Research Article

Approximation of Fixed Points and Best Proximity Points of Relatively Nonexpansive Mappings

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In this article, we study the Agarwal iterative process for finding fixed points and best proximity points of relatively nonexpansive mappings. Using the Von Neumann sequence, we establish the convergence result in a Hilbert space framework. We present a new example of relatively nonexpansive mapping and prove that its Agarwal iterative process is more efficient than the Mann and Ishikawa iterative processes.

1. Introduction

Let E be a nonempty subset of a Banach space X. A self-map T of E is said to be nonexpansive mapping if

$$\|Tu - Tv\| \leq \|u - v\|, \quad \text{for all } u, v \in E. \tag{1}$$

The class of nonexpansive mappings is important as an application point of view. One of the celebrated result of Kirk [1] states that any self nonexpansive mapping of closed bounded convex subset E of a reflexive Banach space has a fixed point provided that E has normal structure. This result was also independently proved in the same year by Browder [2] and Gohde [3] in uniformly convex Banach space (in short UCBS). After this celebrated result, many generalizations of nonexpansive mappings have been published [4–14]. Among the other things, one of the natural generalization of nonexpansive mappings was given by Eldred et al. [15] as follows. Let H and L be two nonempty subsets of a Banach space X. A self-map T of $H \cup L$ is said to be relatively nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \quad \text{for all } u \in H \text{ and } v \in L. \tag{2}$$

Iterative methods played a very important role in variational inequalities and many other areas of applied sciences (e.g., see [16–27] and others). One of the earlier iterative scheme is the Picard iteration process, $u_{n+1} = Tu_n$, which converges very well for Banach contraction mappings. However, this scheme is not suitable for finding fixed points of nonexpansive mappings and hence for the generalized nonexpansive mappings. Let E be a nonempty subset set of a Banach space X. In [30], Eldred and Praveen studied Mann [29] iterative process for finding fixed points and best proximity points of relatively nonexpansive mappings. In [30], Gopi and Pragadeeswara studied Ishikawa [31] iterative process for finding fixed points and best proximity points of relatively nonexpansive mappings.

Motivated by the above work, we study the Agarwal [32] iterative process for finding fixed points and best proximity points of relatively nonexpansive mappings. We present a new example of relatively nonexansive mapping and prove

that its Agarwal iterative process is more efficient than the Mann [29] and Ishikawa [31] iterative processes.

Now, we present some notations which will be used in the sequel:

$$P_H(u) = \{v \in H: \|u - v\| = d(u, H)\},$$
$$d(H, L) = \inf\{\|u - v\|: u \in H, v \in L\},$$
$$H_0 = \{u \in H: \|u - v'\| = d(H, L) \text{ for some } v' \in L\},$$
$$L_0 = \{v \in L: \|u' - v\| = d(H, L) \text{ for some } u' \in H\}. \tag{3}$$

Notice that $P_H(u)$ is singleton, provided that H is closed convex in a reflexive and strictly convex space. Moreover, if H and L are a closed convex in a reflexive space, such that one of the H and L is bounded, then $H_0 \neq \varnothing$.

A handful of definitions and theorems given below correspond to our results.

Definition 1. Suppose that H and L be two nonempty subsets of a metric space. A point $u \in H$ is said to be a best proximity point of the nonself-map $T: H \longrightarrow L$ provided that

$$d(u, Tu) = d(H, L). \tag{4}$$

Theorem 1 (see [15]). *Suppose H and L be two nonempty bounded closed convex subsets of a UCBS. Assume that $T: H \cup L \longrightarrow H \cup L$ satisfies*

(i) $T(H) \subseteq L$ and $T(L) \subseteq H$

(ii) $\|Tu - Tv\| \leq \|u - v\|$ for all $u \in H, v \in L$

Then, there exist $(u, v) \in H \times L$ such that $\|u - Tu\| = \|v - Tv\| = d(H, L)$.

Theorem 2 (see [15]). *Suppose H and L be two nonempty closed bounded convex subsets of a UCBS. If $T: H \cup L \longrightarrow H \cup L$ satisfies the following:*

(i) $T(H) \subseteq H$ and $T(L) \subseteq L$

(ii) $\|Tu - Tv\| \leq \|u - v\|$ for all $u \in H, v \in L$

Then, there exist $u_0 \in H$ and $v_0 \in L$ such that $Tu_0 = u_0, Tv_0 = v_0,$ and $\|u_0 - v_0\| = d(H, L)$.

Theorem 3 (see [1–3]). *Assume that T be a self-map on a closed convex bounded subset of a UCBS. If T is nonexpansive, then T has a fixed point.*

Proposition 1 (see [33]). *Suppose that X is a UCBS, $\alpha \in (0, 1)$ and $\varepsilon > 0$, then for each $d > 0$ and $u, v \in X$ be such that $\|u\| \leq d, \|v\| \leq d, \|u - v\| \geq \varepsilon$, then there exists some $\delta = $*
$\delta(\varepsilon/d) > 0$ *such that* $\|u + (1 - \alpha)v\| \leq (1 - 2\delta(\varepsilon/d)(\alpha, 1 - \alpha))d$.

Lemma 1 (see [34]). *Let $0 < a < t_n < b < 1$ for every $n \geq 1$. Assume that $\{u_n\}$ and $\{v_n\}$ are sequence in a UCBS X such that $\|u_n\| \leq 1, \|v_n\| \leq 1$. Define $\{b_n\}$ in X by $b_n = (1 - t_n)u_n + t_n v_n$. If $\lim_{n \to \infty} \|b_n\| = 1$, then $\lim_{n \to \infty} \|u_n - v_n\| = 0$.*

Now, we are going to show that, under some appropriate assumptions, Agarwal's [32] iteration converges to a fixed point of a given nonexpansive mapping. This result is useful for the upcoming main results.

Theorem 4. *Suppose E be a nonempty bounded closed convex subset of a UCBS X, and assume that T be a self-map nonexpansive map of E. Choose $u_0 \in E$ and set $v_n = (1 - \beta_n)u_n + \beta_n Tu_n, u_{n+1} = (1 - \alpha_n)Tu_n + \alpha_n Tv_n,$ where $\beta_n, \alpha_n \in (\varepsilon, 1 - \varepsilon), n = 0, 1, 2, \ldots$ and $\varepsilon \in (0, 1/2)$ and $\sum \alpha_n \beta_n < \infty$. Then, $\lim_{n \to \infty} \|u_n - Tu_n\| = 0$. Moreover, if $T(E)$ lies in a compact set, then $\{u_n\}$ converges to a fixed point of T.*

Proof. By Theorem 3, there exists some $v \in E$ such that $Tv = v$. Now,

$$\begin{aligned}
\|v_n - v\| &= \|(1 - \beta_n)u_n + \beta_n Tu_n - v\| \\
&\leq (1 - \beta_n)\|u_n - v\| + \beta_n\|Tu_n - v\| \\
&\leq (1 - \beta_n)\|u_n - v\| + \beta_n\|u_n - v\| \\
&= \|u_n - v\|, \\
\|u_{n+1} - v\| &= \|(1 - \alpha_n)Tu_n + \alpha_n Tv_n - v\| \\
&\leq (1 - \alpha_n)\|Tu_n - v\| + \alpha_n\|Tv_n - v\| \\
&\leq (1 - \alpha_n)\|u_n - v\| + \alpha_n\|v_n - v\| \\
&\leq (1 - \alpha_n)\|u_n - v\| + \alpha_n\|u_n - v\| \\
&= \|u_n - v\|.
\end{aligned} \tag{5}$$

It follows that the sequence $\{\|u_n - v\|\}$ is nonincreasing and bounded below by 0. Hence, we have $\|u_n - v\| \longrightarrow d \geq 0$. ☐

Case 1. If $\|u_n - v\| \longrightarrow 0$, then

$$\begin{aligned}
\|u_n - Tu_n\| &\leq \|u_n - v\| + \|v - Tu_n\| \\
&= \|u_n - v\| + \|Tv - Tu_n\| \\
&\leq \|u_n - v\| + \|u_n - v\|.
\end{aligned} \tag{6}$$

If $n \longrightarrow \infty$, then $\|u_n - Tu_n\| \longrightarrow 0$. Now,

$$
\begin{aligned}
\|u_{n+1} - u_n\| &= \|(1 - \alpha_n)Tu_n + \alpha_n Tv_n - u_n\| \\
&= \|Tu_n - \alpha_n Tu_n + \alpha_n Tv_n - u_n\| \\
&\leq \alpha_n\|Tu_n - Tv_n\| + \|Tu_n - u_n\| \\
&\leq \alpha_n\|v_n - u_n\| + \|Tu_n - u_n\| \\
&= \alpha_n\|(1 - \beta_n)u_n + \beta_n Tu_n - u_n\| + \|Tu_n - u_n\| \\
&\leq \alpha_n\beta_n\|Tu_n - u_n\| + \|Tu_n - u_n\| \\
&= (\alpha_n\beta_n + 1)\|Tu_n - u_n\| \\
&\leq (\alpha_n\beta_n + 1)(\|Tu_n - v\| + \|v - u_n\|) \\
&= (\alpha_n\beta_n + 1)(\|Tu_n - Tv\| + \|v - u_n\|) \\
&\leq (\alpha_n\beta_n + 1)(\|u_n - v\| + \|v - u_n\|).
\end{aligned}
\tag{7}
$$

If $n \longrightarrow \infty$, then $\|u_{n+1} - u_n\| \longrightarrow 0$.

Case 2. If $\|u_n - v\| \longrightarrow d > 0$, we need to show that $\|u_n - Tu_n\| \longrightarrow 0$. Suppose not, then one have a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a positive real number ε such that $\|u_{n_k} - Tu_{n_k}\| \geq \varepsilon > 0$ for all k.

Since the modulus of convexity δ of X is continuous as well as increasing function, one can select some $\xi > 0$ as small such that $(1 - c\delta(\varepsilon/(d + \xi)))(d + \xi) < d$, where $c > 0$.

Now, we select k, such that $\|u_{n_k} - v\| \leq d + \xi$. By Proposition 1, we have

$$
\begin{aligned}
\|v - u_{n_k+1}\| &= \|v - ((1 - \alpha_{n_k})Tu_{n_k} + \alpha_{n_k}Tv_{n_k})\| \\
&= \|v - ((1 - \alpha_{n_k})Tu_{n_k} + \alpha_{n_k}T((1 - \beta_{n_k})u_{n_k} + \beta_{n_k}Tu_{n_k}))\| \\
&= \|(1 - \alpha_{n_k})v + \alpha_{n_k}v - ((1 - \alpha_{n_k})Tu_{n_k} + \alpha_{n_k}T((1 - \beta_{n_k})u_{n_k} \\
&\quad + \beta_{n_k}Tu_{n_k}))\| \\
&\leq (1 - \alpha_{n_k})\|v - Tu_{n_k}\| + \alpha_{n_k}\|v - T((1 - \beta_{n_k})u_{n_k} + \beta_{n_k}Tu_{n_k})\| \\
&= (1 - \alpha_{n_k})\|Tv - Tu_{n_k}\| + \alpha_{n_k}\|Tv - T((1 - \beta_{n_k})u_{n_k} + \beta_{n_k}Tu_{n_k})\| \\
&\leq (1 - \alpha_{n_k})\|v - u_{n_k}\| + \alpha_{n_k}\|v - ((1 - \beta_{n_k})u_{n_k} + \beta_{n_k}Tu_{n_k})\| \\
&= (1 - \alpha_{n_k})(d + \xi) + \alpha_{n_k}\|(1 - \beta_{n_k})(v - u_{n_k}) + \beta_{n_k}(v - Tu_{n_k})\| \\
&\leq (1 - \alpha_{n_k})(d + \xi) + \alpha_{n_k}(1 \\
&\quad - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\beta_{n_k}, 1 - \beta_{n_k}\})(d + \xi) \\
&= \left(1 - \alpha_{n_k} + \alpha_{n_k} - 2\alpha_{n_k}\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\beta_{n_k}, 1 - \beta_{n_k}\}\right)(d + \xi) \\
&= \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\alpha_{n_k}\beta_{n_k}, \alpha_{n_k}(1 - \beta_{n_k})\}\right)(d + \xi).
\end{aligned}
\tag{8}
$$

Since there exist $h > 0$, such that $2\min\{\alpha_{n_k}\beta_{n_k}, \alpha_{n_k}(1 - \beta_{n_k})\} \geq h$,

$$
\begin{aligned}
&\left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\alpha_{n_k}\beta_{n_k}, \alpha_{n_k}(1 - \beta_{n_k})\}\right)(d + \xi) \\
&\leq \left(1 - h\delta\left(\frac{\varepsilon}{d + \xi}\right)\right)(d + \xi).
\end{aligned}
\tag{9}
$$

Select very small $\xi > 0$, we have $(1 - h\delta(\varepsilon/d + \xi))(d + t\xi) < d$, which is contradiction. This implies that the $\lim_{n \to \infty}\|u_n - Tu_n\| = 0$.

Now, we prove that $\|u_{n+1} - u_n\| \longrightarrow 0$. We have $\|u_{n+1} - u_n\| \leq (\alpha_n\beta_n + 1)\|Tu_n - u_n\|$.

Now, we define $b_n = (u_{n+1} - v)/(\|u_n - v\|)$, $v_n = (Tv_n - v)/(\|u_n - v\|)$, and $u_n = (Tu_n - v)/(\|u_n - v\|)$. One can note that $\|u_n\| \leq 1$. Now,

$$
\begin{aligned}
\|Tv_n - v\| &= \|Tv_n - Tv\| \\
&\leq \|v_n - v\| \\
&= \|(1 - \beta_n)u_n + \beta_n Tu_n - v\| \\
&= \|(1 - \beta_n)u_n - (1 - \beta_n)v + \beta_n(Tu_n - v)\| \\
&\leq (1 - \beta_n)\|u_n - v\| + \beta_n\|Tu_n - Tv\| \\
&\leq (1 - \beta_n)\|u_n - v\| + \beta_n\|u_n - v\| \\
&= \|u_n - v\|.
\end{aligned}
\tag{10}
$$

Therefore, $\|v_n\| = (\|Tv_n - v\|)/(\|u_n - v\|) \leq (\|u_n - v\|)/(\|u_n - v\|) = 1$. From Agarwal's iteration, we obtain $u_{n+1} - v = (1 - \alpha_n)(Tu_n - v) + \alpha_n(Tv_n - v)$. Dividing by $\|u_n - v\|$, we obtain

$$
\frac{u_{n+1} - v}{\|u_n - v\|} = (1 - \alpha_n)\frac{(Tu_n - v)}{\|u_n - v\|} + \alpha_n\frac{(Tv_n - v)}{\|u_n - v\|}.
\tag{11}
$$

Hence, $b_n = (1 - \alpha_n)u_n + \alpha_n v_n$. Now, we show that $\|b_n\| \longrightarrow 1$. Now,

$$
\lim_{n \to \infty}\|b_n\| = \lim_{n \to \infty}\frac{\|u_{n+1} - v\|}{\|u_n - v\|} = \frac{d}{d} = 1.
\tag{12}
$$

By Lemma 1, $\|u_n - v_n\| \longrightarrow 0$. This implies that $\|u_n - Tu_n\| \longrightarrow 0$. Therefore, $\|u_{n+1} - u_n\| \longrightarrow 0$.

Since $T(E)$ is contained in a compact set, $\{Tu_n\}$ has a subsequence $\{Tu_{n_k}\}$ that converges to point $b \in H$. Also, $\{u_{n_k}\}$ and $\{u_{n_k+1}\}$ converge to b. This implies that $\{u_n\}$ converge to b. Then, $Tu_n \longrightarrow b$. In particular, $Tu_{n_k} \longrightarrow b$. Since T is continuous, implies that $Tu_{n_k} \longrightarrow Tb$. Therefore, $Tb = b$.

Theorem 5 (see [28]). *Let H and L be nonempty closed bounded convex subset of a UCBS. Let $T: H \cup L \longrightarrow H \cup L$ satisfy*

(1) $T(H) \subseteq H$ and $T(L) \subseteq L$

(2) $\|Tu - Tv\| \leq \|u - v\|$ for all $u \in H, v \in L$

Let $u_0 \in H$ and define $u_{n+1} = P^n((1 - \alpha_n)u_n + \alpha_n Tv_n)$, where $\alpha_n \in (\varepsilon, 1 - \varepsilon)$, $n = 0, 1, 2, \ldots$ and $\varepsilon \in (0, 1/2)$.

Then, $\lim_{n \to \infty} \|u_n - Tu_n\| = 0$. Moreover, if $T(H)$ lies in a compact set, $\{u_n\}$ converges to a fixed point of T.

Assume that H be a convex closed subset of a Hilbert space X. Then, for $u \in X$, $P_H(u)$ is the nearest to u and element of H. Furthermore, P_H is nonexpansive and distinguished by Kolmogorove's criterion:

$$\langle u - P_H(u), P_H(u) - b \rangle \geq 0, \quad \text{for all } u \in X \text{ and } b \in H. \tag{13}$$

Assume that H and L are two convex closed subsets of X. Set

$$P(u) = P_H(P_L(u)), \quad \text{for each } u \in X. \tag{14}$$

Then, the sequences $\{P^n(u)\} \subset H$ and $\{P_L(P^n(u))\} \subset L$. When H and L are closed, the convergence of these sequences were established by Von Neumann in [35]. The sequences $\{P^n(u)\}$ and $\{P_L(P^n(u))\}$ are called Von Neumann sequences (sometimes called alternating projection algorithm for two sets).

Definition 2 (see [36]). Suppose H and L are two nonempty convex closed subsets of a Hilbert space X. Then, (H, L) is called boundedly regular provided that, for every bounded subset S of X and for every $\varepsilon > 0$ one can select a $\delta > 0$ such that

$$\max\{d(u, H), d(u, L - z)\} \leq \delta \Rightarrow d(u, L) \leq \varepsilon, \forall u \in X, \tag{15}$$

where $z = P_{\overline{L-H}}(0)$ is the displacement vector from the set H to set L (z is the unique vector such that $\|z\| = d(H, L)$).

Theorem 6 (see [36]). *Suppose (H, L) is boundedly regular; then, the Von Neumann sequence converges in norm.*

Theorem 7 (see [36]). *Assume that one of the H and L is boundedly compact; then, (H, L) is boundedly regular.*

Lemma 2 (see [37]). *Suppose that H be a nonempty convex closed subset and L be a nonempty closed subset of a UCBS. Assume that $\{u_n\}$ and $\{b_n\}$ be sequences in H and $\{v_n\}$ be a sequence in L such that*

(i) $\|u_n - v_n\| \longrightarrow d(H, L)$
(ii) $\|b_n - v_n\| \longrightarrow d(H, L)$

Then, $\|u_n - b_n\|$ converges to 0.

Corollary 1 (see [37]). *Suppose H be a nonempty closed convex subset and L be a nonempty closed subset of a UCBS. Assume that $\{u_n\}$ be a sequence in H and $v_0 \in L$ such that $\|u_n - v_0\| \longrightarrow d(H, L)$. Then, $\{u_n\}$ converges to $P_H(v_0)$.*

Proposition 2 (see [15]). *Suppose H and L be two closed and convex subset of a Hilbert space X. Then, $P_L(H) \subseteq L$, $P_H(L) \subseteq H$, and $\|P_L u - P_H v\| \leq \|u - v\|$ for each $u \in H$ and $v \in L$.*

Lemma 3. *Suppose H and L be two closed and convex subset of a Hilbert space X. Then, for each $u \in X$, we have*

$$\|P^{n+1}(u) - b\| \leq \|P^n(u) - b\|, \quad \text{for } b \in H_0 \cup L_0. \tag{16}$$

2. Main Results

Theorem 8. *Suppose H and L be nonempty bounded closed convex subsets of a UCBS X and assume that $T \colon H \cup L \longrightarrow H \cup L$ such that*

(i) $T(H) \subseteq H$ and $T(L) \subseteq L$

(ii) $\|Tu - Tv\| \leq \|u - v\|$, for all $u \in H, v \in L$

Select $u_0 \in H$ and set $v_n = (1 - \beta_n)u_n + \beta_n Tu_n$, $u_{n+1} = (1 - \alpha_n)Tu_n + \alpha_n Tv_n$, where $\beta_n, \alpha_n \in (\varepsilon, 1 - \varepsilon)$, $n = 0, 1, 2, \ldots$ and $\varepsilon \in (0, 1/2)$ and $\sum \alpha_n \beta_n < \infty$. Suppose $d(u_n, H_0) \longrightarrow 0$, then $\lim_{n \to \infty} \|u_n - Tu_n\| = 0$. Moreover, if $T(H)$ lies in a compact set, then $\{u_n\}$ converges to a fixed point of T.

Proof. If $d(H, L) = 0$, then $H_0 = L_0 = H \cap L$, and by Theorem 4, we can establish the theorem from the fact that $T \colon H \cap L \longrightarrow H \cap L$ is nonexpansive. Let $d(H, L) > 0$. By Theorem 2, there exists $v \in L_0$ such that $Tv = v$. Now,

$$
\begin{aligned}
\|v_n - z\| &= \|(1 - \beta_n)u_n + \beta_n Tu_n - v\| \\
&\leq (1 - \beta_n)\|u_n - v\| + \beta_n\|Tu_n - Tv\| \\
&\leq (1 - \beta_n)\|u_n - v\| + \beta_n\|u_n - v\| \\
&= \|u_n - v\|,
\end{aligned}
\tag{17}
$$

which implies that

$$
\begin{aligned}
\|u_{n+1} - v\| &= \|(1 - \alpha_n)Tu_n + \alpha_n Tv_n - v\| \\
&\leq (1 - \alpha_n)\|Tu_n - v\| + \alpha_n\|Tv_n - v\| \\
&\leq (1 - \alpha_n)\|u_n - v\| + \alpha_n\|v_n - v\| \\
&\leq (1 - \alpha_n)\|u_n - v\| + \alpha_n\|u_n - v\| \\
&= \|u_n - v\|.
\end{aligned}
\tag{18}
$$

Hence, the sequence $\{\|u_n - v\|\}$ is nonincreasing. So, one can select a $d > 0$ such that $\lim_{n \to \infty} \|u_n - v\| = d$. Assume that there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and an $\varepsilon > 0$ such that $\|u_{n_k} - Tu_{n_k}\| \geq \varepsilon > 0$ for every k.

However, the modulus of convexity δ of X is continuous and increasing function, and we choose $\xi > 0$ as small that $(1 - c\delta(\varepsilon/(d + \xi)))(d + \xi) < d$, where $c > 0$.

Now, we choose k, such that $\|u_{n_k} - v\| \le d + \xi$. By Proposition 1, we have

$$
\begin{aligned}
\left\|v - u_{n_k+1}\right\| &= \left\|v - \left((1 - \alpha_{n_k})Tu_{n_k} + \alpha_{n_k}Tv_{n_k}\right)\right\| \\
&= \left\|v - \left((1 - \alpha_{n_k})Tu_{n_k} + \alpha_{n_k}T\left((1 - \beta_{n_k})u_{n_k} + \beta_{n_k}Tu_{n_k}\right)\right)\right\| \\
&= \left\|(1 - \alpha_{n_k})v + \alpha_{n_k}v - \left((1 - \alpha_{n_k})Tu_{n_k} + \alpha_{n_k}T\left((1 - \beta_{n_k})u_{n_k}\right.\right.\right. \\
&\qquad \left.\left.\left. + \beta_{n_k}Tu_{n_k}\right)\right)\right\| \\
&\le \left(1 - \alpha_{n_k}\right)\left\|v - Tu_{n_k}\right\| + \alpha_{n_k}\left\|v - T\left((1 - \beta_{n_k})u_{n_k} + \beta_{n_k}Tu_{n_k}\right)\right\| \\
&= \left(1 - \alpha_{n_k}\right)\left\|Tv - Tu_{n_k}\right\| + \alpha_{n_k}\left\|Tv - T\left((1 - \beta_{n_k})u_{n_k} + \beta_{n_k}Tu_{n_k}\right)\right\| \\
&\le \left(1 - \alpha_{n_k}\right)(d + \xi) + \alpha_{n_k}\left\|v - \left((1 - \beta_{n_k})u_{n_k} + \beta_{n_k}Tu_{n_k}\right)\right\| \\
&= \left(1 - \alpha_{n_k}\right)(d + \xi) + \alpha_{n_k}\left\|(1 - \beta_{n_k})(v - u_{n_k}) + \beta_{n_k}(v - Tu_{n_k})\right\| \\
&\le \left(1 - \alpha_{n_k}\right)(d + \xi) + \alpha_{n_k}\left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\beta_{n_k}, 1 - \beta_{n_k}\}\right)(d + \xi) \\
&= \left(1 - \alpha_{n_k} + \alpha_{n_k} - 2\alpha_{n_k}\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\beta_{n_k}, 1 - \beta_{n_k}\}\right)(d + \xi) \\
&= \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\alpha_{n_k}\beta_{n_k}, \alpha_{n_k}(1 - \beta_{n_k})\}\right)(d + \xi).
\end{aligned}
\tag{19}
$$

Since there exist $h > 0$ such that $2\min\{\alpha_{n_k}\beta_{n_k}, \alpha_{n_k}(1 - \beta_{n_k})\} \ge h$,

$$
\begin{aligned}
&\left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\alpha_{n_k}\beta_{n_k}, \alpha_{n_k}(1 - \beta_{n_k})\}\right)(d + \xi) \\
&\qquad \le \left(1 - h\delta\left(\frac{\varepsilon}{d + \xi}\right)\right)(d + \xi).
\end{aligned}
\tag{20}
$$

Suppose choosing very small $\xi > 0$, we have $(1 - h\delta(\varepsilon/(d + \xi)))(d + \xi) < d$, which is a contradiction. This implies that $\lim_{n \to \infty}\|u_n - Tu_n\| = 0$.

Now, we prove that $\|u_{n+1} - u_n\| \to 0$. We have $\|u_{n+1} - u_n\| \le (\alpha_n\beta_n + 1)\|Tu_n - u_n\|$. Now, we define $b_n = (u_{n+1} - z)/(\|u_n - v\|)$, $v_n = (Tv_n - v)/(\|u_n - v\|)$, and $u_n = (Tu_n - v)/(\|u_n - v\|)$. One can note that $\|u_n\| \le 1$. Now,

$$
\begin{aligned}
\|Tv_n - v\| &= \|Tv_n - Tv\| \\
&\le \|v_n - v\| \\
&= \|(1 - \beta_n)u_n + \beta_nTu_n - v\| \\
&= \|(1 - \beta_n)u_n - (1 - \beta_n)v + \beta_n(Tu_n - v)\| \\
&\le (1 - \beta_n)\|u_n - v\| + \beta_n\|Tu_n - Tv\| \\
&\le (1 - \beta_n)\|u_n - v\| + \beta_n\|u_n - v\| \\
&= \|u_n - v\|.
\end{aligned}
\tag{21}
$$

Therefore, $\|v_n\| = (\|Tv_n - v\|)/(\|u_n - v\|) \le (\|u_n - v\|)/(\|u_n - v\|) = 1$. From Agarwal's iteration, we obtain $u_{n+1} - v = (1 - \alpha_n)(Tu_n - v) + \alpha_n(Tv_n - v)$. Dividing by $\|u_n - v\|$, we obtain

$$\frac{u_{n+1} - v}{\|u_n - v\|} = (1 - \alpha_n)\frac{(Tu_n - v)}{\|u_n - v\|} + \alpha_n\frac{(Tv_n - v)}{\|u_n - v\|}. \tag{22}$$

Then, $b_n = (1 - \alpha_n)u_n + \alpha_n v_n$. Now, we prove that $\|b_n\| \longrightarrow 1$. Now,

$$\lim_{n\longrightarrow\infty}\|b_n\| = \lim_{n\longrightarrow\infty}\frac{\|u_{n+1} - v\|}{\|u_n - v\|} = \frac{d}{d} = 1. \tag{23}$$

By Lemma 1, $\|u_n - v_n\| \longrightarrow 0$. This shows that $\|u_n - Tu_n\| \longrightarrow 0$. Therefore, $\|u_{n+1} - u_n\| \longrightarrow 0$.

Since $T(H)$ is contained in a compact set, $\{Tu_n\}$ has a subsequence $\{Tu_{n_k}\}$ that converges to point $b \in H$. Also, $\{u_{n_k}\}$ and $\{u_{n_{k+1}}\}$ converge to b.

Since $d(u_n, H_0) \longrightarrow 0$, there exists $\{b_n\} \subseteq H_0$, such that $\|u_n - b_n\| \longrightarrow 0$. Therefore, $b_{n_k} \longrightarrow b$, which gives that $b \in H_0$.

Let $D = d(H, L)$ and choose $p \in L_0$ such that $\|b - p\| = D$.

We have $\|u_{n_k} - p\| \longrightarrow \|b - p\| = D$, and $\|u_{n_k} - p\| \geq \|Tu_{n_k} - Tp\| \longrightarrow \|b - Tp\|$. So, $\|b - Tp\| = D$. By strict convexity of the norm, $Tp = p$. It follows that $Tb = b$. □

Corollary 2. *Suppose H and L are two nonempty bounded closed convex subsets of a UCBS X, and assume that $T: H \cup L \longrightarrow H \cup L$ is such that*

(1) $T(H) \subseteq H$ and $T(L) \subseteq L$

(2) $\|Tu - Tv\| \leq \|u - v\|$ for all $u \in H, v \in L$

Choose $u_0 \in H$ and set $v_n = (1 - \beta_n)u_n + \beta_n Tu_n, u_{n+1} = (1 - \alpha_n)Tu_n + \alpha_n Tv_n$, where $\beta_n, \alpha_n \in (\varepsilon, 1 - \varepsilon), n = 0, 1, 2, \dots$ and $\varepsilon \in (0, 1/2)$ and $\sum \alpha_n\beta_n < \infty$. Then, $\lim_{n\longrightarrow\infty}\|u_n - Tu_n\| = 0$. Moreover, if $T(H)$ contained in a compact set, then $\{u_n\}$ converges to a fixed point of T.

Corollary 3. *Suppose H and L are two nonempty bounded closed convex subsets of a Hilbert space X, and assume that $T: H \cup L \longrightarrow H \cup L$ be a relatively nonexpansive mapping such that*

(1) $T(H) \subseteq H$ and $T(L) \subseteq L$

(2) $\|Tu - Tv\| \leq \|u - v\|$ for all $u \in H, v \in L$

Choose $u_0 \in H$ and set $v_n = (1 - \beta_n)u_n + \beta_n Tu_n, u_{n+1} = P^n((1 - \alpha_n)Tu_n + \alpha_n Tv_n)$, where $\beta_n, \alpha_n \in (\varepsilon, 1 - \varepsilon), n = 0, 1, 2, \dots$ and $\varepsilon \in (0, 1/2)$ and $\sum \alpha_n\beta_n < \infty$. Then, $\lim_{n\longrightarrow\infty}\|u_n - Tu_n\| = 0$. Moreover, if $T(H)$ is mapped into a compact subset of L, then $\{u_n\}$ converges to a fixed point of T.

Proof. One can note that $P^n((1 - \alpha_n)Tu_n + \alpha_n Tv_n) = (1 - \alpha_n)Tu_n + \alpha_n Tv_n$, and by Theorem 8, the result follows.

Now, we present a new example of relatively non-expansive mappings and prove that its Agarwal [32] iterative process is better than the Mann [29] and Ishikawa [31] iterative processes. □

Example 1. Take $X = \mathbb{R}^2$,

$$\begin{aligned} H &= \{(u, 0): \ -4 \leq u \leq -3\}, \\ L &= \{(u', 0): \ 3 \leq u' \leq 4\}. \end{aligned} \tag{24}$$

Define

$$\begin{aligned} T: H &\longrightarrow H \text{ by } T(u, 0) = \left(\frac{u - 4}{5}, 0\right), \\ T: L &\longrightarrow L \text{ by } T(u', 0) = \left(\frac{u' + 4}{5}, 0\right). \end{aligned} \tag{25}$$

Let $(u, 0) \in H$ and $(u', 0) \in L$. Then,

$$\begin{aligned} \|T(u, 0) - T(u', 0)\| &= \left\|\left(\frac{u - 4}{5}, 0\right) - \left(\frac{u' + 4}{5}, 0\right)\right\| \\ &= \left\|\left(\frac{u - u' - 8}{5}, 0\right)\right\| \\ &= \sqrt{\left(\frac{u - u' - 8}{5}\right)^2 + 0} \\ &\leq \sqrt{(u - u')^2} \\ &= \|(u, 0) - (u', 0)\|. \end{aligned} \tag{26}$$

From the above process, we get T is relatively non-expansive mapping. The iterative values for $u_0 = -1.5$ are shown below in Table 1 and Figure 1.

Remark 1. From Table 1 and Figure 1, we see that Agarwal iterates converges faster to -1 than the Ishikawa and Mann iterates for the class of relatively nonexpansive mappings.

The stronger version for the approximation of fixed point by using Von Neumann sequences are follows.

Theorem 9. *Let H and L be nonempty bounded closed convex subsets of a Hilbert space X and suppose $T: H \cup L \longrightarrow H \cup L$ is such that*

(1) $T(H) \subseteq H$ and $T(L) \subseteq L$

(2) $\|Tu - Tv\| \leq \|u - v\|$ for all $u \in H, v \in L$

Let $u_0 \in H$ and define $v_n = (1 - \beta_n)u_n + \beta_n Tu_n, u_{n+1} = P^n((1 - \alpha_n)Tu_n + \alpha_n Tv_n)$, where $\beta_n, \alpha_n \in (\varepsilon, 1 - \varepsilon), n = 0, 1, 2, \dots$ with restriction $\sum_{n\longrightarrow\infty}\alpha_n\beta_n < \infty$ and $\varepsilon \in (0, 1/2)$ and $\sum \alpha_n\beta_n < \infty$. Then, $\lim_{n\longrightarrow\infty}\|u_n - Tu_n\| = 0$. Moreover, if $T(H)$ is lies in a compact set and $\|u_n - Tv_n\| \longrightarrow 0$, then $\{u_n\}$ converges to a fixed point of T.

TABLE 1: Sequences generated by Agarwal, Ishikawa, and Mann iterative algorithms.

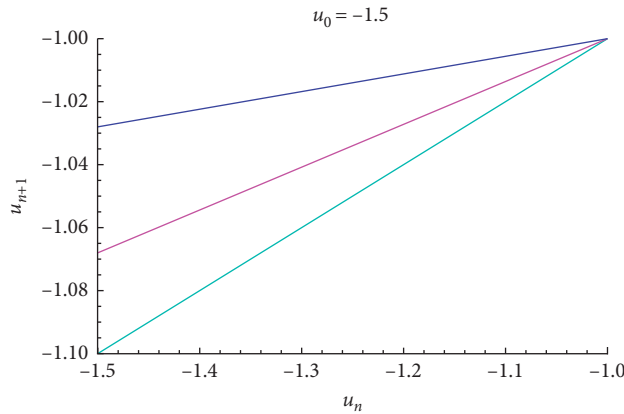|	Agarwal	Ishikawa	Mann
u_0	−1.5	−1.5	−1.5
u_1	−1.028000072000000	−1.068000072000000	−1.100000400000000
u_2	−1.001568008064010	−1.009248019584011	−1.020000160000320
u_3	−1.000087808677378	−1.001257731995140	−1.004000048000192
u_4	−1.000004917298577	−1.000171051732452	−1.000800012800077
u_5	−1.000000275369428	−1.000023263060245	−1.000160003200025
u_6	−1.000000015420727	−1.000003163779543	−1.000032000768007
u_7	−1.000000000863563	−1.000000430274473	−1.000006400179202
u_8	−1.000000000048359	−1.000000058517390	−1.000001280040961
u_9	−1.000000000002708	−1.000000007958373	−1.000000256009216
u_{10}	−1.000000000000151	−1.000000001082346	−1.000000051202048
u_{11}	−1.000000000000008	−1.000000000147198	−1.000000010240450
u_{12}	−1	−1.000000000020019	−1.000000002048098
u_{13}	−1	−1.000000000002722	−1.000000000409621
u_{14}	−1	−1.000000000000370	−1.000000000081924
u_{15}	−	−1.000000000000050	−1.000000000016385
u_{16}	−1	−1.000000000000007	−1.000000000003277
u_{17}	−	−1.000000000000001	−1.000000000000655
u_{18}	−1	−1	−1.000000000000131
u_{19}	−1	−1	−1.000000000000026
u_{20}	−1	−1	−1.000000000000005

FIGURE 1: Convergence behavior of Agarwal (blue), Ishikawa (magenta), and Mann (cyan) iterates to the fixed point $v = -1$ of the mapping T.

Proof. If $d(H, L) = 0$, then $H_0 = L_0 = H \cap L$ and $T: H \cap L \longrightarrow H \cap L$ is nonexpansive with $u_{n+1} = P^n((1 - \alpha_n)Tu_n + \alpha_n T((1 - \beta_n)u_n + \beta_n Tu_n)) = (1 - \alpha_n)Tu_n + \alpha_n T((1 - \beta_n)u_n + \beta_n Tu_n)$, the usual Agarwal's iteration. So, let us take that $d(H, L) > 0$. By Theorem 2, there exist $v \in L_0$ such that $Tv = v$. Now,

$$\begin{aligned}
\|v_n - v\| &= \|(1 - \beta_n)u_n + \beta_n Tu_n - v\| \\
&\leq (1 - \beta_n)\|u_n - v\| + \beta_n \|Tu_n - v\| \\
&\leq (1 - \beta_n)\|u_n - v\| + \beta_n \|u_n - v\| \\
&= \|u_n - v\|,
\end{aligned} \tag{27}$$

which implies that

$$\begin{aligned}
\|u_{n+1} - v\| &= \|P^n((1 - \alpha_n)Tu_n + \alpha_n Tv_n) - v\| \\
&\leq \|(1 - \alpha_n)Tu_n + \alpha_n Tv_n - v\| \\
&\leq (1 - \alpha_n)\|Tu_n - v\| + \alpha_n \|Tv_n - v\| \\
&\leq (1 - \alpha_n)\|Tu_n - Tv\| + \alpha_n \|Tv_n - Tv\| \\
&\leq (1 - \alpha_n)\|u_n - v\| + \alpha_n \|v_n - v\| \\
&\leq (1 - \alpha_n)\|u_n - v\| + \alpha_n \|u_n - v\| \\
&= \|u_n - v\|.
\end{aligned} \tag{28}$$

Hence, the e sequence $\{\|u_n - v\|\}$ is nonincreasing. So, one can choose some $d > 0$ with $\lim_{n \to \infty}\|u_n - v\| = d$. Assume that one can find a subsequence, namely, $\{u_{n_k}\}$ of $\{u_n\}$ and some positive ε such that $\|u_{n_k} - Tu_{n_k}\| \geq \varepsilon > 0$ for every k.

Since the modulus of convexity of δ of X is continuous as well as increasing function, one may choose $\xi > 0$ as small such that $(1 - c\delta(\varepsilon/(d + \xi)))(d + \xi) < d$, where $c > 0$.

Now, we select k, such that $\|u_{n_k} - v\| \leq d + \xi$. By Proposition 1, we have

$$
\begin{aligned}
\left\|v - u_{n_k+1}\right\| &= \left\|v - P^{n_k}\left(\left(\left(1 - \alpha_{n_k}\right)Tu_{n_k} + \alpha_{n_k}Tv_{n_k}\right)\right)\right\| \\
&\leq \left\|v - \left(\left(1 - \alpha_{n_k}\right)Tu_{n_k} + \alpha_{n_k}Tv_{n_k}\right)\right\| \\
&= \left\|v - \left(\left(\left(1 - \alpha_{n_k}\right)Tu_{n_k} + \alpha_{n_k}T\left(\left(1 - \beta_{n_k}\right)u_{n_k} + \beta_{n_k}Tu_{n_k}\right)\right)\right)\right\| \\
&= \left\|\left(1 - \alpha_{n_k}\right)v + \alpha_{n_k}v - \left(\left(1 - \alpha_{n_k}\right)Tu_{n_k} + \alpha_{n_k}T\left(\left(1 - \beta_{n_k}\right)u_{n_k} + \beta_{n_k}Tu_{n_k}\right)\right)\right\| \\
&\leq \left(1 - \alpha_{n_k}\right)\left\|v - Tu_{n_k}\right\| + \alpha_{n_k}\left\|v - T\left(\left(1 - \beta_{n_k}\right)u_{n_k} + \beta_{n_k}Tu_{n_k}\right)\right\| \\
&= \left(1 - \alpha_{n_k}\right)\left\|Tv - Tu_{n_k}\right\| + \alpha_{n_k}\left\|Tv - T\left(\left(1 - \beta_{n_k}\right)u_{n_k} + \beta_{n_k}Tu_{n_k}\right)\right\| \\
&\leq \left(1 - \alpha_{n_k}\right)\left\|v - u_{n_k}\right\| + \alpha_{n_k}\left\|v - \left(\left(1 - \beta_{n_k}\right)u_{n_k} + \beta_{n_k}Tu_{n_k}\right)\right\| \\
&= \left(1 - \alpha_{n_k}\right)(d + \xi) + \alpha_{n_k}\left\|\left(1 - \beta_{n_k}\right)\left(v - u_{n_k}\right) + \beta_{n_k}\left(v - Tu_{n_k}\right)\right\| \\
&\leq \left(1 - \alpha_{n_k}\right)(d + \xi) + \alpha_{n_k}\left(\left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\beta_{n_k}, 1 - \beta_{n_k}\}\right)d + \xi\right) \\
&= \left(1 - \alpha_{n_k} + \alpha_{n_k} - 2\alpha_{n_k}\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\beta_{n_k}, 1 - \beta_{n_k}\}\right)(d + \xi) \\
&= \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\alpha_{n_k}\beta_{n_k}, \alpha_{n_k}\left(1 - \beta_{n_k}\right)\}\right)(d + \xi).
\end{aligned}
\tag{29}
$$

Since there exist $h > 0$ such that $2\min\{\alpha_{n_k}\beta_{n_k}, \alpha_{n_k}(1 - \beta_{n_k})\} \geq h$:

$$
\begin{aligned}
&\left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\alpha_{n_k}\beta_{n_k}, \alpha_{n_k}\left(1 - \beta_{n_k}\right)\}\right)(d + \xi) \\
&\leq \left(1 - h\delta\left(\frac{\varepsilon}{d + \xi}\right)\right)(d + \xi).
\end{aligned}
\tag{30}
$$

If we select a small $\xi > 0$, then $(1 - h\delta(\varepsilon/d + \xi))(d + \xi) < d$ which is clearly a contradiction. This implies that $\lim_{n \longrightarrow \infty}\|u_n - Tu_n\| = 0$

Since the set $T(H)$ is contained in a compact set, so the sequence $\{Tu_n\}$ has a subsequence $\{Tu_{n_k}\}$ such that it converges to a some point $z_0 \in H$. Also, $\{u_{n_k}\}$ converge to z_0. From the given sequence, one has

$$
\begin{aligned}
\left\|u_{n_k+1} - u_{n_k}\right\| &= \left\|P^{n_k}\left(\left(1 - \alpha_{n_k}\right)Tu_{n_k} + \alpha_{n_k}Tv_{n_k}\right) - u_{n_k}\right\| \\
&\leq \left\|\left(1 - \alpha_{n_k}\right)Tu_{n_k} + \alpha_{n_k}Tv_{n_k} - u_{n_k}\right\| \\
&= \left\|Tu_{n_k} - \alpha_{n_k}Tu_{n_k} + \alpha_{n_k}Tv_{n_k} - u_{n_k}\right\| \\
&\leq \alpha_{n_k}\left\|Tu_{n_k} - Tv_{n_k}\right\| + \left\|Tu_{n_k} - u_{n_k}\right\| \\
&\leq \alpha_{n_k}\left\|v_{n_k} - u_{n_k}\right\| + \left\|Tu_{n_k} - u_{n_k}\right\| \\
&= \alpha_{n_k}\left\|\left(1 - \beta_n\right)u_{n_k} + \beta_nTu_{n_k} - u_{n_k}\right\| + \left\|Tu_{n_k} - u_{n_k}\right\| \\
&= \alpha_{n_k}\beta_{n_k}\left\|Tu_{n_k} - u_{n_k}\right\| + \left\|Tu_{n_k} - u_{n_k}\right\| \\
&= \left(\alpha_{n_k}\beta_{n_k} + 1\right)\left\|Tu_{n_k} - u_{n_k}\right\|.
\end{aligned}
\tag{31}
$$

Since $\|Tu_{n_k} - u_{n_k}\| \longrightarrow 0$ which implies that $\|u_{n_k+1} - u_{n_k}\| \longrightarrow 0$. Therefore, $u_{n_k+1} \longrightarrow z_0$, which implies that $u_n \longrightarrow z_0$. Also, we have $Tv_{n_k} \longrightarrow z_0$ as $k \longrightarrow \infty$.

Now, $\|Tu_{n_k} - T(P_L(z_0))\| \le \|u_{n_k} - P_L(z_0)\|$, which gives that $\|z_0 - T(P_L(z_0))\| \le \|z_0 - P_L(z_0)\|$. Therefore, $T(P_L(z_0)) = P_L(z_0)$.

Also, $\|T(P(z_0)) - P_L(z_0)\| = \|T(P(z_0)) - T(P_L(z_0))\| \le \|P(z_0) - P_L(z_0)\|$. So, $T(P(z_0)) = P(z_0)$

Now, $\|T(P_L(P(z_0))) - P(z_0)\| = \|T(P_L(P(z_0))) - T(P(z_0))\| \le \|P_L(P(z_0)) - P(z_0)\|$. Thus, $T(P_L(P(z_0))) = P_L(P(z_0))$.

For any n, $T(P^n(z_0)) = P^n(z_0)$ and $T(P_L(P^n(z_0))) = P_L(P^n(z_0))$. By Theorem 6, for each $u \in H$, the sequence $\{P^n(u)\}$ converges to some $w(u) \in H_0$. Now,

$$\begin{aligned}
\|T(w(z_0)) - P_L(w(z_0))\| &\le \lim_{n \to \infty} \|T(w(z_0)) - P_L(P^n(z_0))\| \\
&= \lim_{n \to \infty} \|T(w(z_0)) - T(P_L(P^n(z_0)))\| \\
&\le \lim_{n \to \infty} \|w(z_0) - P_L(P^n(z_0))\| \\
&= \|w(z_0) - P_L(w(z_0))\|.
\end{aligned} \tag{32}$$

So, $\|T(w(z_0)) - P_L(w(z_0))\| \le \|w(z_0) - P_L(w(z_0))\|$.

Therefore, $T(w(z_0)) = w(z_0)$, and similarly $T(P_L(w(z_0))) = P_L(w(z_0))$.

Now, we define $g_n: H \longrightarrow \mathbb{R}$ by $g_n(u) = \|P^n(u) - w(u)\|$.

Since $\|w(u) - w(v)\| = \lim_{n \to \infty} \|P^n(u) - P^n(v)\| \le \|u - v\|$, then we conclude that w is continuous. Therefore, $g_n(u)$ is continuous and converges pointwise to zero. Since $w(u) \in H_0$, by Lemma 3, we obtain $g_{n+1} \le g_n$. Therefore, g_n converges uniformly on the compact set:

$$S = \left\{\left(1 - \alpha_{n_k}\right)Tu_{n_k} + \alpha_{n_k}Tv_{n_k}\right\} \cup \{z_0\}. \tag{33}$$

Therefore,

$$\lim_{n \to \infty} \left\|P^{n_k}\left(\left(1 - \alpha_{n_k}\right)Tu_{n_k} + \alpha_{n_k}Tv_{n_k}\right) - w\left(\left(1 - \alpha_{n_k}\right)Tu_{n_k} + \alpha_{n_k}Tv_{n_k}\right)\right\| = 0. \tag{34}$$

Since $w((1 - \alpha_{n_k})Tu_{n_k} + \alpha_{n_k}Tv_{n_k}) \longrightarrow w(z_0)$, we get $u_{n_{k+1}} \longrightarrow w(z_0)$, which gives that $w(z_0) = z_0$. Therefore, $Tz_0 = T(w(z_0)) = w(z_0) = z_0$, which completes the proof.

Suppose X is a Hilbert space and assume that T be as in Theorem 1. Consider $P_H T: H \longrightarrow H$ and $P_L T: L \longrightarrow L$.

From Proposition 2, $\|P_H T(u) - P_L T(v)\| \le \|u - v\|$ for $u \in H$ and $v \in L$, by Theorems 8 and 9, we give the following results on convergence of best proximity points. □

Corollary 4. *Suppose H and L are two nonempty bounded closed convex subsets of a Hilbert space X. Assume that T be as in Theorem 1. If $T(H)$ is mapped into a compact subset of L, then for every $u_0 \in H_0$ the sequence generated by by $u_{n+1} = (1 - \alpha_n)Tu_n + \alpha_n P_H(T((1 - \beta_n)u_n + \beta_n P_H Tu_n))$ converges to u in H_0 such that $\|u - Tu\| = d(H, L)$*

Corollary 5. *Suppose H and L are two nonempty bounded closed convex subsets of a Hilbert space X. Assume that T be as in Theorem 1. If $T(H)$ is mapped into a compact subset of L, then for any $u_0 \in H_0$ the sequence defined by $u_{n+1} = (1 - \alpha_n)Tu_n + \alpha_n P_H(T((1 - \beta_n)u_n + \beta_n P_H Tu_n))$ converges to u in H_0 such that $\|u - Tu\| = d(H, L)$ provided $d(u_n, H_0) \longrightarrow 0$.*

Corollary 6. *Suppose H and L are two nonempty bounded closed convex subsets of a Hilbert space X. Assume that T be as in Theorem 1. If $T(H)$ is mapped into a compact subset of L, then for every $u_0 \in H_0$ the sequence generated by $u_{n+1} = P^n((1 - \alpha_n)Tu_n + \alpha_n P_H(T((1 - \beta_n)u_n + \beta_n P_H Tu_n)))$ converges to u in H_0 such that $\|u - Tu\| = d(H, L)$.*

Proof. The result follows by Corollary 4. □

Corollary 7. *Suppose H and L are two nonempty bounded closed convex subsets of a Hilbert space X. Assume that T be as in Theorem 1. Choose $u_0 \in H$ and set $v_n = (1 - \beta_n)u_n + \beta_n Tu_n, u_{n+1} = P^n((1 - \alpha_n)Tu_n + \alpha_n Tv_n)$, where $\beta_n, \alpha_n \in (\varepsilon, 1 - \varepsilon), n = 0, 1, 2, \ldots$ and $\varepsilon \in (0, 1/2)$ and $\sum \alpha_n \beta_n < \infty$. If $T(H)$ is mapped into a compact subset of L and $\|u_n - P_H Tv_n\| \longrightarrow 0$, then $\{u_n\}$ converges to u in H_0 such that $\|u - Tu\| = d(H, L)$.*

Proof. The result follows by Theorem 9. □

3. Conclusions

In this article, we have used the Agarwal iterative process for finding fixed points and best proximity points of relatively nonexpansive mappings. Using the Von Neumann sequence, we have established the convergence result in a Hilbert space framework. We have offered a new example of relatively nonexpansive mapping and proved that its Agarwal iterative process is more efficient than the Mann and Ishikawa iterative processes.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgments

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