$$

where the radial and angular windows W and V satisfy their respective admissibility conditions (15) and (16).

Proof. We note that the novel curvelet transform $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ defined in (20) can be expressed via the convolution $\circledast$ as follows:

$$\begin{aligned}\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta) &= \int_{\mathbb{R}^2} f(\mathbf{t}) \overline{\Psi_a^M}(R_\theta(\mathbf{t}-\mathbf{b})) \mathrm{d}\mathbf{t} \\ &= \int_{\mathbb{R}^2} f(\mathbf{t}) \overline{\Psi_{a,0,\theta}^M}(-(\mathbf{b}-\mathbf{t})) \mathrm{d}\mathbf{t} \\ &= \left(f \circledast \widetilde{\Psi}_{a,0,\theta}^M\right)(\mathbf{b}), \widetilde{\Psi}^M(\mathbf{t}) = \overline{\Psi^M}(-\mathbf{t}).\end{aligned} \tag{32}$$

Next, we define a function

$$F_{a,\theta}^M(\mathbf{t}) = \int_{\mathbb{R}^2} \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta) \Psi_{a,b,\theta}^M(\mathbf{t}) \mathrm{d}\mathbf{b}, \tag{33}$$

Invoking (32), we can express (33) as follows:

$$F_{a,\theta}^M(\mathbf{t}) = \left(\left(f \circledast \widetilde{\Psi}_{a,0,\theta}^M\right)(\mathbf{b}) \circledast \Psi_{a,0,\theta}^M(\mathbf{b})\right)(\mathbf{t}). \tag{34}$$

Applying the convolution theorem (30), we can compute the Fourier transform of the function $F_{a,\theta}(\mathbf{t})$ as

$$\begin{aligned}\mathscr{F}\left[F_{a,\theta}^M\right](\xi) &= 2\pi \mathscr{F}\left[\left(f \circledast \widetilde{\Psi}_{a,0,\theta}^M\right)\right](\xi) \mathscr{F}\left[\Psi_{a,0,\theta}^M\right](\xi) \\ &= (2\pi)^2 \mathscr{F}[f](\xi) \left|\mathscr{F}\left[\Psi_{a,0,\theta}^M\right](\xi)\right|^2.\end{aligned} \tag{35}$$

Consequently, we have

$$\int_0^{a_0} \int_0^{2\pi} \mathscr{F}\left[F_{a,\theta}^M\right](\xi) \frac{\mathrm{d}\theta \, \mathrm{d}a}{a^3} = (2\pi)^2 \mathscr{F}[f](\xi) \int_0^{a_0} \int_0^{2\pi} \left|\mathscr{F}\left[\Psi_{a,0,\theta}^M\right](\xi)\right|^2 \frac{\mathrm{d}\theta \, \mathrm{d}a}{a^3}. \tag{36}$$

Next, we shall evaluate the integral on the right-hand side of (36). To do so, we shall use the polar coordinates of ξ and invoke the admissibility conditions (15) and (16). For $r \geq 2/a_0 B, a_0 < \pi^2$, we have

$$\begin{aligned}&(2\pi)^2 \int_0^{a_0} \int_0^{2\pi} \left|\mathscr{F}\left[\Psi_{a,0,\theta}^M\right](r, \omega)\right|^2 \frac{\mathrm{d}\theta \mathrm{d}a}{a^3} \\ &= (2\pi B)^2 \int_0^{a_0} \int_0^{2\pi} |W(aBr)|^2 \left|V\left(\frac{\omega-\theta}{\sqrt{a}}\right)\right|^2 \frac{\mathrm{d}\theta \, \mathrm{d}a}{a^{3/2}} \\ &= B^2 \int_0^{a_0} |W(aBr)|^2 \left\{(2\pi)^2 \int_0^{2\pi} \left|V\left(\frac{\omega-\theta}{\sqrt{a}}\right)\right|^2 \mathrm{d}\theta\right\} \frac{\mathrm{d}a}{a^{3/2}} \\ &= B^2 \int_0^{a_0} |W(aBr)|^2 \frac{\mathrm{d}a}{a} \\ &= B^2 \int_0^{a_0 Br} \left|W(r)'\right|^2 \frac{\mathrm{d}r'}{r'} = 1.\end{aligned} \tag{37}$$

Implementing (37) in (36), we obtain

$$\mathscr{F}[f](\xi) = \int_0^{a_0} \int_0^{2\pi} \mathscr{F}\left[F_{a,\theta}^M\right](\xi) \frac{\mathrm{d}\theta \mathrm{d}a}{a^3}. \tag{38}$$

That is,

$$f(\mathbf{t}) = \int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta) \Psi_{a,b,\theta}^M(\mathbf{t}) \frac{\mathrm{d}a \mathbf{d}\mathbf{b} \mathrm{d}\theta}{a^3}. \tag{39}$$

This completes the proof of Theorem 1. □

Theorem 2 (Rayleigh's Energy Formula). *For any $f \in L^2(\mathbb{R}^2)$ satisfying $\mathscr{F}[f](\xi) = 0, \forall |\xi| < 2/a_0 B, \quad a_0 < \pi^2$, we have*

$$\left\|\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right\|_2^2 = \|f\|_2^2. \tag{40}$$

That is, the total energy of the signal is preserved from the natural domain $L^2(\mathbb{R}^2)$ to transformed domain $L^2((0, a_0) \times \mathbb{R}^2 \times [0, 2\pi))$, where $a_0 < \pi^2$.

Proof. Invoking the well-known Parseval's formula and using (30), we have

$$\begin{aligned}\left\|\left[\Gamma_\Psi^M f\right](a,\mathbf{b},\theta)\right\|_2^2 &= \int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}\left|\left[\Gamma_\Psi^M f\right](a,\mathbf{b},\theta)\right|^2\frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}\\ &= \int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}\left|\left(f\circledast\tilde{\Psi}_{a,0,\theta}^M\right)(\mathbf{b})\right|^2\frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}\\ &= \int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}\left|\mathcal{F}\left[\left(f\circledast\tilde{\Psi}_{a,0,\theta}^M\right)\right](\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\xi\mathrm{d}\theta}{a^3}\\ &= (2\pi)^2\int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|\mathcal{F}[f](\xi)|^2\left|\mathcal{F}\left[\Psi_{a,0,\theta}^M\right](\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\xi\mathrm{d}\theta}{a^3}\\ &= \int_{\mathbb{R}^2}|\mathcal{F}[f](\xi)|^2\left\{(2\pi)^2\int_0^{a_0}\int_0^{2\pi}\left|\mathcal{F}\left[\Psi_{a,0,\theta}^M\right](\xi)\right|^2\frac{\mathrm{d}\theta\mathrm{d}a}{a^3}\right\}\mathrm{d}\xi\\ &= \int_{\mathbb{R}^2}|\mathcal{F}[f](\xi)|^2\mathrm{d}\xi = \|f\|_2^2,\end{aligned} \tag{41}$$

which evidently completes the proof. □

Remark 2. From (40), we infer that the novel curvelet transform defined in (20) is an isometry from the space of signals $L^2(\mathbb{R}^2)$ to the space of transforms $L^2((0,a_0)\times\mathbb{R}^2\times[0,2\pi))$, where $a_0<\pi^2$.

We note that the reconstruction formula (31) is concerned for those signals $f\in L^2(\mathbb{R}^2)$ satisfying $\mathcal{F}[f](\xi)=0, \forall|\xi|<2/a_0B, a_0<\pi^2$. In order to have a complete reconstruction formula, we need to take care of the other frequency components as well. To facilitate the narrative, we consider an arbitrary square integrable function f on $\mathbb{R}^2$ and define

$$\begin{aligned}(T_1f)(\mathbf{t}) &= \int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}\left[\Gamma_\Psi^M f\right](a,\mathbf{b},\theta)\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})\frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}\\ &= \int_0^{2\pi}\int_0^{a_0}\left(\left(f\circledast\tilde{\Psi}_{a,0,\theta}^M\right)(\mathbf{b})\circledast\Psi_{a,0,\theta}^M(\mathbf{b})\right)(\mathbf{t})\frac{\mathrm{d}a\mathrm{d}\theta}{a^3},\end{aligned} \tag{42}$$

$$(T_0f)(\mathbf{t}) = f(\mathbf{t})-(T_1f)(\mathbf{t}). \tag{43}$$

Here, we note that

$$\begin{aligned}\mathcal{F}[(T_1f)](\xi) &= (2\pi)^2\int_0^{2\pi}\int_0^{a_0}\mathcal{F}[f](\xi)\left|\mathcal{F}\left[\Psi_{a,0,\theta}^M\right](\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\theta}{a^3}\\ &= B^2\mathcal{F}[f](\xi)\int_0^{a_0B|\xi|}|W(a)|^2\frac{\mathrm{d}a}{a}\\ &= (2\pi)^2\mathcal{F}[f](\xi)\left(\mathcal{F}\left[\Omega^M\right](\xi)\right)^2,\end{aligned} \tag{44}$$

where $(\mathcal{F}[\Omega^M](\xi))^2 = B^2/(2\pi)^2\int_0^{a_0B|\xi|}|W(a)|^2\mathrm{d}a/a$.

Furthermore, using additivity of the Fourier transform, we observe that

$$\begin{aligned}\mathcal{F}[(T_0f)](\xi) &= \mathcal{F}[f](\xi)-\mathcal{F}[(T_1f)](\xi)\\ &= \mathcal{F}[f](\xi)\left(1-(2\pi)^2\left(\mathcal{F}\left[\Omega^M\right](\xi)\right)^2\right)\\ &= (2\pi)^2\mathcal{F}[f](\xi)\left(\frac{1}{(2\pi)^2}-\left(\mathcal{F}\left[\Omega^M\right](\xi)\right)^2\right)\\ &= (2\pi)^2\mathcal{F}[f](\xi)\left(\mathcal{F}\left[\Phi^M\right](\xi)\right)^2,\end{aligned} \tag{45}$$

where $(\mathcal{F}[\Phi^M](\xi))^2 = 1/(2\pi)^2-(\mathcal{F}[\Omega^M](\xi))^2$.

Also, thanks to the convolution theorem (30), we infer from (44) and (45) that

$$\begin{aligned}(T_0f)(\mathbf{t}) &= \left(f\circledast\Phi^M\circledast\Phi^M\right)(\mathbf{t}),\\ (T_1f)(\mathbf{t}) &= \left(f\circledast\Omega^M\circledast\Omega^M\right)(\mathbf{t}).\end{aligned} \tag{46}$$

Moreover, we note that

$$(2\pi)^2\left[\left(\mathcal{F}\left[\Omega^M\right](\xi)\right)^2+\left(\mathcal{F}\left[\Phi^M\right](\xi)\right)^2\right] = 1. \tag{47}$$

Also,

$$\begin{aligned}\mathcal{F}\left[\Phi^M\right](\xi) &= 0, \quad |\xi|>\frac{2}{a_0B},\\ \mathcal{F}\left[\Phi^M\right](\xi) &= \frac{1}{2\pi}, \quad |\xi|<\frac{1}{2a_0B}.\end{aligned} \tag{48}$$

Finally, we define the father wavelet $\Phi_{\mathbf{b}}^M(\mathbf{t}) = \Phi^M(\mathbf{t}-\mathbf{b})$, so that

$$\left(T_0 f\right)(\mathbf{t}) = \int_{\mathbb{R}^2} \langle f, \Phi_{\mathbf{b}}^M \rangle_2 \Phi_{\mathbf{b}}^M(\mathbf{t}) d\mathbf{b}. \tag{49}$$

Consequently, (43) implies that

$$f(\mathbf{t}) = \int_{\mathbb{R}^2} \langle f, \Phi_{\mathbf{b}}^M \rangle_2 \Phi_{\mathbf{b}}^M(\mathbf{t}) d\mathbf{b} + \int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta) \Psi_{a,\mathbf{b},\theta}^M(\mathbf{t}) \frac{da d\mathbf{b} d\theta}{a^3}. \tag{50}$$

Therefore, we conclude that the complete reconstruction formula for the novel curvelet transform (20) is composed of both curvelet waveforms and isotropic father wavelets. The above discussion can be summarized into the following theorem:

Theorem 3 (Complete Reconstruction Formula). *For any $f \in L^2(\mathbb{R}^2)$, the reproducing formula for the novel curvelet transform $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ defined in [20] is given by*

$$f(\mathbf{t}) = \int_{\mathbb{R}^2} \langle f, \Phi_{\mathbf{b}}^M \rangle_2 \Phi_{\mathbf{b}}^M(\mathbf{t}) d\mathbf{b} + \int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta) \Psi_{a,\mathbf{b},\theta}^M(\mathbf{t}) \frac{da\mathbf{b}d\theta}{a^3}, \tag{51}$$

where the radial and angular windows W and V satisfy their respective admissibility conditions (15) and (16).

The classical Heisenberg's uncertainty principle in harmonic analysis gives information about the spread of a signal and its Fourier transform by asserting that a signal cannot be sharply localized in both the time and frequency domains [29]. That is, if we limit the behaviour of one, we lose control over the other. The essence of the uncertainty principle is that it provides a lower bound for optimal resolution of a signal in both the time and frequency domains. This classical uncertainty inequality has been extended in different settings and, as of now, many analogues have appeared in the literature [28–31]. In analogy to the uncertainty principles governing the simultaneous localization of a function f and its Fourier transform, a different class of uncertainty principles comparing the localization of f with the localization of its Gabor or wavelet transform were studied by Wilczok [28]. Motivated by this fact, we shall also obtain an uncertainty inequality comparing the localization of the Fourier transform of a function f with the corresponding novel curvelet transform $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$, regarded as a function of the translation variable $\mathbf{b}$.

Theorem 4 (Heisenberg-Type Uncertainty Principle). *If $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ is the novel curvelet transform of any nontrivial function $f \in L^2(\mathbb{R}^2)$, satisfying $\mathcal{F}[f](\xi) = 0, \forall |\xi| < 2/a_0 B, a_0 < \pi^2$, the following uncertainty inequality holds:*

$$\left\{\int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |\mathbf{b}|^2 \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)|^2 \frac{da d\mathbf{b} d\theta}{a^3} |\right\}^{1/2} \cdot \left\{\int_{\mathbb{R}^2} |\xi|^2 |\mathcal{F}[f](\xi)|^2 d\xi\right\}^{1/2} \geq \frac{1}{2} \|f\|_2^2. \tag{52}$$

Proof. The classical Heisenberg-Pauli-Weyl inequality is given by [29]

$$\left\{\int_{\mathbb{R}^2} |\mathbf{t}|^2 |f(\mathbf{t})|^2 dt\right\}^{1/2} \left\{\int_{\mathbb{R}^2} \xi^2 |\mathcal{F}[f](\xi)|^2 d\xi\right\}^{1/2} \geq \frac{1}{2} \left\{\int_{\mathbb{R}^2} |f(\mathbf{t})|^2 d\mathbf{t}\right\}. \tag{53}$$

Identifying $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ as a function of the translation variable $\mathbf{b}$ and invoking (53), we have

$$\left\{\int_{\mathbb{R}^2} |\mathbf{b}|^2 \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)|^2 d\mathbf{b}\right\}^{1/2} \cdot \left\{\int_{\mathbb{R}^2} |\xi|^2 \left|\mathcal{F}\left(\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right)(\xi)\right|^2 d\xi\right\}^{1/2} \geq \frac{1}{2} \left\{\int_{\mathbb{R}^2} \left|\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right|^2 d\mathbf{b}\right\}. \tag{54}$$

Integrating (54) with respect to the measure $da d\theta / a^3$, we obtain

$$\int_0^{2\pi} \int_0^{a_0} \left\{\int_{\mathbb{R}^2} |\mathbf{b}|^2 \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)|^2 d\mathbf{b}\right\}^{1/2} \cdot \left\{\int_{\mathbb{R}^2} |\xi|^2 \left|\mathcal{F}\left(\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right)(\xi)\right|^2 d\xi\right\}^{1/2} \frac{da d\theta}{a^3} \geq \frac{1}{2} \left\{\int_0^{2\pi} \int_0^{a_0} \int_{\mathbb{R}^2} \left|\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right|^2 \frac{d\mathbf{b} da d\theta}{a^3}\right\}. \tag{55}$$

As a consequence of the Cauchy-Schwartz's inequality, Fubini's theorem, and (40), the above inequality can be expressed as

$$\left\{\int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |\mathbf{b}|^2 \left|\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right|^2 \frac{da d\mathbf{b} d\theta}{a^3}\right\}^{1/2} \times \left\{\int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |\xi|^2 \left|\mathcal{F}\left(\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right)(\xi)\right|^2 \frac{da d\xi d\theta}{a^3}\right\}^{1/2} \geq \frac{1}{2} \|f\|_2^2. \tag{56}$$

Invoking (21) and noting that $f \in L^2(\mathbb{R}^2)$ satisfies $\mathcal{F}[f](\xi) = 0, \forall r < 2/a_0 B, a_0 < \pi^2$, we have

$$\int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|\xi|^2\left|\mathscr{F}\left(\left[\Gamma_\Psi^M f\right](a,\mathbf{b},\theta)\right)(\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\xi\mathrm{d}\theta}{a^3}$$
$$=(2\pi)^2\int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|\xi|^2|\mathscr{F}[f](\xi)|^2\left|\mathscr{F}\left[\Psi_a^M\right](R_\theta\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\xi\mathrm{d}\theta}{a^3}$$
$$=(2\pi)^2\int_{\mathbb{R}^2}|\xi|^2|\mathscr{F}[f](\xi)|^2\left\{\int_0^{2\pi}\int_0^{a_0}\left|\mathscr{F}\left[\Psi_a^M\right](R_\theta\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\theta}{a^3}\right\}\mathrm{d}\xi$$
$$=\int_{\mathbb{R}^2}|\xi|^2|\mathscr{F}[f](\xi)|^2\left\{B^2\int_0^{a_0}|W(aBr)|^2\frac{\mathrm{d}a}{a}\right\}\mathrm{d}\xi$$
$$=\int_{\mathbb{R}^2}|\xi|^2|\mathscr{F}[f](\xi)|^2\left\{B^2\int_0^{a_0Br}W(r')|^2\frac{\mathrm{d}r'}{r'}\right\}\mathrm{d}\xi$$
$$=\int_{\mathbb{R}^2}|\xi|^2|\mathscr{F}[f](\xi)|^2\mathrm{d}\xi. \tag{57}$$

Plugging (57) in (56), we obtain the desired Heisenberg-type uncertainty inequality as

$$\left\{\int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|\mathbf{b}|^2\left|\left[\Gamma_\Psi^M f\right](a,\mathbf{b},\theta)\right|^2\frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}\right\}^{1/2}\cdot\left\{\int_{\mathbb{R}^2}|\xi|^2|\mathscr{F}[f](\xi)|^2\mathrm{d}\xi\right\}^{1/2}\geq\frac{1}{2}\|f\|_2^2. \tag{58}$$

This completes the proof of Theorem 4. □

4. Novel Semidiscrete and Discrete Curvelet Transforms

In this section, our main aim is to study both the semidiscrete and discrete analogues of the proposed novel curvelet transform defined in [20]. In the beginning of the section, we formulate the definition of the novel semidiscrete curvelet transform, wherein the spatial variable $\mathbf{b}$ is continuous, whereas the scalings and orientations vary over a discrete grid. In the sequel, we obtain a reconstruction formula associated with the novel semidiscrete curvelet transform. Towards the culmination, we introduce the notion of the novel discrete curvelet transform by extending the aforementioned discretization to the spatial variable $\mathbf{b}$.

4.1. Novel Semidiscrete Curvelet Transform.

To formulate the semidiscrete analogue of the proposed transform (20), we shall discretize the scaling parameter a and the rotation parameter θ in the following manner:

(i) For $\lambda>1$, we choose the j^{th} scale as $a_j=\lambda^{-j}$, $j\geq 0$, and $j\in\mathbb{Z}$.

(ii) For a fixed $L_0\in\mathbb{Z}$, we sample the rotation parameter θ into L_0 equispaced pieces as

$$\theta_\ell=\frac{2\pi\ell}{L_0},\quad\text{where }\ell\in\mathbb{Z}_{L_0}=\{0,1,2,\ldots,L_0-1\}. \tag{59}$$

To prevent the expansion of the angular part as the radial parameter moves away from origin, it is desirable to make the spacing between the consecutive angles scale-dependent. As such, we choose $L_0=\lambda^{\lfloor j/2\rfloor}$, where $\lfloor j/2\rfloor$ denotes the integer part of $\lfloor j/2\rfloor$. Consequently, the scale-dependent angular discretization is given below:

$$\theta_{\ell_j}=\frac{2\pi\ell}{\lambda^{\lfloor j/2\rfloor}},\quad\text{where }\ell\in\mathbb{Z}_{\lambda^{\lfloor j/2\rfloor}}=\left\{0,1,2,\ldots,\lambda^{\lfloor j/2\rfloor}-1\right\}. \tag{60}$$

Now, for a given unimodular matrix $M=(A,B\colon C,D)$, with $B>0$, the radial and angular windows W and V are chosen to satisfy the discrete admissibility conditions:

$$B^2\sum_{j=-\infty}^{\infty}\left|W\left(\lambda^j r\right)\right|^2=1,\quad\lambda>1,r>0, \tag{61}$$

$$(2\pi)^2\sum_{\ell=-\infty}^{\infty}|V(y-\ell)|^2=1,\quad y\in\mathbb{R}. \tag{62}$$

Having discretized the scale and angular parameters, we define a semidiscrete family of linear canonical curvelets as

$$\Psi_{j,b,\ell}^M(\mathbf{t})=\Psi_j^M\left(R_{\theta_{\ell_j}}(\mathbf{t}-\mathbf{b})\right),\quad t\in\mathbb{R}^2, \tag{63}$$

where the novel basic waveform $\Psi_j^M(\mathbf{t})$ is defined in the polar coordinate setting as

$$\mathscr{F}\left[\Psi_j^M\right](r,\omega):=\lambda^{-3j/4}B\exp\left\{-\frac{iDBr^2}{2}\right\}W\left(\frac{Br}{\lambda^j}\right)V\left(\frac{\omega}{\theta_{1_j}}\right)$$
$$=\lambda^{-3j/4}B\exp\left\{-\frac{iDBr^2}{2}\right\}W\left(\frac{Br}{\lambda^j}\right)V\left(\frac{\lambda^{\lfloor j/2\rfloor}\omega}{2\pi}\right), \tag{64}$$

with $\Psi_j^M(\mathbf{t})=\exp\{iA|\mathbf{t}|^2/2B\}\Psi_j(\mathbf{t})$.

With the semidiscrete family of novel curvelets $\Psi_{j,\mathbf{b},\ell}^M(\mathbf{t})$ at hand, we have the following definition.

Definition 5. Given a real, unimodular matrix $M=(A,B\colon C,D)$, with $B>0$, the novel semidiscrete curvelet transform corresponding to any $f\in L^2(\mathbb{R}^2)$ is defined as

$$\left[\Gamma_\Psi^M f\right](j,\mathbf{b},\ell)=\langle f,\Psi_{j,\mathbf{b},\ell}^M\rangle_2=\int_{\mathbb{R}^2}f(\mathbf{t})\overline{\Psi_{j,\mathbf{b},\ell}^M}(\mathbf{t})\mathrm{d}\mathbf{t}, \tag{65}$$

where the novel semidiscrete family $\Psi_{j,\mathbf{b},\ell}^M(\mathbf{t})$ is given by (63).

We now intend to establish a reconstruction formula associated with the novel semidiscrete curvelet transform defined in (65).

Theorem 5 (Reconstruction Formula). *For any $f\in L^2(\mathbb{R}^2)$ satisfying $\mathscr{F}[f](\xi)=0,\forall|\xi|<2/a_0B, a_0<\pi^2$, we have*

$$f(\mathbf{t})=\sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{j/2-1}}\int_{\mathbb{R}^2}\left[\Gamma_\Psi^M f\right](j,b,\ell)\Psi_{j,b,\ell}^M(\mathbf{t})\frac{\mathrm{d}\mathbf{b}}{\lambda^{-3j/2}}, \tag{66}$$

where the radial and angular windows W and V satisfy their respective admissibility conditions (61) and (62).

Proof. For $\lambda > 1$, we define the function

$$F_{j,\ell}^{M}(\mathbf{t}) = \int_{\mathbb{R}^2} \left[\Gamma_{\Psi}^{M} f\right](j, \mathbf{b}, \ell)\Psi_{j,\mathbf{b},\ell}^{M}(\mathbf{t})\frac{d\mathbf{b}}{\lambda^{-3j/2}}. \quad (67)$$

Then, we observe that

$$\begin{aligned}\mathscr{F}\left[F_{j,\ell}^{M}\right](\xi) &= (2\pi)^2\mathscr{F}[f](\xi)\left|\mathscr{F}\left[\Psi_{j,0,\ell}^{M}\right](\xi)\right|^2\\ &= (2\pi B)^2\mathscr{F}[f](\xi)\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\\ &\cdot\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\left(\omega-\theta_{\ell_j}\right)}{2\pi}\right)\right|^2,\end{aligned} \quad (68)$$

Noting that $\mathscr{F}[f](\xi) = 0, \forall|\xi| < 2/a_0 B, a_0 < \pi^2$, and invoking the admissibility condition (61), we have

$$\begin{aligned}B^2\sum_{j\geq 0}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2 &= B^2\sum_{j=-\infty}^{-1}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2 + B^2\sum_{j=0}^{\infty}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\\ &= B^2\sum_{j=-\infty}^{\infty}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\\ &= B^2\sum_{j=-\infty}^{\infty}\left|W\left(\lambda^j Br\right)\right|^2 = 1.\end{aligned} \quad (69)$$

Invoking the admissibility condition (62) yields

$$\begin{aligned}&(2\pi)^2\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\left(\omega-\theta_{\ell_j}\right)}{2\pi}\right)\right|^2,\\ &= (2\pi)^2\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\omega}{2\pi}-\ell\right)\right|^2,\\ &= (2\pi)^2\sum_{\ell=-(\lambda^{j/2}/2)}^{(\lambda^{\lfloor j/2\rfloor}/2)-1}|V(y-\ell)|^2 = 1,\end{aligned} \quad (70)$$

where y is proportional to the distance from ω to the nearest θ_{ℓ_j}. Thus, we have

$$\begin{aligned}&\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\mathscr{F}\left[F_{j,\ell}^{M}\right](\xi),\\ &= B^2\mathscr{F}[f](\xi)\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\left((2\pi)^2\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\left(\omega-\theta_{\ell_j}\right)}{2\pi}\right)\right|^2\right),\\ &= B^2\mathscr{F}[f](\xi)\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2.\end{aligned} \quad (71)$$

Hence,

$$\begin{aligned}\sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\mathscr{F}\left[F_{j,\ell}^{M}\right](\xi) &= \mathscr{F}[f](\xi)\left(B^2\sum_{j\geq 0}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\right)\\ &= \mathscr{F}[f](\xi).\end{aligned} \quad (72)$$

From (72), we obtain the desired reconstruction formula as

$$f(\mathbf{t}) = \sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\int_{\mathbb{R}^2}\left[\Gamma_{\Psi}^{M} f\right](j, b, \ell)\Psi_{j,b,\ell}^{M}(t)\frac{db}{\lambda^{-3j/2}}. \quad (73)$$

This completes the proof of Theorem 5. □

Corollary 1. *Invoking (69) and (70), we observe that*

$$\begin{aligned}&\sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\lambda^{3j/2}\left\|\left[\Gamma_{\Psi}^{M} f\right](j,\mathbf{b},\ell)\right\|_2^2,\\ &= \sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\lambda^{3j/2}\left\|\mathscr{F}\left(\left[\Gamma_{\Psi}^{M} f\right](j,\mathbf{b},\ell)\right)(\xi)\right\|_2^2,\\ &= \sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\int_{\mathbb{R}^2}\left|\mathscr{F}\left(\left[\Gamma_{\Psi}^{M} f\right](j,\mathbf{b},\ell)\right)(\xi)\right|^2\frac{d\xi}{\lambda^{-3j/2}},\\ &= \sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}\lambda^{\lambda\lfloor j/2\rfloor-1}}\int_{\mathbb{R}^2}(2\pi B)^2\lambda^{-3j/2}|\mathscr{F}[f](\xi)|^2\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\\ &\cdot\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\left(\omega-\theta_{\ell_j}\right)}{2\pi}\right)\right|^2\frac{d\xi}{\lambda^{-3j/2}},\\ &\cdot(j/2) = \int_{\mathbb{R}^2}||\mathscr{F}[f](\xi)|^2\left(B^2\sum_{j\geq 0}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\right)\\ &\cdot\left((2\pi)^2\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\left(\omega-\theta_{\ell_j}\right)}{2\pi}\right)\right|^2\right)d\xi,\\ &= \int_{\mathbb{R}^2}|\mathscr{F}[f](\xi)|^2 d\xi = \|f\|_2^2.\end{aligned} \quad (74)$$

4.2. Novel Discrete Curvelet Transform. In this subsection, we shall present a complete discrete analogue of the proposed novel curvelet transform defined in (20). Having formulated the semidiscrete analogue, we need to discretize the spatial variable $\mathbf{b}$ by taking both the previous discretizations of the scale and angular parameters into consideration. For $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $\beta_1, \beta_2 > 0$, we sample the spatial variable $\mathbf{b}$ as

$$\mathbf{b}_{\mathbf{m}}^{j\ell} \coloneqq R_{-\theta_{\ell_j}} \mathscr{K}(\mathbf{m}, \beta_1, \beta_2, j) = R_{-\theta_{\ell_j}} \left(\frac{\beta_1 m_1}{\lambda^j}, \frac{\beta_2 m_2}{\lambda^{j/2}} \right)^T. \tag{75}$$

Consequently, the novel discrete family of curvelets takes the following form:

$$\Psi_{j,\mathbf{m},\ell}^M(\mathbf{t}) = \Psi_j^M \left(R_{\theta_{\ell_j}} \mathbf{t} - \mathscr{K}(\mathbf{m}, \beta_1, \beta_2, j) \right), \quad t \in \mathbb{R}^2, \tag{76}$$

where the basic waveform $\Psi_j^M(\mathbf{t})$ is given by (64). Moreover, an easy computation yields that

$$\begin{aligned} \mathscr{F}\left[\Psi_{j,\mathbf{m},\ell}^M\right](\xi) &= \exp\left\{-i\left(\mathbf{b}_{\mathbf{m}}^{j\ell}\right)^T \xi\right\} \mathscr{F}\left[\Psi_j^M\right]\left(R_{\theta_{\ell_j}} \xi\right), \\ &= \lambda^{-3j/4} B \quad \exp\left\{-i\left(\mathbf{b}_{\mathbf{m}}^{j\ell}\right)^T \xi\right\} \exp\left\{-\frac{i\, DB\, r^2}{2}\right\} \\ &\quad \cdot W\left(\frac{Br}{\lambda^j}\right) V\left(\frac{\lambda^{\lfloor j/2 \rfloor}\left(\omega - \theta_{\ell_j}\right)}{2\pi}\right). \end{aligned} \tag{77}$$

The formal definition of the novel discrete curvelet transform is given below.

Definition 6. Given a real, unimodular matrix $M = (A, B: C, D)$, with $B > 0$, the novel discrete curvelet transform corresponding to any $f \in L^2(\mathbb{R}^2)$ is defined as

$$\left[\Gamma_\Psi^M f\right](j, \mathbf{m}, \ell) = \langle f, \Psi_{j,\mathbf{m},\ell}^M \rangle_2 = \int_{\mathbb{R}^2} f(\mathbf{t}) \overline{\Psi_{j,\mathbf{m},\ell}^M}(\mathbf{t}) d\mathbf{t}, \tag{78}$$

where the novel discrete family of curvelets $\Psi_{j,\mathbf{m},\ell}^M(\mathbf{t})$ is given by (76).

By implementing Parseval's formula for the Fourier transform and taking the benefit of (77), we can express the above definition as

$$\begin{aligned} \left[\Gamma_\Psi^M f\right](j, \mathbf{m}, \ell) &= \langle f, \Psi_{j,\mathbf{m},\ell}^M \rangle_2 = \langle \mathscr{F}[f], \mathscr{F}\left[\Psi_{j,\mathbf{m},\ell}^M\right] \rangle_2 \\ &= \int_{\mathbb{R}^2} \exp\left\{i\left(\mathbf{b}_{\mathbf{m}}^{j\ell}\right)^T \xi\right\} \mathscr{F}[f](\xi) \overline{\mathscr{F}\left[\Psi_j^M\right]}\left(R_{\theta_{\ell_j}} \xi\right) d\xi. \end{aligned} \tag{79}$$

In analogy to the continuous case, we need to take care of the low-frequency signals. We introduce another radial window $W_0(r)$ satisfying

$$\left|W_0(Br)\right|^2 + \sum_{j \geq 0} W\left(\frac{Br}{\lambda^j}\right) = \frac{1}{(2\pi)^2}. \tag{80}$$

And, for $\mathbf{m} \in \mathbb{Z}^2$, the father wavelet $\Phi_{\mathbf{m}}^M$ is defined by

$$\Phi_{\mathbf{m}}^M(\mathbf{t}) = \Phi^M(\mathbf{t} - \mathbf{m}), \text{where } \mathscr{F}\left[\Phi^M\right](\xi) = W_0(B|\xi|). \tag{81}$$

These father wavelets are nondirectional in nature. Therefore, the complete family of novel discrete curvelets $\mathscr{F}_{\Phi,\Psi}$ takes the following form:

$$\mathscr{F}_{\Phi,\Psi} \coloneqq \left\{\Phi_{\mathbf{m}}^M(t) \colon \mathbf{m} \in \mathbb{Z}^2\right\} \bigcup \left\{\Psi_{j,\mathbf{m},\ell}^M(\mathbf{t}) \colon j \geq 0, \mathbf{m} \in \mathbb{Z}^2, \ell \in \mathbb{Z}_{\lambda^{j/2}}\right\}. \tag{82}$$

5. Construction of Novel Curvelets: An Example

In this section, we shall present a lucid construction of the radial and angular window functions W and V satisfying the prescribed admissibility conditions. As is evident from (18) and (64), the construction of basic curvelet waveforms is governed by the admissible radial and angular window functions W and V; therefore, the upcoming example also guides the construction of novel basic curvelet waveforms. Consequently, the family of novel curvelets can be obtained by appropriately translating and rotating the basic waveform. It is pertinent to mention that our approach is motivated by [19].

Example 1. Given a 2×2 real, unimodular matrix $M = (A, B: C, D)$, with $B > 0$, we consider the following window functions:

$$\begin{aligned} W(r) &= \begin{cases} \dfrac{1}{B} \cos\left[\dfrac{\pi}{2}(\nu(5 - 6r))\right], & 2/3 \leq r \leq 5/6, \\ \dfrac{1}{B}, & 5/6 \leq r \leq 4/3, \\ \dfrac{1}{B} \cos\left[\dfrac{\pi}{2}(\nu(3r - 4))\right], & 4/3 \leq r \leq 5/3, \\ 0, & \text{elsewhere}, \end{cases} \\ V(\omega) &= \begin{cases} \dfrac{1}{2\pi} \cos\left[\dfrac{\pi}{2}(\nu(-3\omega - 1))\right], & -2/3 \leq \omega \leq -1/3, \\ \dfrac{1}{2\pi}, & -1/3 \leq \omega \leq 1/3, \\ \dfrac{1}{2\pi} \cos\left[\dfrac{\pi}{2}(\nu(3\omega - 1))\right], & -1/3 \leq \omega \leq 2/3, \\ 0, & \text{elsewhere}, \end{cases} \end{aligned} \tag{83}$$

where ν is a smooth function, such that

$$\begin{aligned} \nu(y) &= \begin{cases} 0, & y \leq 0, \\ 1, & y \geq 1, \end{cases} \\ \nu(y) + \nu(1 - y) &= 1, \quad y \in \mathbb{R}. \end{aligned} \tag{84}$$

Certain choices of the function ν include $\nu(y) = y$ or even smoother polynomials like $\nu(y) = 3y^2 - 2y^3$ and $\nu(y) = y^5 - 5y^4 + 5y^3$. We note that the smoothness of the window functions W and V is governed by the function ν. The smoother ν is, the smoother the window functions are and consequently the faster the decay of curvelets is. As an example, one of the sufficiently smooth functions is given below:

$$\nu(y)=\begin{cases}-0, & y\le 0,\\ \dfrac{\alpha(y-1)}{\alpha(y-1)+\alpha(y)}, & 0<y<1\text{, where,}\quad \alpha(y)=\exp\left\{-\dfrac{1}{(1+y)^2}-\dfrac{1}{(1-y)^2}\right\}.\\ -1, & y\ge 1.\end{cases} \tag{85}$$

Next, we show that the aforementioned window functions obey the admissibility conditions (61) and (62). By definition, we have $\operatorname{supp}V\subseteq[-2/3,2/3]$. Firstly, we shall compute the sum $\sum_{\ell=-\infty}^{\infty}|V(\omega-\ell)|^2$, where $\omega\in\mathbb{R}$. For a fixed $\omega\in\mathbb{R}$, the aforementioned sum contains only two nonvanishing terms, and for $t\in[1/3,2/3]$ we have

$$\begin{aligned}(2\pi)^2\sum_{\ell=-\infty}^{\infty}|V(\omega-\ell)|^2&=(2\pi)^2\left(|V(\omega)|^2+|V(\omega-1)|^2\right),\\ &=\cos^2\left[\frac{\pi}{2}(\nu(3\omega-1))\right]+\cos^2\left[\frac{\pi}{2}(\nu(-3\omega+2))\right]\\ &=\cos^2\left[\frac{\pi}{2}(\nu(x))\right]+\cos^2\left[\frac{\pi}{2}(\nu(1-x))\right]\\ &=\cos^2\left[\frac{\pi}{2}(\nu(x))\right]+\cos^2\left[\frac{\pi}{2}(1-\nu(x))\right]\\ &=\cos^2\left[\frac{\pi}{2}(\nu(x))\right]+\sin^2\left[\frac{\pi}{2}(\nu(x))\right]=1.\end{aligned} \tag{86}$$

In order to show that the admissibility condition (61) holds for the window function W we choose the scale $\lambda=2$. Since $\operatorname{supp}W\subset[1/2,2]$, it follows that $\operatorname{Supp}W(2^jr)\subseteq[2^{-j-1},2^{1-j}]$. Consequently, the sum on the left-hand side of (61) has only two nonvanishing terms corresponding to $r\in[1/2,1]$, namely, $|W(r)|^2$ and $|W(2r)|^2$. Thus, for $r\in[1/2,1]$, we have

$$\begin{aligned}B^2\sum_{j=-\infty}^{\infty}\left|W\left(2^jr\right)\right|^2&=B^2\left(|W(r)|^2+|W(2r)|^2\right),\\ &=\begin{cases}1, & 1/2\le r\le 2/3,\\ \cos^2\left[\dfrac{\pi}{2}(\nu(6r-4))\right]+\cos^2\left[\dfrac{\pi}{2}(\nu(5-6r))\right], & 2/3\le r\le 5/6,\\ 1, & 5/6\le r\le 1,\\ 0, & \text{elsewhere.}\end{cases}\end{aligned} \tag{87}$$

Moreover, we observe that

$$\begin{aligned}
&\cos^2\left[\frac{\pi}{2}\left(\nu(6r-4)\right)\right]+\cos^2\left[\frac{\pi}{2}\left(\nu(5-6r)\right)\right]\\
&\quad=\cos^2\left[\frac{\pi}{2}\left(\nu(z)\right)\right]+\cos^2\left[\frac{\pi}{2}\left(\nu(1-z)\right)\right]\\
&\quad=\cos^2\left[\frac{\pi}{2}\left(\nu(z)\right)\right]+\cos^2\left[\frac{\pi}{2}\left(1-\nu(z)\right)\right]\\
&\quad=\cos^2\left[\frac{\pi}{2}\left(\nu(z)\right)\right]+\sin^2\left[\frac{\pi}{2}\left(\nu(z)\right)\right]=1,
\end{aligned}\tag{88}$$

Plugging equation (88) in equation (87), we obtain

$$B^2\sum_{j=-\infty}^{\infty}\left|W\left(2^j r\right)\right|^2=1.\tag{89}$$

Finally, if we choose $\ln 2\, W'(r)=W(r)$, then we shall demonstrate that the window functions W' and V satisfy the admissibility conditions (15) and (16). We proceed with

$$\begin{aligned}
1&=(2\pi)^2\sum_{\ell=-\infty}^{\infty}|V(\omega-\ell)|^2=(2\pi)^2\int_0^1\sum_{\ell=-\infty}^{\infty}|V(\omega-\ell)|^2\mathrm{d}\omega\\
&=(2\pi)^2\int_{-\infty}^{\infty}|V(\omega)|^2\mathrm{d}\omega.
\end{aligned}\tag{90}$$

Finally, for $r\in(0,\infty)$, we take $r=2^x, x\in(-\infty,\infty)$ so that we have

$$\begin{aligned}
1&=B^2\sum_{j=-\infty}^{\infty}\left|W\left(2^j r\right)\right|^2=B^2\sum_{j=-\infty}^{\infty}\left|W\left(2^{j+x}\right)\right|^2\\
&=B^2\int_0^1\sum_{j=-\infty}^{\infty}\left|W\left(2^{j+x}\right)\right|^2\mathrm{d}x\\
&=B^2\sum_{j=-\infty}^{\infty}\frac{1}{\ln 2}\int_{2^j}^{2^{j+1}}|W(y)|^2\frac{\mathrm{d}y}{y}\\
&=B^2\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}|W'(y)|^2\frac{\mathrm{d}y}{y}=B^2\int_0^{\infty}|W'(y)|^2\frac{\mathrm{d}y}{y}.
\end{aligned}\tag{91}$$

6. Conclusion and Future Work

In the present study, we intertwined the advantages of the curvelet and linear canonical transforms and introduced the notion of the novel curvelet transform. The prime advantage of this intertwining lies in the fact that the novel curvelet transform enjoys certain degrees of freedom and the new radial window achieves higher flexibility, which in turn can be employed in optimizing the concentration of the curvelet spectrum. As such, the proposed transform serves as a significant addition to the contemporary tools of signal and image processing. Nevertheless, the present study, in itself, appeals several ramifications and developments thereon. An immediate concern is to study the frame theory associated with the novel discrete curvelet transform.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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References

[1] L. Debnath and F. A. Shah, *Wavelet Transforms and Their Applications*, Birkhäuser, New York, NY, USA, 2015.

[2] L. Debnath and F. A. Shah, *Lectuer Notes on Wavelet Transforms*, Birkhäuser, New York, NY, USA, 2017.

[3] J. P. Antoine, R. Murenzi, P. Vandergheynst, and S. T. Ali, *Two-Dimensional Wavelets and Their Relatives*, Cambridge University Press, Cambridge, UK, 2009.

[4] F. A. Shah and A. Y. Tantary, "Polar wavelet transform and the associated uncertainty principles," *International Journal of Theoretical Physics*, vol. 57, no. 6, pp. 1774–1786, 2018.

[5] F. A. Shah and A. Y. Tantary, "Non-isotropic angular Stockwell transform and the associated uncertainty principles," *Applicable Analysis*, 2019.

[6] H. M. Srivastava, F. A. Shah, and A. Y. Tantary, "A family of convolution-based generalized Stockwell transforms," *Journal of Pseudo-Differential Operators and Applications*, vol. 59, 2020.

[7] E. J. Candès and D. Donoho, "Ridgelets: a key to higher-dimensional intermittency," *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, vol. 357, no. 1760, pp. 2495–2509, 1999.

[8] E. J. Candés and D. L. Donoho, "Continuous curvelet transform I. resolution of the wavefront set," *Applied and Computational Harmonic Analysis*, vol. 19, Article ID 162197, 2005.

[9] E. J. Candés and D. L. Donoho, "Continuous curvelet transform II. discretization and frames," *Applied and Computational Harmonic Analysis*, vol. 19, Article ID 198222, 2005.

[10] M. N. Do and M. Vetterli, "The contourlet transform: an efficient directional multiresolution image representation," *IEEE Transactions on Image Processing*, vol. 14, no. 12, pp. 2091–2106, 2005.

[11] G. Kutyniok and D. Labate, "Resolution of the wavefront set using continuous shearlets," *Transactions of the American Mathematical Society*, vol. 361, pp. 2719–2754, 2009.

[12] E. J. Candés and L. Demanet, "The curvelet representation of wave propagators is optimally sparse," *Communications on Pure and Applied Mathematics*, vol. 58, Article ID 14721528, 2005.

[13] E. J. Candès and D. L. Donoho, "Recovering edges in ill-posed inverse problems: optimality of curvelet frames," *The Annals of Statistics*, vol. 30, no. 3, pp. 784–842, 2002.

[14] E. J. Candés and F. Guo, "New multiscale transforms, minimum total variation synthesis: applications to edge-preserving

image reconstruction," *Signal Processing*, vol. 82, Article ID 15191543, 2002.

[15] E. J. Candés and D. L. Donoho, "New tight frames of curvelets and optimal representations of objects with piecewise-C^2 singularities," *Communications on Pure and Applied Mathematics*, vol. 57, Article ID 219266, 2004.

[16] E. Candès, L. Demanet, D. Donoho, and L. Ying, "Fast discrete curvelet transforms," *Multiscale Modeling & Simulation*, vol. 5, no. 3, pp. 861–899, 2006.

[17] J. Li and M. W. Wong, "Localization operators for curvelet transforms," *Journal of Pseudo-Differential Operators and Applications*, vol. 3, Article ID 121143, 2012.

[18] H. A. Marzouqia and G. A. Regibb, "Curvelet transform with learning-based tiling," *Signal Processing: Image Communication*, vol. 53, p. 2439, 2017.

[19] J. Ma and G. Plonka, "The curvelet transform," *IEEE Signal Processing Magazine*, vol. 27, no. 2, pp. 118–133, 2010.

[20] J. J. Healy, M. A. Kutay, H. M. Ozaktas, and J. T. Sheridan, *Linear Canonical Transforms: Theory and Applications*, Springer, Berlin, Germany, 2016.

[21] S. A. Collins, "Lens-system diffraction integral written in terms of matrix optics," *Journal of the Optical Society of America A*, vol. 60, pp. 1772–1780, 1970.

[22] M. Moshinsky and C. Quesne, "Linear canonical transformations and their unitary representations," *Journal of Mathematical Physics*, vol. 12, no. 8, pp. 1772–1780, 1971.

[23] T. Z. Xu and B. Z. Li, *Linear Canonical Transform and its Applications*, Science Press, Beijing, China, 2013.

[24] Q. Feng and B.-Z. Li, "Convolution and correlation theorems for the two-dimensional linear canonical transform and its applications," *IET Signal Processing*, vol. 10, no. 2, pp. 125–132, 2016.

[25] A. I. Zayed, "Sampling of signals bandlimited to a disc in the linear canonical transform domain," *IEEE Signal Processing Letters*, vol. 25, no. 12, pp. 1765–1769, 2018.

[26] D. Wei, W. Yang, and Y.-M. Li, "Lattices sampling and sampling rate conversion of multidimensional bandlimited signals in the linear canonical transform domain," *Journal of the Franklin Institute*, vol. 356, no. 13, pp. 7571–7607, 2019.

[27] F. A. Shah and A. Y. Tantary, "Linear canonical Stockwell transform," *Journal of Mathematical Analysis and Applications*, vol. 443, pp. 1–28, 2020.

[28] E. Wilczok, "New uncertainty principles for the continuous Gabor transform and the continuous wavelet transform," *Documenta Mathematica*, vol. 5, pp. 201–226, 2000.

[29] G. B. Folland and A. Sitaram, "The uncertainty principle: a mathematical survey," *The Journal of Fourier Analysis and Applications*, vol. 3, no. 3, pp. 207–238, 1997.

[30] W. Beckner, "Pitt's inequality and the uncertainty principle," *Proceedings of the American Mathematical Society*, vol. 123, no. 6, pp. 1897–1905, 1995.

[31] M. Bahri, F. A. Shah, and A. Y. Tantary, "Uncertainty principles for the continuous shearlet transforms in arbitrary space dimensions," *Integral Transforms and Special Function*, vol. 31, 2020.