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Research Article

S-Semiprime Submodules and *S*-Reduced Modules

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This article introduces the concept of S-semiprime submodules which are a generalization of semiprime submodules and S-prime submodules. Let M be a nonzero unital R-module, where R is a commutative ring with a nonzero identity. Suppose that S is a multiplicatively closed subset of R. A submodule P of M is said to be an S-semiprime submodule if there exists a fixed $s \in S$, and whenever $r^n m \in P$ for some $r \in R, m \in M$, and $n \in \mathbb{N}$, then $srm \in P$. Also, M is said to be an S-reduced module if there exists (fixed) $s \in S$, and whenever $r^n m = 0$ for some $r \in R, m \in M$, and $n \in \mathbb{N}$, then $srm = 0$. In addition, to give many examples and characterizations of S-semiprime submodules and S-reduced modules, we characterize a certain class of semiprime submodules and reduced modules in terms of these concepts.

1. Introduction

In this article, all rings are assumed to be commutative with a nonzero identity, and all modules are assumed to be nonzero unital. Let R always denote such a ring and M always denote such an R-module. Recalling from [1], an R-module M is said to be a reduced module if $a^2 m = 0$ for each $a \in R$ and $m \in M$ implying that $am = 0$. Note that M is a reduced module if and only if $a^n m = 0$ for some $a \in R, m \in M$, and $n \in \mathbb{N}$ implying that $am = 0$. Let P be a submodule of M. P is said to be a semiprime submodule; if $a^2 m \in P$, where $a \in R$ and $m \in M$, then $am \in P$ [2]. It is easy to see that M is a reduced module if and only if the zero submodule is semiprime. Also, it is clear that a submodule P of M is semiprime if and only if $a^n m \in P$ for some $a \in R, m \in M$, and $n \in \mathbb{N}$ implying that $am \in P$. As a generalization of the prime submodule (torsion-free module), the notion of the semiprime submodule (reduced module) has been widely studied in many papers. See, for example, [1–5]. Our aim, in this paper, is to introduce S-semiprime submodules and S-reduced modules which are generalizations of semiprime submodules and reduced modules, respectively. For the sake of completeness, we begin by giving some notions and notations which will be used throughout the paper. $\mathrm{Spec}(R)$ and $\mathrm{Max}(R)$ denote the set of all prime ideals and maximal ideals of R, respectively. A nonempty subset S of R is said to be a multiplicatively closed set (briefly, m.c.s) if $1 \in S$ and S is a subsemigroup of R under multiplication. Let P be a submodule of M, L be a nonempty subset of M, and J be a nonempty subset of R; the residuals of P by L and J are defined as follows:

$$\begin{aligned}(P:_R L) &= \{x \in R\colon xL \subseteq P\},\\ (P:_M J) &= \{m \in M\colon Jm \subseteq P\}.\end{aligned} \tag{1}$$

In particular, if $J = \{a\}$ is the singleton, where $a \in R$, we use $(P:_R a)$ instead of $(P:_R \{a\})$, and also, we use $\mathrm{ann}(M)$ to denote $(0:_R M)$. For each $x \in R$, the set $(0:_M x)$ is denoted by $\mathrm{ann}_M(x)$, and if $\mathrm{ann}_M(x) \neq 0$, $x \in R$ is called a zero divisor on M. Furthermore, the set of all zero divisors on M is denoted by $z(M) = \{x \in R\colon \mathrm{ann}_M(x) \neq 0\}$. It is clear that the set of all units $u(R)$ of R and $R - z(M)$ are always a m.c.s of R.

The concepts of prime ideals/submodules have a distinguished place in commutative algebra. Since certain class of rings and modules are characterized in terms of prime ideals/submodules, they have been widely studied by many authors. See, for example, [6–11]. Recently, Sevim et al., in [12], introduced S-prime submodules and S-torsion-free modules and used them to characterize certain prime submodules and torsion-free modules. Let S be a m.c.s of R.

A submodule P of M is said to be an S-prime submodule if there exists a fixed $s \in S$, and whenever $am \in P$, then either $sa \in (P:_R M)$ or $sm \in P$ for each $a \in R$ and $m \in M$. Note that if $(P:_R M) \cap S \neq \varnothing$, then P is (trivially) an S-prime submodule, and so, the authors in [12] defined S-prime submodules with the condition that $(P:_R M) \cap S = \varnothing$ to avoid the trivial case. Similarly, an R-module M with $\operatorname{ann}(M) \cap S = \varnothing$ is said to be an S-torsion-free module if there exists a fixed $s \in S$, and whenever $am = 0$ for some $a \in R$ and $m \in M$, then either $sa = 0$ or $sm = 0$. They showed in [12], Theorem 2.26, that R-module M with $\operatorname{ann}(M) \cap S = \varnothing$ is a simple module if and only if its each proper submodule is an S-prime submodule.

Let S be a m.c.s of R and P be a submodule of M. Then, we call P an S-semiprime submodule if there exists a fixed $s \in S$, and whenever $a^n m \in P$ for some $a \in R, m \in M$, and $n \in \mathbb{N}$, then $sam \in P$. Also, M is said to be an S-reduced module if there exists (fixed) $s \in S$, and whenever $a^n m = 0$ for some $a \in R, m \in M$, and $n \in \mathbb{N}$, then $sam = 0$. To avoid the trivial case, we assume that $(P:_R M) \cap S = \varnothing$ for each S-semiprime submodule P of M and $\operatorname{ann}(M) \cap S = \varnothing$ for each S-reduced module M. Among other results in this paper, we show that the class of S-semiprime submodules properly contains the class of semiprime submodules and the class of S-prime submodules (see Proposition 1 and Examples 1 and 2). Also, we show that, in Proposition 2, if P is an S-semiprime submodule of M, then $S^{-1}P$ is a semiprime submodule of the quotient module $S^{-1}M$ of M. Also, we investigate the behaviour of S-semiprime submodules under homomorphism, in factor modules, and in Cartesian products of modules (see Proposition 5, Corollary 3, and Theorems 2 and 3). An R-module M is said to be a multiplication module if each submodule P of M has the form $P = IM$ for some ideal I of R. In Theorem 1, we determine all S-semiprime submodules of finitely generated multiplication modules. Also, we characterize certain semiprime submodules of an arbitrary module in terms of S-semiprime submodules (see Theorem 5). Using Theorem 5, we determine all semiprime submodules of modules over quasi-local rings in terms of S-semiprime submodules (see Corollary 4). Finally, we characterize reduced modules in terms of S-reduced modules (see Theorem 6).

2. Characterization of S-Semiprime Submodules

Definition 1. Let P be a submodule of M with $(P:_R M) \cap S = \varnothing$, where $S \subseteq R$ is a m.c.s of R. P is said to be an S-semiprime submodule if there exists a fixed $s \in S$, and whenever $r^n m \in P$ for some $r \in R, m \in M$, and $n \in \mathbb{N}$, then srm $\in P$.

Let S be a m.c.s of R. If we consider the ring R as a module over itself, then we say that P is an S-semiprime ideal if it is an S-semiprime submodule of R. Note that an ideal P of R with $P \cap S = \varnothing$ is an S-semiprime ideal if and only if there exists (fixed) $s \in S$, and whenever $a^n \in P$ for some $a \in R$ and $n \in \mathbb{N}$, then $sa \in P$.

Proposition 1. *Let $S \subseteq R$ be a m.c.s of R and M be an R-module. The following statements are satisfied:*

(i) If P is a semiprime submodule of M provided that $(P:_R M) \cap S = \varnothing$, then P is an S-semiprime submodule of M

(ii) If P is an S-semiprime submodule of M and $S \subseteq u(R)$, then P is a semiprime submodule of M

(iii) Every S-prime submodule is also an S-semiprime submodule

Proof

(i) Let P be a semiprime submodule of M and $(P:_R M) \cap S = \varnothing$. Now, we will show that P is an S-semiprime submodule of M. To see this, take $r \in R$ and $m \in M$ such that $r^n m \in P$ for some $n \in \mathbb{N}$. Since P is a semiprime submodule, we have rm $\in P$, and this implies that srm $\in P$ for each $s \in S$. Then, P is an S-semiprime submodule of M.

(ii) Let P be an S-semiprime submodule of M. Then, we know that $r^n m \in P$, where $r \in R, m \in M$, and $n \in \mathbb{N}$, implies that $srm \in P$ for a fixed $s \in S$. Since $S \subseteq u(R)$, there is a $t \in R$ such that $st = 1$, and so, $rm = t(srm) \in P$. Therefore, P is a semiprime submodule of M.

(iii) Suppose that P is an S-semiprime submodule of M. Let $r^n m \in P$ for some $r \in R, m \in M$, and $n \in \mathbb{N}$. Since P is an S-prime submodule, there exists a fixed $s \in S$ such that $sr \in (P:_R M)$ or $sr^{n-1}m \in P$. If sr $\in (P:_R M)$, then $srm \in P$, and so, the proof is completed. Now, assume that sr $\notin (P:_R M)$, that is, $sr^{n-1}m = r(sr^{n-2}m) \in P$. This implies that $s^2 r^{n-2} m \in P$. If we continue in this manner, we conclude that $s^n m \in P$. As P is an S-prime submodule, we get either $s^{n+1} \in (P:_R M)$ or $sm \in P$. As $(P:_R M) \cap S = \varnothing$ and $s^{n+1} \in S$, we have $sm \in P$, and so, $srm \in P$.

The converses of Proposition 1 (i) and (iii) need not be true. See the following examples. □

Example 1. Consider the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}_4$ and the zero submodule $P = 0 \times \overline{0}$. First, note that $(P: \mathbb{Z} \times \mathbb{Z}_4) = 0$ and $2^2(0, \overline{1}) = (0, \overline{0}) \in P$. Since $2(0, \overline{1}) \notin P$, P is not a semiprime submodule of $\mathbb{Z} \times \mathbb{Z}_4$. Now, take the m.c.s $S = \mathbb{Z}^* = \mathbb{Z} - \{0\}$ of $\mathbb{Z}$, and put $s = 4 \in S$. Now, we will show that P is an S-semiprime submodule. To see this, let $a^k(x, \overline{m}) \in P$ for some $a, x, m \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then, we have $a^k x = 0$. If $a = 0$, then $sa(x, \overline{m}) = (0, \overline{0}) \in P$. Otherwise, we have $x = 0$, and so, $sa(x, \overline{m}) = (0, \overline{0})$. Therefore, P is an S-semiprime submodule.

Example 2. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{pqr}$, where p, q, and r are distinct prime numbers. Consider the multiplicatively closed subset $S = \{p^n : n \in \mathbb{N}\} \cup \{1\}$ of R. Take the submodule $P = (\overline{0})$. Then, note that $q(\overline{pr}) = \overline{0}$, and also, $p^n q \notin (P:_R M) = pqr\mathbb{Z}$ and $p^n(\overline{pr}) \notin P$ for any $p^n \in S$. Thus, P is not an

S-prime submodule. Also, it is clear that P is an S-semiprime submodule of M.

Let $S \subseteq R$ be a m.c.s of R. Then, $S^{-1}R = \{\lambda\colon \lambda = a/s, \exists a \in R, s \in S\}$ is called the quotient ring of R. For any m.c.s S of R, the saturation S^* of S is defined as $\{a \in R\colon a/1$ is a unit of $S^{-1}R\}$ [13]. Note that S^* is a m.c.s of R containing S.

Proposition 2. *Let $S \subseteq R$ be a m.c.s of R and M be an R-module. The following statements hold:*

(i) If $S_1 \subseteq S_2$ is a m.c.s of R and P is an S_1-semiprime submodule of M, then P is an S_2-semiprime submodule of M in case $(P\colon {}_RM) \cap S_2 = \varnothing$

(ii) P is an S-semiprime submodule of M if and only if P is an S^*-semiprime submodule of M

(iii) If P is an S-semiprime submodule of M, then $S^{-1}P$ is a semiprime submodule of $S^{-1}M$

Proof

(i) It is straightforward.

(ii) Let $(P\colon {}_RM) \cap S = \varnothing$. It is clear that $S \subseteq S^*$. Then, we need to show that $(P\colon {}_RM) \cap S^* = \varnothing$. Assume that $(P\colon {}_RM) \cap S^* \neq \varnothing$. So, there exists $a \in (P\colon {}_RM) \cap S^*$. Since $a \in S^*$, $a/1$ is a unit of $S^{-1}R$, and so, $(a/1)(x/y) = 1$ for some $(x/y) \in S^{-1}R$. This yields that $axu = yu$ for some $u \in S$. Now, put $s' = yu \in S$. Then, $s' = uax = yu \in (P\colon {}_RM) \cap S$, a contradiction. Thus, $(P\colon {}_RM) \cap S^* = \varnothing$. So, by (i), P is an S^*-semiprime submodule of M. For the converse, let P be an S^*-semiprime submodule of M. Then, $(P\colon {}_RM) \cap S^* = \varnothing$, and thus, $(P\colon {}_RM) \cap S = \varnothing$. Let $r^n m \in P$ for some $r \in R, m \in M$, and $n \in \mathbb{N}$. Since P is an S^*-semiprime submodule, there is an $s' \in S^*$ such that $s'rm \in P$. As $s' \in S^*$, there exists $x/y \in S^{-1}R$ such that $(s'/1)(x/y) = 1$. Then, we conclude that $us'x = yu$ for some $u \in S$. Now, take $s = yu \in S$. Then, we get $srm = yurm = uxs'rm \in P$, and hence, P is an S-semiprime submodule of M.

(iii) Suppose that P is an S-semiprime submodule of M. Let $(r/s)^k(m/t) = (r^k m/s^k t) \in S^{-1}P$ for some $(r/s) \in S^{-1}R$, $(m/t) \in S^{-1}M$, and $k \in \mathbb{N}$. Then, there exists $u \in S$ such that $ur^k m = r^k(um) \in P$. As P is an S-semiprime submodule of M, we get $us'rm \in P$ for some $s' \in S$. This implies that $(r/s)(m/t) = (us'rm/us'st) \in S^{-1}P$. Hence, $S^{-1}P$ is a semiprime submodule of $S^{-1}M$.

The following example shows that the converse of Proposition 2 (iii) is not true in general. □

Example 3. Let $R = \mathbb{Z}$ and $M = \mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. Take the submodule $P = \mathbb{Z} \times (0)$ and the m.c.s $S = \mathbb{Z}^* = \mathbb{Z} - \{0\}$ of $\mathbb{Z}$. It is easy to see that $S^{-1}M$ is a vector space over $S^{-1}R$, and thus, $S^{-1}P$ is a prime (semiprime) submodule of $S^{-1}M$. Now, we will show that P is not S-semiprime. Let s be an arbitrary element of S. Choose a prime number p with $\gcd(p, s) = 1$. Then, note that $p^2(1/p^2, 0) = (1, 0) \in P$ and $sp(1/p^2, 0) = (sp/p^2, 0) \notin P$. Thus, P is not an S-semiprime submodule of M.

Proposition 3. *Let M be an R-module and $S = \{s_1, s_2, \ldots, s_n\}$ be a finite m.c.s of R. Suppose that P is a submodule of M provided that $(P\colon {}_RM) \cap S = \varnothing$. Then, P is an S-semiprime submodule of M if and only if $S^{-1}P$ is a semiprime submodule of $S^{-1}M$.*

Proof. Suppose that P is an S-semiprime submodule of M. Then, by Proposition 2 (iii), $S^{-1}P$ is a semiprime submodule of $S^{-1}M$. For the converse, take a semiprime submodule $S^{-1}P$ of $S^{-1}M$. Let $r^n m \in P$ for some $r \in R, m \in M$, and $n \in \mathbb{N}$. Then, we have $(r/1)^n(m/1) = (r^n m/1) \in S^{-1}P$. Since $S^{-1}P$ is a semiprime submodule of $S^{-1}M$, we conclude that $(r/1)(m/1) = (rm/1) \in P$, and this yields that $s_i rm \in P$ for some $s_i \in S$. Now, put $s = s_1 s_2 \cdots s_n \in S$. Then, we conclude that $srm \in P$, and so, P is an S-semiprime submodule of M. □

Lemma 1. *Suppose P is a submodule of M and S is a m.c.s of R provided that $(P\colon {}_RM) \cap S = \varnothing$. The following statements are equivalent:*

(i) P is an S-semiprime submodule of M

(ii) There is a fixed $s \in S$ and $J^k N \subseteq P$ for some $k \in \mathbb{N}$ implying that $sJN \subseteq P$ for each ideal J of R and submodule N of M

Proof. (i) $\Longrightarrow$ (ii): let P be an S-semiprime submodule of M. Suppose that $J^k N \subseteq P$ for some ideal J of R, some submodule N of M, and $k \in \mathbb{N}$. Now, we will show that $sJN \subseteq P$. Suppose to the contrary. Then, there exist $a \in J$ and $m \in N$ such that $sam \notin P$. Since $a^k m \in J^k N \subseteq P$ and P is an S-semiprime submodule of M, we conclude that $sam \in P$, a contradiction. Therefore, $sJN \subseteq P$. (ii) $\Longrightarrow$ (i): conversely, let $r^n m \in P$ for some $r \in R, m \in M$, and $k \in \mathbb{N}$. Now, put $J = Rr$ and $N = Rm$. Then, we have $J^n N = Rr^n m \subseteq P$. Hence, by assumption, $sJN = Rsrm \subseteq P$ for a fixed $s \in S$, and so, $srm \in P$. Then, P is an S-semiprime submodule of M.

As immediate consequences of the previous lemma, we give the following corollary which will be used in the sequel. □

Corollary 1. *Suppose that S is a m.c.s of R and P is an ideal of R with $P \cap S = \varnothing$. The following statements are equivalent:*

(i) P is an S-semiprime ideal of R

(ii) There is a (fixed) $s \in S$ and $I^k J \subseteq P$ for some ideals I, J of R and $k \in \mathbb{N}$ implying that $sIJ \subseteq P$

Proposition 4. *Let M be an R-module and S be a m.c.s of R. Suppose that P is a submodule of M with $(P\colon {}_RM) \cap S = \varnothing$. The following statements hold:*

(i) If P is an S-semiprime submodule of M, then $(P\colon {}_RM)$ is an S-semiprime ideal of R

(ii) If M is a multiplication module and $(P:_R M)$ is an S-semiprime ideal of R, then P is an S-semiprime submodule of M

Proof

(i) Let $x^k \in (P:_R M)$ for some $x \in R$ and $k \in \mathbb{N}$. Then, we have $x^k m \in P$ for each $m \in M$. Since P is an S-semiprime submodule, we conclude that $sxm \in P$, and this yields that $sx \in (P:_R M)$. Therefore, $(P:_R M)$ is an S-semiprime ideal of R.

(ii) Assume that M is a multiplication module and $(P:_R M)$ is an S-semiprime ideal of R. Let $J^k N \subseteq P$ for some ideal J of R, submodule N of M, and $k \in \mathbb{N}$. Then, we conclude that $J^k(N:_R M) \subseteq (J^k N:_R M) \subseteq (P:_R M)$. Also, note that, by Corollary 1, there exists a fixed $s \in S$ such that $sJ(N:_R M) \subseteq (P:_R M)$. Since M is a multiplication module, we have $sJN = sJ(N:_R M)M \subseteq (P:_R M)M = P$. Then, by Lemma 1, P is an S-semiprime submodule of M. □

Corollary 2. *Suppose that P is a submodule of a multiplication R-module M and S is a m.c.s of R such that $(P:_R M) \cap S = \emptyset$. Then, the following statements are equivalent:*

(i) P is an S-semiprime submodule of M

(ii) There exists a fixed $s \in S$ such that $L^k N \subseteq P$ for some submodules L, N of M and $k \in \mathbb{N}$ implying $sLN \subseteq P$

Proof

(i) $\Longrightarrow$ (ii): suppose that P is an S-semiprime submodule of M. Let $L^k N \subseteq P$ for some submodules L, N of M and $k \in \mathbb{N}$. Since M is a multiplication module, $L = IM$ and $K = JM$ for some ideal I, J of R. Also, note that $L^k N = I^k JM = I^k N \subseteq P$. Since P is an S-semiprime submodule, by Lemma 1, there exists $s \in S$ such that $sIN \subseteq P$, and this yields that $sIJM = sLN \subseteq P$.

(ii) $\Longrightarrow$ (i): suppose that $J^k N \subseteq P$ for some ideal J of R, submodule N of M, and $k \in \mathbb{N}$. Then, we have $J^k N = J^k(N:_R M)M$. Now, put $L = JM$, and note that $L^k = J^k M$. This implies that $J^k N = J^k(N:_R M)M = L^k N \subseteq P$. Then, by assumption, there exists a fixed $s \in S$ such that $sLN = sJ(N: M)M = sJN \subseteq P$. Then, by Lemma 1, P is an S-semiprime submodule of M. □

Theorem 1. *Let P be a submodule of a finitely generated multiplication R-module M and S be a m.c.s of R with $(P:_R M) \cap S = \emptyset$. The following statements are equivalent:*

(i) P is an S-semiprime submodule of M

(ii) $(P:_R M)$ is an S-semiprime ideal of R

(iii) $P = IM$ for some S-semiprime ideal I of R with $\mathrm{ann}(M) \subseteq I$

Proof

(i) $\Longrightarrow$ (ii): follows from Proposition 4 (i).

(ii) $\Longrightarrow$ (iii): it is straightforward.

(iii) $\Longrightarrow$ (i): suppose that $P = IM$ for some S-semiprime ideal I of R with $\mathrm{ann}(M) \subseteq I$. Assume that $J^k N \subseteq P$ for some ideal J of R, some submodule N of M, and $k \in \mathbb{N}$. Then, we obtain $J^k(N:_R M)M \subseteq IM$. As M is a finitely generated multiplication module, by [14], Theorem 9 Corollary, we have $J^k(N:_R M) \subseteq I + \mathrm{ann}(M) = I$. Since I is an S-semiprime ideal of R, by Corollary 1, for a fixed $s \in S$ such that $sJ(N:_R M) \subseteq I$, this yields that $sJN = sJ(N:_R M)M \subseteq IM = P$. Then, by Lemma 1, P is an S-semiprime submodule of M. □

Proposition 5. *Let $h: M \longrightarrow M'$ be an R-homomorphism. The following statements are satisfied:*

(i) If P' is an S-semiprime submodule of M' such that $(h^{-1}(P'):_R M) \cap S = \emptyset$, then $h^{-1}(P')$ is an S-semiprime submodule of M

(ii) If h is an epimorphism and P is an S-semiprime submodule of M such that $\mathrm{Ker}(h) \subseteq P$, then $h(P)$ is an S-semiprime submodule of M'

Proof

(i) Let $r^n m \in h^{-1}(P')$ for some $r \in R, m \in M$, and $n \in \mathbb{N}$. Then, we conclude that $h(r^n m) = r^n h(m) \in P'$. Since P' is an S-semiprime submodule, we have $srh(m) = h(srm) \in P'$ for some $s \in S$. Then, we get $srm \in h^{-1}(P')$, and thus, $h^{-1}(P')$ is an S-semiprime submodule of M.

(ii) Let $r^n m' \in h(P)$ for some $r \in R, m' \in M'$, and $n \in \mathbb{N}$. As h is an epimorphism, we can write $m' = h(m)$ for some $m \in M$, and so, $r^n m' = r^n h(m) = h(r^n m) \in h(P)$. Since $\mathrm{Ker}(h) \subseteq P$, we conclude that $r^n m \in P$. Since P is an S-semiprime submodule, there exists $s \in S$ such that $srm \in P$, and so, we obtain $h(srm) = srh(m) = srm' \in h(P)$. Consequently, $h(P)$ is an S-semiprime submodule of M'. □

Corollary 3. *Suppose that S is a m.c.s of R and L is a submodule of M. Then, the following statements are satisfied:*

(i) If P' is an S-semiprime submodule of M with $(P':_R L) \cap S = \emptyset$, then $L \cap P'$ is an S-semiprime submodule of L.

(ii) Suppose that P is a submodule of M with $L \subseteq P$. Then, P is an S-semiprime submodule of M if and only if P/L is an S-semiprime submodule of M/L.

Proof

(i) Consider the injection $i: L \longrightarrow M$ defined by $i(m) = m$ for all $m \in L$. Then, note that $i^{-1}(P') = L \cap P'$.

Now, we will show that $(i^{-1}(P')\colon {}_R L)\cap S=\varnothing$. Assume that $s\in (i^{-1}(P')\colon {}_R L)\cap S$. Then, we have $sL\subseteq i^{-1}(P')=L\cap P'\subseteq P'$, and thus, $s\in (P'\colon {}_R L)\cap S$, a contradiction. By Proposition 5 (i), we can say that $L\cap P'$ is an S-semiprime submodule of L.

(ii) Assume that P is an S-semiprime submodule of M. Consider the natural epimorphism $\pi\colon M\longrightarrow M/L$, defined by $\pi(m)=m+L$, for all $m\in M$. By Proposition 5 (ii), P/L is an S-semiprime submodule of M/L. For the converse, let P/L be an S-semiprime submodule of M/L. Take $r\in R$ and $m\in M$ with $r^k m\in P$ for some $k\in\mathbb{N}$. Then, we get $r^k(m+L)=r^k m+L\in P/L$. Since P/L is an S-semiprime submodule of M/L, for a fixed $s\in S$, we conclude that $sr(m+L)=srm+L\in P/L$. This implies that $srm\in P$, and hence, P is an S-semiprime submodule of M.

Let M_i be an R_i module and S_i be a m.c.s of R_i for each $i=1,2,\ldots,n$, where $n\in\mathbb{N}$. Suppose that $M=M_1\times M_2\times\cdots\times M_n$, $R=R_1\times R_2\times\cdots\times R_n$ and $S=S_1\times S_2\times\cdots\times S_n$. Then, M is an R-module with componentwise addition and scalar multiplication, and note that S is a m.c.s of R. Also, each submodule P of M has the form $P=P_1\times P_2\times\cdots\times P_n$, where P_i is a submodule of M_i for each $i=1,2,\ldots,n$. □

Theorem 2. *Suppose that M_i is an R_i-module, P_i is a submodule of M_i, and S_i is a m.c.s of R_i for each $i=1,2$. Let $M=M_1\times M_2$, $R=R_1\times R_2$ and $S=S_1\times S_2$. The following statements are equivalent for $P=P_1\times P_2$:*

(i) P is an S-semiprime submodule of M.

(ii) P_1 is an S_1-semiprime submodule of M_1 and $(P_2\colon {}_{R_2}M_2)\cap S_2\neq\varnothing$ or $(P_1\colon {}_{R_1}M_1)\cap S_1\neq\varnothing$, and P_2 is an S_2-semiprime submodule of M_2 or P_i is an S_i-semiprime submodule of M_i for each $i=1,2$.

Proof.

(i)$\Longrightarrow$(ii): let P be an S-semiprime submodule of M. Then, we have $(P\colon {}_R M)\cap S=\varnothing$, and this yields that $(P_1\colon {}_{R_1}M_1)\cap S_1=\varnothing$ or $(P_2\colon {}_{R_2}M_2)\cap S_2=\varnothing$. Without loss of generality, we may assume that $(P_2\colon {}_{R_2}M_2)\cap S_2=\varnothing$ and $(P_1\colon {}_{R_1}M_1)\cap S_1\neq\varnothing$. We must show that P_2 is an S_2-semiprime submodule of M_2. To prove this, take $r\in R_2$ and $m\in M_2$ such that $r^k m\in P_2$ for some $k\in\mathbb{N}$. Then, $(1,r)^k(0,m)=(0,r^k m)\in P$. Since P is an S-semiprime submodule of M, there exists a fixed $s=(s_1,s_2)\in S$ such that $s(1,r)(0,m)=(0,s_2rm)\in P$. This implies that $s_2rm\in P_2$. Hence, P_2 is an S_2-semiprime submodule of M_2. One can similarly show that if $(P_2\colon {}_{R_2}M_2)\cap S_2\neq\varnothing$, then P_1 is an S_1-semiprime submodule of M_1. Also, if $(P_1\colon {}_{R_1}M_1)\cap S_1=(P_2\colon {}_{R_2}M_2)\cap S_2=\varnothing$, then a similar argument shows that P_i is an S_i-semiprime submodule of M_i for each $i=1,2$.

(ii)$\Longrightarrow$(i): let $(P_1\colon {}_{R_1}M_1)\cap S_1\neq\varnothing$ and P_2 be an S_2-semiprime submodule of M_2. Then, we have $s_1\in(P_1\colon {}_{R_1}M_1)\cap S_1$. Let $(r_1,r_2)^n(m_1,m_2)=(r_1^n m_1, r_2^n m_2)\in P$ for some $r_i\in R_i$ and $m_i\in M_i$, where $i=1,2$. This implies that $r_2^n m_2\in P_2$, and so, $s_2r_2m_2\in P_2$ for a fixed $s_2\in S_2$ since P_2 is an S_2-semiprime submodule of M_2. Now, take $s=(s_1,s_2)\in S$. Then, $s(r_1,r_2)(m_1,m_2)=(s_1r_1m_1,s_2r_2m_2)\in P$. Similarly, we can show that P is an S-semiprime submodule of M in other cases. □

Theorem 3. *Let $n\geq 1$, M_i be an R_i module and S_i be a m.c.s of R_i for each $i=1,2,\ldots,n$. Suppose that $M=M_1\times M_2\times\cdots\times M_n$, $R=R_1\times R_2\times\cdots\times R_n$, and $S=S_1\times S_2\times\cdots\times S_n$. Let $P=P_1\times P_2\times\cdots\times P_n$, where P_i is a submodule of M_i for each $i=1,2,\ldots,n$. The following statements are equivalent:*

(i) P is an S-semiprime submodule of M

(ii) P_i is an S_i-semiprime submodule of M_i for each $i\in\{t_1,t_2,\ldots,t_k\colon 1\leq k\leq n\}$, and $(P_j\colon {}_{R_j}M_j)\cap S_j\neq\varnothing$ for each $j\in\{1,2,\ldots n\}-\{t_1,t_2,\ldots,t_k\}$

Proof. We use mathematical induction to prove the claim (i)$\Longleftrightarrow$(ii). For $n=1$, the result is clear. If $n=2$, the claim (i)$\Longleftrightarrow$(ii) follows from Theorem 2. Suppose that (i) and (ii) are equivalent for each $k<n$. Now, we will show that the claim (i)$\Longleftrightarrow$(ii) is true for $k=n$. Let $P'=P_1\times P_2\times\cdots\times P_{n-1}$, $M'=M_1\times M_2\times\cdots\times M_{n-1}$ and also $S'=S_1\times S_2\times\cdots\times S_{n-1}$ and $R'=R_1\times R_2\times\cdots\times R_{n-1}$. Note that $P=P'\times P_n$, $M=M'\times M_n$ and also $S=S'\times S_n$, $R=R'\times R_n$. Then, by Theorem 2, P is an S-semiprime submodule of M if and only if P' is an S'-semiprime submodule of M' and $(P_n\colon {}_{R_n}M_n)\cap S_n\neq\varnothing$ or $(P'\colon {}_{R'}M')\cap S'\neq\varnothing$ and P_n is an S_n-semiprime submodule of M_n or P' is an S'-semiprime submodule of M', and P_n is an S_n-semiprime submodule of M_n. The rest follows from induction hypothesis. □

Theorem 4. *Let P be a submodule of M and S be a m.c.s of R such that $(P\colon {}_R M)\cap S=\varnothing$. Then, P is an S-semiprime submodule of M if and only if $(P\colon {}_M s')$ is a semiprime submodule of M for some $s'\in S$.*

Proof. Let P be an S-semiprime submodule of M, and put $s'=s^2\in S$. Now, we will show that $(P\colon {}_M s')$ is a semiprime submodule of M. Let $a^n m\in(P\colon {}_M s')$ for some $a\in R$, $m\in M$ and $n\in\mathbb{N}$. Then, we get $s'a^n m=s^2a^n m\in P$. If $n=1$, then we have $am\in(P\colon {}_M s\prime)$. So, assume that $n\geq 2$. Then, clearly we have $s^n a^n m=(sa)^n m\in P$, and this gives $s(sa)m=s^2am\in P$. Then, we conclude that $am\in(P\colon {}_M s^2)=(P\colon {}_M s')$. Hence, $(P\colon {}_M s')$ is a semiprime submodule of M. Conversely, assume that $(P\colon {}_M s')$ is a semiprime submodule of M for some $s'\in S$. Let $a^n m\in P$ for some $a\in R$, $m\in M$, and $n\in\mathbb{N}$. Since $(P\colon {}_M s')$ is a semiprime submodule and $a^n m\in P\subseteq(P\colon {}_M s')$, we conclude that $am\in(P\colon {}_M s')$, and hence, $s'am\in P$. Therefore, P is an S-semiprime submodule of M. □

Theorem 5. *Let P be a submodule of M such that $(P\colon {}_R M)\subseteq Jac(R)$, where $Jac(R)$ is the Jacobson radical of R. The following statements are equivalent:*

(i) P is a semiprime submodule of M

(ii) P is an $S_m = (R - m)$-semiprime submodule of M for each maximal ideal m of R

Proof

(i) $\Longrightarrow$ (ii): suppose that P is a semiprime submodule of M. Then, by Proposition 1, P is an $S_m = (R-m)$-semiprime submodule of M for each maximal ideal m of R.

(ii) $\Longrightarrow$ (i): let $r^n m \in P$ for some $r \in R, m \in M$, and $n \in \mathbb{N}$. As P is an $S_m = (R - m)$-semiprime submodule of M, there exists an $s_m \notin m$ such that $s_m rm \in P$. Now, consider the set $\Omega = \{s_m\colon \exists m \in \mathrm{Max}(R), s_m \notin m$ such that $s_m rm \in P\}$. Now, we will show that $\langle\Omega\rangle = R$. Assume that $\langle\Omega\rangle \neq R$. Then, there exists a maximal ideal m^* of R containing Ω. By the definition of Ω, there exists $s_{m^*} \in \Omega$ such that $s_{m^*} \notin m^*$. Since $\Omega \subseteq \langle\Omega\rangle \subseteq m^*$, we have $s_{m^*} \in m^*$ which is a contradiction. Thus, $\langle\Omega\rangle = R$, and so, there exists $s_{m_1}, s_{m_2}, \ldots, s_{m_n} \in \Omega$ such that $1 = x_1 s_{m_1} + x_2 s_{m_2} + \cdots + x_n s_{m_n}$ for some $x_1, x_2, \ldots, x_n \in R$. Since $s_{m_i} rm \in P$ for each $i = 1, 2, \ldots, n$, we conclude that $rm = x_1(s_{m_1} rm) + x_2(s_{m_1} rm) + \cdots + x_n(s_{m_n} rm) \in P$. Therefore, P is an S-semiprime submodule of M.

As immediate consequence of the previous theorem, we give the following result. □

Corollary 4. *Let M be a module over a quasi-local ring (R, m). Suppose that P is a submodule of M. The following statements are equivalent:*

(i) P is a semiprime submodule of M

(ii) P is an $S_m = (R - m)$-semiprime submodule of M

Definition 2. Let M be an R-module and S be a m.c.s of R. M is said to be an S-reduced module if there exists $s \in S$, and whenever $r^n m = 0$, where $r \in R, m \in M$, and $n \in \mathbb{N}$, then $srm = 0$.

Proposition 6. *Suppose that M is an R-module and S is a m.c.s of R. The following statements are satisfied:*

(i) If M is a reduced module, then M is an S-reduced module. In particular, the converse holds if $S \subseteq R - z(M)$, where $z(M) = \{x \in R\colon \mathrm{ann}_M(x) \neq 0_M\}$.

(ii) If M is an S-torsion-free module, then M is an S-reduced module.

(iii) M is an S-reduced module if and only if the zero submodule is an S-semiprime submodule.

(iv) Let P be a submodule of M with $(P\colon_R M) \cap S = \varnothing$. Then, P is an S-semiprime submodule if and only if R-module M/P is an S-reduced module.

(v) If M is an S-reduced module, then $S^{-1}M$ is a reduced module.

Proof

(i) The claim "reduced module implies the S-reduced module" is obvious. Let M be an S-reduced module such that $S \subseteq R - z(M)$. Let $a^n m = 0$ for some $a \in R$, $m \in M$, and $n \in \mathbb{N}$. Since M is an S-reduced module, there exists $s \in R - z(M)$ such that $sam = 0$. As $\mathrm{ann}_M(s) = 0_M$, we have $am \in \mathrm{ann}_M(s) = 0$, and so, $am = 0$. Hence, M is a reduced module.

(ii) Let M be an S-torsion-free module and $a^n m = a(a^{n-1}m) = 0$ for some $a \in R, m \in M$, and $n \in \mathbb{N}$. Since M is an S-torsion-free module, there exists $s \in S$ such that $sa = 0$ or $sa^{n-1}m = 0$. If $sa = 0$, then $sam = 0$ which completes the proof. So, assume that $sa \neq 0$. Since $sa^{n-1}m = a(sa^{n-2}m) = 0$, we conclude that $s(sa^{n-2}m) = s^2 a^{n-2}m = 0$ since M is an S-torsion-free module. If we continue in the previous way, we conclude that $s^n m = 0$. Since M is an S-torsion-free module, we get either $s^{n+1} = 0$ or $sm = 0$. If $s^{n+1} = 0$, then $s^{n+1} = 0 \in S$, which is a contradiction so that $sm = 0$, and this yields $sam = 0$.

(iii) It follows from Definitions 1 and 2.

(iv) It follows from (iii).

(v) Let M be an S-reduced module. Then by (iii), 0 is an S-semiprime submodule of M. Again by Proposition 2 (iii), $S^{-1}0$ is a semiprime submodule of $S^{-1}M$. Thus, $S^{-1}M$ is a reduced module.

Now, we will characterize reduced modules in terms of S-reduced modules. □

Theorem 6. *The following statements are equivalent for any R-module M:*

(i) M is a reduced module

(ii) M is an $S_{\wp} = (R - \wp)$-reduced module for each $P \in Spec(R)$

(iii) M is an $S_m = (R - m)$-reduced module for each $m \in Max(R)$

Proof

(i) $\Longrightarrow$ (ii): it follows from Proposition 6.

(ii) $\Longrightarrow$ (iii): it follows from the fact that $\mathrm{Max}(R) \subseteq \mathrm{Spec}(R)$.

(iii) $\Longrightarrow$ (i): let M be an $S_m = (R - m)$-reduced module for each $m \in \mathrm{Max}(R)$. Choose $a \in R$ and $m \in M$ such that $a^2 m = 0$. Since M is an $S_m = (R - m)$-reduced module for each $m \in \mathrm{Max}(R)$, there exists $s_m \notin m$ such that $s_m(am) = 0$. Now, consider the set $\Omega = \{s_m\colon \exists m \in \mathrm{Max}(R), s_m \notin m$ and $s_m(am) = 0\}$. Note that Ω is not empty since M is an $S_m = (R - m)$-reduced module. Similar argument in Theorem 5 shows that $\langle\Omega\rangle = R$, and so, there exists $s_1, s_2, \ldots, s_n \in \Omega$ such that $r_1 s_1 + r_2 s_2 + \cdots + r_n s_n = 1$ and $s_i(am) = 0$. This yields that $am = (r_1 s_1 + r_2 s_2 + \cdots + r_n s_n)(am) =$

$r_1(s_1am) + r_2(s_2\ am) + \cdots + r_n(s_nam) = 0$. Hence, M is a reduced module. □

3. Conclusion

This paper is mainly concerned with S-semiprime submodules of modules over commutative rings. We first investigate some properties of S-semiprime submodules similar to semiprime submodules. Then, we introduce S-reduced modules and give some new characterizations of semiprime submodules and reduced modules in terms of these concepts.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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